Evolution & Learning in Games
Econ 243B

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Lecture 2.
Foundations of
Evolution & Learning in Games II
In this lecture, we shall:

- Take a first look at local stability. In particular, we shall define an *evolutionary stable state* and explore its relationship to Nash equilibrium.

- Apply what we have learned so far to analyze iterated play of the prisoners’ dilemma.
Evolutionary Stable States (ESS)

Maynard Smith and Price (1973) defined the notion of an evolutionary stable strategy:

- Their focus was on monomorphic populations: every member plays the same strategy, which can be a mixed strategy.

- We are concerned with a polymorphic population of agents each programmed with a pure strategy.
Evolutionary Stable States (ESS)

Mathematically these problems are identical. For example:

\[ F_1(x) = x_1 u(1, 1) + x_2 u(1, 2) = u(1, x). \]

Being randomly matched with a population of agents proportion \( x_1 \) of which are programmed with pure strategy 1 and \( x_2 \) of which are programmed with pure strategy 2 is the same as playing a single individual who plays strategy 1 with probability \( x_1 \) and strategy 2 with probability \( x_2 \).
Evolutionary Stable States (ESS)

Hence we can adapt the concept of an evolutionary stable strategy to a population setting:

- The term we shall use is evolutionary stable state (ESS).

As we shall eventually see, ESS provides a sufficient condition for local stability under a wide range of evolutionary dynamics, including the replicator dynamic.
Invasion

Let the state be $x = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$.

Consider a game $F$, where $F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ \ldots \\ F_n(x) \end{pmatrix}$.

Consider an invasion of mutants who make up a fraction $\varepsilon$ of the post-entry population.

The shares of each strategy in the mutant population are represented by $y = \begin{pmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{pmatrix}$.
Therefore, the post-entry population state is:

\[ x_\varepsilon = (1 - \varepsilon)x + \varepsilon y = \begin{pmatrix} (1 - \varepsilon)x_1 + \varepsilon y_1 \\ (1 - \varepsilon)x_2 + \varepsilon y_2 \\ \vdots \\ (1 - \varepsilon)x_n + \varepsilon y_n \end{pmatrix}. \]

The average payoff in the incumbent population in the post-entry state is \( x'F((1 - \varepsilon)x + \varepsilon y) \).

The average payoff in the mutant population in the post-entry state is \( y'F((1 - \varepsilon)x + \varepsilon y) \).
Uniform Invasion Barrier

The average payoff in the incumbent population is higher if:

\[(y - x)'F((1 - \varepsilon)x + \varepsilon y) < 0.\]  \hspace{1cm} (1)

State \(x\) is said to admit a **uniform invasion barrier** if there exists an \(\bar{\varepsilon} > 0\) such that (1) holds for all \(y \in X - \{x\}\) and \(\varepsilon \in (0, \bar{\varepsilon})\).

That is, for all possible mutations \(y\), as long as the mutant population is less than fraction \(\bar{\varepsilon}\) of the postentry population, the incumbent population receives a higher average payoff.
Invasion Barriers

Define the invasion barrier of $x$ against $y$ as:

$$b_x(y) = \inf \left( \{ \varepsilon \in (0, 1) : (y - x)'F((1 - \varepsilon)x + \varepsilon y) \geq 0 \} \cup \{1\} \right).$$

(2)

If $x$ admits a uniform invasion barrier, then there exists an $\bar{\varepsilon} > 0$ such that $b_x(y) \geq \bar{\varepsilon} > 0$ for all $y \in X - \{x\}$. 
**ESS**

**Definition.** State \( x \in X \) is an **evolutionary stable state** (ESS) of \( F \) if there exists a neighborhood \( O \) of \( x \) such that:

\[
(y - x)'F(y) < 0 \quad \text{for all } y \in O - \{x\}. \tag{3}
\]

In other words, if \( x \) is an ESS, then for any state \( y \) sufficiently close to \( x \), a population playing \( x \) will receive a larger average payoff in state \( y \) than a population playing \( y \) (i.e. \( x \) is a better reply to \( y \) than \( y \) is to itself).

Note that this considers invasions of other states \( y \) by \( x \) rather than invasions of \( x \) by other states. Hence it is not clear, at present, why this should be a stability condition.
Theorem 2.1. State $x \in X$ is an evolutionary stable state (ESS) if and only if it admits a uniform invasion barrier.

Thus if $x$ is stable in the face of an arbitrarily large population of entrants who mutate to a nearby state, then it is stable in the face of a sufficiently small population of entrants who mutate to an arbitrary state.
What is the relationship between ESS and NE?

**Definition.** Suppose that $x \in X$ is a NE. Then $(y - x)'F(x) \leq 0$ for all $y \in X$.

In addition, suppose there exists a neighborhood of $x$ that does not contain any other NE.

Then $x$ is an isolated NE.

**Proposition 2.1.** Every ESS is an isolated NE.
Proof

Let $x$ be an ESS of $F$, $O$ be the nhd posited in (3) and $y \in X - \{x\}$ (not necessarily in $O$).

Then for all $\varepsilon > 0$ sufficiently small, the postentry state $x_\varepsilon = \varepsilon y + (1 - \varepsilon)x$ is in $O$.

This implies that:

$$(x_\varepsilon - x)'F(x_\varepsilon) < 0$$

$$(\varepsilon y + (1 - \varepsilon)x - x)'F(x_\varepsilon) < 0$$

$$\varepsilon(y - x)'F(x_\varepsilon) < 0$$

$$(y - x)'F(x_\varepsilon) < 0.$$  

(4)
Proof

Taking $\varepsilon \to 0$ yields:

$$(y - x)'F(x) \leq 0.$$  

That is, $x$ is a NE.

To establish that $x$ is isolated, note that if $w \in O - \{x\}$ were a NE then $(w - x)'F(w) \geq 0$, contradicting the supposition that $x$ is an ESS [by (3)]. □

The converse of Proposition 2.1 is not true.

- The mixed equilibrium of a two-strategy coordination game is a counterexample.
Therefore, *ESS is stronger than NE.*

In particular, an ESS satisfies the additional property:

Suppose there exists a state $y$ which is an alternative best reply to $x$, i.e. $(y - x)'F(x) = 0$.
—Then $(y - x)'F(y) < 0$, i.e. $x$ is a better reply to $y$ than $y$ is to itself.

Therefore:

- A strict NE is an ESS.
- A polymorphic population state (equivalent to a mixed NE) cannot be strict and hence must satisfy the additional property.
More on ESS and Nash

In the case in which agents are matched uniformly at random to play a normal form game (the case we have been focusing on), then it is easy to see why the additional property is required.

Suppose \((y - x)'F(x) = 0\), i.e. \(y\) is an alternative best reply to \(x\).

Then:

\[
(y - x)'F(\varepsilon y + (1 - \varepsilon)x) = \varepsilon(y - x)'F(y) + (1 - \varepsilon) (y - x)'F(x)
\]

\[
= \varepsilon(y - x)'F(y).
\]

(5)

Therefore, \((y - x)'F(y)\) must be negative for (1) to hold and hence, by Theorem 2.1, for \(x\) to be an ESS.
Example: Hawk Dove

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<tr>
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<th>Hawk</th>
<th>Dove</th>
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<tr>
<td>Hawk</td>
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<td>0</td>
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<tr>
<td>Dove</td>
<td>4</td>
<td>0</td>
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ESS: $x = \left( \frac{2}{3}, \frac{1}{3} \right)$. ESS payoff = 0.
Example: Hawk Dove

- Consider a mutation \( y \) such that \( y_1 > x_1 = \frac{2}{3} \).
- Check that \((y - x)'F((1 - \varepsilon)x + \varepsilon y) < 0\) for all such \( y \):
  \[
  (y_1 - x_1)[-2((1 - \varepsilon)x_1 + \varepsilon y_1) + 4((1 - \varepsilon)(1 - x_1) + \varepsilon(1 - y_1))].
  \]
- This equals:
  \[
  (y_1 - x_1)\varepsilon[-2y_1 + 4(1 - y_1)]
  \]
  because \(-2 \times \frac{2}{3} + 4 \times (1 - \frac{2}{3}) = 0\). This in turn equals:
  \[
  (y_1 - x_1)\varepsilon[4 - 6y_1]
  \]
  which is negative because \( y_1 > \frac{2}{3} \) by hypothesis.
- A similar argument can be applied to the case \( y_1 < x_1 \). Hence \( x \) is an ESS.
The Prisoners’ Dilemma

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<th>C</th>
<th>D</th>
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<tbody>
<tr>
<td>C</td>
<td>3</td>
<td>5</td>
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<tr>
<td>D</td>
<td>5</td>
<td>1</td>
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**NE/ESS:** $x = (0, 1)$.

Therefore, an ESS is not necessarily efficient.
Not Every Game has an ESS

\[
x = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)
\] is the unique NE and therefore the only possible ESS.
Not Every Game has an ESS

Note that $x$ is a polymorphic population state (equivalent to a mixed strategy), so any basis vector (pure strategy) is an alternative best reply to $x$.

Check that the additional property holds: $(e_1 - x)'F(e_1) < 0$, where $e_1 = (1, 0, 0)$, i.e. the pure-strategy $A$.

This is not the case: $x'F(e_1) = e'_1 F(e_1) = 1$. 
The Iterated Prisoners’ Dilemma

► Two players engage in a series of PD games.

► The engagement ends after the current round with probability \( \delta < \frac{1}{2} \). We call this the stopping probability.

► The expected number of rounds per engagement is:

\[
1 + (1 - \delta) + (1 - \delta)^2 + (1 - \delta)^3 + \ldots = \frac{1}{1 - (1 - \delta)} = \frac{1}{\delta}.
\]

► Consider a population in which three strategies are present:

► \( C \)—always cooperate,

► \( D \)—always defect,

► \( T \)—tit-for-tat, i.e. start by cooperating, thenceforth cooperate in period \( t \) if partner cooperated in \( t - 1 \).
## Expected Payoffs Within Each Pairing

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<tbody>
<tr>
<td>C</td>
<td>$\frac{3}{\delta}$</td>
<td>0</td>
<td>$\frac{3}{\delta}$</td>
</tr>
<tr>
<td>D</td>
<td>$\frac{5}{\delta}$</td>
<td>$\frac{1}{\delta}$</td>
<td>$4 + \frac{1}{\delta}$</td>
</tr>
<tr>
<td>T</td>
<td>$\frac{3}{\delta}$</td>
<td>$\frac{1}{\delta} - 1$</td>
<td>$\frac{3}{\delta}$</td>
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**Note:**

Payoff from playing $T$ against $D$ is $0 + (1 - \delta)\frac{1}{\delta} = \frac{1}{\delta} - 1$.

Payoff from playing $D$ against $T$ is $5 + (1 - \delta)\frac{1}{\delta} = 4 + \frac{1}{\delta}$. 
Expected Payoffs Over All Pairings

\[ F_C(x) = \left( x_C + x_T \right)^{\frac{3}{\delta}} \]
\[ F_D(x) = x_C^{\frac{5}{\delta}} + x_D^{\frac{1}{\delta}} + x_T \left( 4 + \frac{1}{\delta} \right) \]
\[ F_T(x) = \left( x_C + x_T \right)^{\frac{3}{\delta}} + x_D \left( \frac{1}{\delta} - 1 \right) \]
Replicator Dynamics

\[
\frac{d}{dt} \left[ \frac{x_T}{x_C} \right] = \frac{x_T}{x_C} (F_T(x) - F_C(x)) \\
= \frac{x_T}{x_C} \left[ x_D \left( \frac{1}{\delta} - 1 \right) \right],
\]

which is positive because \( \delta < \frac{1}{2} \).

\[
\frac{d}{dt} \left[ \frac{x_T}{x_D} \right] = \frac{x_T}{x_D} (F_T(x) - F_D(x)) \\
= \frac{x_T}{x_D} \left[ - x_C \frac{2}{\delta} - x_D + x_T \left( \frac{2}{\delta} - 4 \right) \right],
\]

which is positive for \( x_T \) sufficiently large.
Vector Field
All-\(T\) is not an ESS

Let \(x = (x_D, x_C, x_T) = (0, 0, 1)\).

Consider any alternative state \(y\) such that \(y_D = 0\).

\[
(y - x)'F(y) = (0 \ y_C \ y_T - 1) \begin{pmatrix} F_D(y) \\ F_C(y) \\ F_T(y) \end{pmatrix}
\]
\[
= y_C F_C(y) + (y_T - 1) F_T(y)
= [y_C + (y_T - 1)] \frac{3}{\delta} \quad \text{(recall that } y_D = 0) \\
= [y_C + (1 - y_C - 1)] \frac{3}{\delta} \quad \text{(because } y_T = 1 - y_C) \\
= 0.
\]

This violates (3). Hence all-\(T\) is not an ESS.

This is a case of evolutionary drift.