Lecture 10.
Nonconvergence of Evolutionary Dynamics
In this lecture, we shall further explore the conditions under which an evolutionary dynamic is locally stable.

Once again we shall study the linear approximation to a (non-linear) evolutionary dynamic around a rest point $x^*$. We will be particularly interested in the conditions under which this linear dynamic does not converge to $x^*$ from states in the neighborhood of $x^*$.

This will essentially be a review of some standard results for systems of linear ordinary differential equations.
We shall then:

- Explore a method for studying the long-run behavior of nonconvergent dynamics,
- Define a class of games in which nonconvergence is bound to occur, and
- Introduce the concept of chaotic dynamics.
Linear Approximations around Rest Points

- The single-population dynamic $\dot{x} = V(x)$, which we shall refer to as (D), describes the evolution of the population state through the simplex $X$.

- Recall that near $x^*$, the dynamic (D) can typically be well approximated by the linear dynamic:

  $$\dot{z} = DV(x^*)z,$$  

  $(L)$ in a neighborhood of the origin.

- Note that (L) is a dynamic on the tangent space $TX$. 
Eigenvalues and Stability

- (L) approximates the motion of deviations from $x^*$ following a small displacement $z$.

- To check for local stability of $x^*$ under (D), we need to check whether the origin is stable under (L).

- The stability of the origin under (L) is completely determined by the eigenvalues of the Jacobian matrix $DV(x^*)$:
  - The origin is stable if all the eigenvalues have negative real part.
  - The origin is unstable if at least one eigenvalue has positive real part.
  - The origin is a saddle if some of each.
Note that if $DV(x^*)$ is positive definite, i.e. $z'DV(x^*)z > 0$ for all nonzero $z \in TX$, then all eigenvalues have positive real part.

—Therefore, $x^*$ is unstable; all solutions that start near $x^*$ are repelled.

If $DV(x^*)$ is negative definite, i.e. $z'DV(x^*)z < 0$ for all nonzero $z \in TX$, then all eigenvalues have negative real part and the opposite is true.
To fully characterize the behavior of (L), we need to first reduce it to a simpler form.

Consider an arbitrary matrix $A \in \mathbb{R}^{n \times n}$. We can find a matrix with the same eigenvalues as $A$, i.e. a matrix that is similar to $A$, but is easier to work with.

The matrix $A$ is similar to matrix $B \in \mathbb{R}^{n \times n}$ if there exists an invertible matrix $S \in \mathbb{C}^{n \times n}$, called a similarity matrix, such that:

$$B = S^{-1}AS.$$
Real Jordan Matrices

- We shall now find a simple class of matrices with the property that every matrix is similar to a unique representative from this class.

- A real Jordan matrix is block diagonal matrix whose diagonal blocks—Jordan blocks—are of the following four types:

\[
J_1 = (\lambda); \quad J_2 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix};
\]

\[
J_3 = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}; \quad J_4 = \begin{pmatrix} J_2 & I & 0 & 0 & 0 \\ 0 & J_2 & I & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & J_2 & I \\ 0 & 0 & 0 & 0 & J_2 \end{pmatrix}.
\]
Theorem 10.1. Every matrix $A \in \mathbb{R}^{n \times n}$ is similar to a real Jordan matrix $J = S^{-1}AS$. The latter is unique up to an ordering of the Jordan blocks.

- Each $J_1$ block corresponds to a real eigenvalue $\lambda$.

- Each $J_2$ block corresponds to a pair of complex eigenvalues $a \pm ib$.

- Each $J_3$ block corresponds to a real eigenvalue with less than full geometric multiplicity (i.e. $\lambda$ corresponds to at least two eigenvectors that are not linearly independent).

- Each $J_4$ block corresponds to a pair of complex eigenvalues with less than full geometric multiplicities.
Linear Dynamics on the Plane

There are three generic types of $2 \times 2$ matrices:

1. Diagonalizable matrices with two real eigenvalues—their real Jordan form is a diagonal matrix containing two $J_1$ blocks.

2. Diagonalizable matrices with two complex eigenvalues—their real Jordan form is a $J_2$ matrix.

3. Nondiagonalizable matrices with one real eigenvalue—their real Jordan form is a $J_3$ matrix.
1. When $A$ has two real eigenvalues, $\dot{z} = Az$ and its solution from initial condition $z_0 = \xi$ are of the following form:

$$\dot{z} = Az = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}; \quad z_t = \begin{pmatrix} \xi_1 e^{\lambda t} \\ \xi_2 e^{\mu t} \end{pmatrix}.$$

- If $\lambda$ and $\mu$ are both negative, then the origin is a stable node,
- If both are positive, then the origin is an unstable node, and
- If the signs differ, then the origin is a saddle.
2. When $A$ has two complex eigenvalues $a \pm ib$, then:

\[
\dot{z} = Az = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}; \quad z_t = \begin{pmatrix} \xi_1 e^{at} \cos bt + \xi_2 e^{at} \sin bt \\ \xi_1 e^{at} \sin bt + \xi_2 e^{at} \cos bt \end{pmatrix}.
\]

The stability of the origin is determined by the real part of the eigenvalues:

- If $a < 0$, then the origin is a *stable spiral*,
- If $a > 0$, then the origin is an *unstable spiral*, and
- If $a = 0$, then the origin is a *center*, with each solution following a closed orbit around the origin.
Linear Dynamics on the Plane

3. When $A$ has lone eigenvalue $\lambda$:

$$\dot{z} = Az = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}; \quad z_t = \begin{pmatrix} \xi_1 e^{\lambda t} + \xi_2 te^{\lambda t} \\ \xi_2 e^{\lambda t} \end{pmatrix}.$$

The origin is:

- stable if $\lambda < 0$,
- unstable if $\lambda > 0$. 

Characterizing Long-Run Behavior of Nonconvergent Dynamics

- This is often an impossible task.

- But as we have seen, the replicator dynamic in certain contexts has useful conservative properties which allow us to characterize its long-run behavior even when the dynamic does not converge.

- In particular, we have seen that in null stable games, all interior solutions of the replicator dynamic preserve the value of the strict Lyapunov function:

\[ h_{x^*}(x) = \sum_{i \in S} x_i^* \log \frac{x_i^*}{x_i}. \]
Characterizing Long-Run Behavior of Nonconvergent Dynamics

- We know that standard RPS is a null stable game (good RPS is strictly stable and bad RPS is unstable).

- Let $x^*$ be the unique Nash equilibrium $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

- Then:

$$h_{x^*}(x) = \sum_{i \in S} \frac{1}{3} \log \frac{1}{x_i}$$

$$= \frac{1}{3} \sum_{i \in S} \left[ \log(1/3) - \log(x_i) \right]$$

$$= \log(1/3) - \frac{1}{3} \sum_{i \in S} \log(x_i)$$

$$= \log(1/3) - \frac{1}{3} \log(1/3) \log(x_1x_2x_3).$$

(1)
Characterizing Long-Run Behavior of Nonconvergent Dynamics

Therefore, if every solution trajectory preserves $h_{x^*}(x)$, then it preserves $x_1x_2x_3$ (an affine transformation of $h_{x^*}(x)$).

That is, the level sets of $x_1x_2x_3$ form closed orbits around $x^*$.
Convergence of Time Averages

- Even if the process itself does not converge, the average population share over time for each strategy $i \in S$ could converge to its Nash equilibrium share.

- Let the average value of the state over the time interval $[0, t]$ be:

$$\bar{x}_t = \frac{1}{t} \int_0^t x_s ds.$$ 

- One can show that in standard RPS under the replicator dynamic, $\{\bar{x}_t\}_{t \geq 0}$ converges to the Nash equilibrium $x^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ as $t$ approaches infinity:

$$\lim_{t \to \infty} |\bar{x}_t - x^*| = 0.$$
Games with Nonconvergent Dynamics

- We have focussed on RPS in much of our discussion so far, but we can generalize these insights to a broader class of games in which convergence can fail, called **circulant games** of which RPS is a member.

- The matrix $A \in \mathbb{R}^{n \times n}$ is called a **circulant matrix** if it is of the form:

$$
A = \begin{pmatrix}
    a_1 & a_2 & \cdots & a_{n-1} & a_n \\
    a_n & a_1 & a_2 & \cdots & a_{n-1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_3 & \cdots & a_n & a_1 & a_2 \\
    a_2 & a_3 & \cdots & a_n & a_1
\end{pmatrix}.
$$
Circulant Games

- When $A$ is a payoff matrix for a symmetric normal form game, then $A$ is a circulant game.

- The barycenter $x^* = \frac{1}{n}1$ is always in the set of Nash equilibria of such games.

- RPS is a circulant game with $n = 3, a_1 = 0, a_2 = -\ell$, and $a_3 = w$. 
The $\omega$ limit sets we have focused on are fairly simple, mainly rest points and closed orbits of a dynamic.

In one-dimensional systems, all continuous-time dynamics converge to equilibrium.

In two dimensional systems rest points, closed orbits, chains of rest points and connecting orbits exhaust the possibilities.
For flows in three or more dimensions, however, $\omega$-limit sets can be complicated sets known as **chaotic (or strange) attractors**.

In addition, chaotic dynamics are defined by sensitive dependence on initial conditions:

—solution trajectories starting from nearby points on the attractor move apart at an exponential rate.