Lecture 7.
Global Convergence of Evolutionary Dynamics I
Global Convergence

▶ Last week, we examined the connection between the rest points of various dynamics and Nash equilibria of the underlying game.

▶ This week we shall study the limiting behavior of various evolutionary dynamics when set in motion from arbitrary initial conditions.

▶ In particular, we shall derive conditions on games and dynamics under which behavior converges to equilibrium from all (or almost all) initial states.

▶ Our focus will be on potential, stable and supermodular games, though we will also touch upon dominance solvable games.

▶ Positive correlation (PC) will play some role in our out-of-equilibrium analysis.
Limit Sets

- Let us characterize the limiting behavior of deterministic dynamics as follows.

- The $\omega$-limit of trajectory $\{x_t\}_{t \geq 0}$ is the set of all points that the trajectory approaches arbitrarily closely infinitely often:

  $$\omega(\{x_t\}) = \left\{ y \in X : \text{there exists } \{t_k\}_{k=1}^\infty \text{ with } \lim_{k \to \infty} t_k = \infty \text{ such that } \lim_{k \to \infty} x_{t_k} = y \right\}.$$  

- If $\omega(\{x_t\}) = x^*$, a singleton, then $x^*$ is called an absorbing state.

- Otherwise, $\omega(\{x_t\})$ is called a recurrence class or $\omega$-limit set of the dynamic.
Limit Sets

- For dynamics that admit a unique forward solution trajectory from each initial condition, \( \omega(\xi) \) denotes the \( \omega \)-limit set of the trajectory starting from state \( \xi \).

- The set of all limit \( \omega \)-limit points of all solution trajectories is:

\[
\Omega(V_F) = \bigcup_{\xi \in X} \omega(\xi).
\]

- The notion of recurrence (or the set of recurrence classes) of deterministic dynamic is captured by \( \Omega(V_F) \).
Stability Concepts

- Let $A \subseteq X$ be a closed set, and call $O \subseteq X$ a neighborhood of $A$ if it is open relative to $X$ and contains $A$.

- $A$ is **Lyapunov stable** if for every neighborhood $O$ of $A$, there exists a neighborhood $O'$ of $A$ such that every solution $\{x_t\}$ that starts in $O'$ is contained in $O$, that is, $x_0 \in O'$ implies that $x_t \in O$ for all $t \geq 0$.

- Intuitively, this requires that all solutions that start near $A$, stay near $A$ at all points in time.

- Any displacement from $A$ does not lead the process to go ‘very far’ from $A$ at any point in time.
Stability Concepts

- **A is attracting** if there is a neighborhood $Y$ of $A$ such that every solution that starts in $Y$ converges to $A$, that is, $x_0 \in Y$ implies $\omega(\{x_t\}) \subseteq A$.

- **A is globally attracting** if it is attracting with $Y = X$.

- Intuitively, this requires that given any displacement from $A$, the process returns to $A$ in the limit.
Stability Concepts

- *A* is **asymptotically stable** if it is Lyapunov stable and attracting.

- *A* is **globally asymptotically stable** if it is Lyapunov stable and globally attracting.

- Intuitively, this requires that given any displacement from *A*, the process never travels ‘very far’ from *A* and returns to *A* in the limit.
Lyapunov Functions

The most common method for proving global convergence in dynamical systems is by constructing a **strict Lyapunov function**:

▶ A scalar-valued function.
▶ The value of the function changes monotonically along every solution trajectory.
▶ For many ODEs, the existence of a Lyapunov function is a necessary and sufficient condition for stability.
▶ The Lyapunov function allows us to (partially) characterize the evolution of play without requiring explicit solutions to the differential equation (or inclusion).

**Definition.** The $C^1$ function $L : X \rightarrow \mathbb{R}$ is a (decreasing) strict Lyapunov function for the differential equation $\dot{x} = V_F(x)$ if $\dot{L}(x) = \nabla L(x)'V_F(x) \leq 0$ for all $x \in X$, with equality only at rest points of $V_F$. 
Lyapunov Functions and Stability

**Theorem 7.1. (Lyapunov Stability)** Let $A \subseteq X$ be closed and let $Y \subseteq X$ be a neighborhood of $A$. Let $L : Y \to \mathbb{R}_+$ be Lipschitz continuous with $L^{-1}(0) = A$. If each solution $\{x_t\}$ of $V_F$ satisfies $\dot{L}(x_t) \leq 0$ for almost all $t \geq 0$, then $A$ is Lyapunov stable under $V_F$.

**Theorem 7.2. (Asymptotic Stability)** Let $A \subseteq X$ be closed and let $Y \subseteq X$ be a neighborhood of $A$. Let $L : Y \to \mathbb{R}_+$ be $C^1$ with $L^{-1}(0) = A$. If each solution $\{x_t\}$ of $V_F$ satisfies $\dot{L}(x_t) < 0$ for all $x \in Y - A$, then $A$ is asymptotically stable under $V_F$. If in addition, $Y = X$, then $A$ is globally asymptotically stable under $V_F$. 
Let us turn to global convergence in potential games.

In potential games, the natural candidate for a strict (increasing) Lyapunov function is the potential function.

Recall that in a potential game $F : X \to \mathbb{R}^n$, the potential function $f : X \to \mathbb{R}$ summarizes all information about incentives:

$$\nabla f(x) = \Phi F(x) \quad \text{for all } x \in X.$$
**Lemma 7.1.** Let $F$ be a potential game with potential function $f$. Suppose the evolutionary dynamic $\dot{x} = V_F(x)$ satisfies positive correlation (PC). Then $f$ is a strict Lyapunov function for $V_F$.

**Proof.**
$$\dot{f}(x) = \nabla f(x)' \dot{x} = (\Phi F(x))' V_F(x) = F(x)' V_F(x).$$

The result then follows immediately from PC. □
Theorem 7.3. Let $F$ be a potential game, and let $\dot{x} = V_F(x)$ be an evolutionary dynamic for $F$ that admits a unique forward solution from each initial condition and that satisfies PC. Then $\Omega(V_F) = RP(V_F)$.

For example, if $V_F$ is an imitative dynamic, then $\Omega(V_F) = RE(F)$, the set of restricted equilibria of $F$. 
What about convergence of the best response dynamic?

Recall that the best response dynamic is:

$$\dot{x} \in M(F(x)) - x, \text{ where } M(\pi) = \arg \max_{i \in S} \pi_i.$$ 

To state the appropriate result one must account for the fact that the dynamic is multivalued.
Theorem 7.4. Let $F$ be a potential game with potential function $f$, and let $\dot{x} \in V_F(x)$ be the best response dynamic for $F$. Then:

$$\frac{\partial f}{\partial z}(x) = \max_{j \in S} \hat{F}_j(x) \quad \text{for all } z \in V_F(x), x \in X.$$ 

Therefore, every solution trajectory $\{x_t\}$ of $V_F$ satisfies $\omega(\{x_t\}) \subseteq NE(F)$. That is, the set of Nash equilibria of $F$ is globally asymptotically stable.
Stable Games

- Recall that the population game $F$ is stable if it satisfies:

\[(y - x)'(F(y) - F(x)) \leq 0 \quad \text{for all } x, y \in X.\]

- When $F$ is $C^1$ this is equivalent to self-defeating externalities:

\[z'DF(x)z \leq 0 \quad \text{for all } z \in TX, x \in X.\]

- The set of Nash equilibria of a stable game is convex and usually a singleton.

- Uniqueness itself does not guarantee convergence (as we shall see later).
Lyapunov Functions for Stable Games

- Once again, convergence proofs rely upon construction of a Lyapunov function.

- But unlike potential games, there is no natural candidate for a Lyapunov function; a distinct one must be constructed for each dynamic.

- We shall now write the Lyapunov function as decreasing over time.

**Definition.** A $C^1$ function $L$ is a (decreasing) strict Lyapunov function for the dynamic $\dot{x} = V_F(x)$ if $\dot{L}(x) \leq 0$ for all $x \in X$, with equality only at rest points of $V_F$. 
For convergence of the replicator dynamics, we need to confine attention to strictly stable games.

We also need to restrict attention to a subset of all initial conditions $\xi \in X$, because if $\xi$ places no mass on a strategy in the support of a Nash equilibrium $x^*$, then the dynamic cannot converge to $x^*$ from $\xi$.

Let the support of $x$ be $S(x) = \{i \in S : x_i > 0\}$. Then $X_y = \{x \in X : S(y) \subseteq S(x)\}$ is the set of states in $X$ whose supports contain the support of $y$. 
The Lyapunov function (in the single-population case) is $h_y : X_y \rightarrow \mathbb{R}$ where:

$$h_y(x) = \sum_{i \in S(y)} y_i \log \frac{y_i}{x_i}.$$ 

$h_y$ is known as the relative entropy of $y$ given $x$. 

Replicator Dynamics in Stable Games
Replicator Dynamics in Stable Games

**Theorem 7.5.** Let $F$ be a strictly stable game with unique Nash equilibrium $x^*$, and let $\dot{x} = V_F(x)$ be the replicator dynamic for $F$.

Then $h_{x^*}^{-1}(0) = \{x^*\}$ and $h_{x^*}(x)$ approaches infinity whenever $x$ approaches $X - X_{x^*}$.

Moreover, $\dot{h}_{x^*}(x) \leq 0$, with equality only when $x = x^*$. Therefore, $x^*$ is globally asymptotically stable with respect to $X_{x^*}$.

If $F$ is simply a stable game, then $x^*$ is Lyapunov stable.
Best Response Dynamics in Stable Games

- Recall that the best response dynamic is:

\[ \dot{x} \in M(\hat{F}(x)) - x, \]

where:

\[ M(\hat{\pi}) = \arg \max_{y \in X} y' \hat{\pi}, \]

i.e. the set of maximizers of (excess) payoffs.
**Theorem 7.6.** Let $F$ be a $C^1$ stable game, and let $\dot{x} \in V_F(x)$ be the best response dynamic for $F$. Define the Lipschitz continuous function $G : X \to \mathbb{R}_+$ by:

$$G(x) = \max_{i \in S} \hat{F}_i(x),$$

which is non-negative and satisfies $G^{-1}(0) = NE(F)$.

Moreover, if $\{x_t\}_{t \geq 0}$ is a solution to $V_F$ then $\dot{G}(x_t) \leq -G(x_t)$ for almost all $t \geq 0$, and so $NE(F)$ is globally asymptotically stable under $V_F$. 