Evolution & Learning in Games
Econ 243B

Jean-Paul Carvalho

Lecture 9.
Local Stability
Local Stability

- Where global convergence does not occur (or cannot be proved), we can at least say something about the local stability of the rest points of an evolutionary dynamic.

- We can immediately state some results about local stability under imitative dynamics and in potential games.

- We will then explore the relationship between evolutionary stable states (in a multiple population setting) and locally stable states.

- Finally, we shall examine two methods of analyzing local stability, via:
  - Lyapunov functions,
  - Linearization of dynamics.
Non-Nash Rest Points of Imitative Dynamics

We can now formalize the argument that such rest points of imitative dynamics are not plausible predictions of play.

**Theorem 9.1.** Let $V_F$ be an imitative dynamic for population game $F$, and let $\hat{x}$ be a non-Nash rest point of $V_F$. Then $\hat{x}$ is not Lyapunov stable under $V_F$, and no interior solution trajectory of $V_F$ converges to $\hat{x}$. 
Recall that imitative dynamics exhibit *monotone percentage growth rates*:

\[ G^p_i(x) \geq G^p_j(x) \quad \text{if and only if} \quad F^p_i(x) \geq F^p_j(x), \]

where \( G^p_i(x) \) is the percentage growth rate of strategy \( i \) in population \( p \) in state \( x \).

The result follows from this fact.
Local Stability in Potential Games

- In Lecture 7, we used the fact that the potential function is a strict Lyapunov function for any evolutionary dynamic satisfying PC to prove global convergence (to rest points of $V_F$).

- This fact is also important to local stability.

- $A \subseteq X$ is a local maximizer set of the potential function $f$ if:
  
  - $A$ is connected,
  
  - $f$ is constant on $A$, and
  
  - there exists a neighborhood $O$ of $A$ such that $f(x) > f(y)$ for all $x \in A$ and all $y \in O - A$. 

Local Stability in Potential Games

For a potential game, a local maximizer set $A$ consists entirely of Nash equilibria.

$A \subseteq NE(F)$ is **isolated** if there is a neighborhood of $A$ that does not contain any Nash equilibria other than $A$.

**Theorem 9.2.** Let $F$ be a potential game with potential function $f$, let $V_F$ be an evolutionary dynamic operating on $F$, and suppose that $A \subseteq NE(F)$ is a local maximizer set of $f$.

(i) If $V_F$ satisfies PC, then $A$ is Lyapunov stable under $V_F$.

(ii) If in addition $V_F$ satisfies NS, and $A$ is isolated, then $A$ is an asymptotically stable set under $V_F$. 
PC and NS are not only sufficient for a local maximizer set to be asymptotically stable, they are also necessary.

**Theorem 9.3.** Let $F$ be a potential game with potential function $f$, and let $V_F$ be an evolutionary dynamic that satisfies PC and NS. Suppose that $A \subseteq NE(F)$ is a smoothly connected asymptotically stable set under $V_F$. Then $A$ is an isolated local maximizer set of $f$. 
Local Stability in Potential Games

The best response dynamic does not satisfy PC because of a lack of smoothness. Nevertheless the following theorem applies.

**Theorem 9.4.** Let $F$ be a potential game with potential function $f$, let $V_F$ be the best response dynamic, and let $A \subseteq NE(F)$ be smoothly connected. Then $A$ is an isolated local maximizer set of $f$ if and only if $A$ is asymptotically stable under $V_F$. 
Evolutionarily Stable States

- We have already introduced the notion of evolutionarily stable states (ESS) in a single population setting.

- Suppose \( x \) is an ESS. Consider a fraction \( \varepsilon \) of mutants who switch to \( y \neq x \). Then the average post-entry payoff in the incumbent population is higher than that in the mutant population, for \( \varepsilon \) sufficiently small.

- We showed that this is equivalent to:

Suppose \( x \) is an ESS. Consider a fraction \( \varepsilon \) of mutants who switch to \( y \). Then the average post-entry payoff in the incumbent population is higher than that in the mutant population, for \( y \) sufficiently close to \( x \).
Evolutionarily Stable States

- Thus an ESS is defined with respect to population averages and explicitly it says nothing about dynamics.

- We shall now extend the ESS concept to a multipopulation setting and relate it to the local stability of evolutionary dynamics.
Taylor ESS

**Definition.** If $F$ is a game played by $p \geq 1$ populations, we call $x \in X$ a Taylor ESS of $F$ if:

There is a neighborhood $O$ of $x$ such that $(y - x)'F(y) < 0$ for all $y \in O - \{x\}$.

This is the same as the statement for single-population games, except $F$ can now be a multipopulation game.

Note that in the multipopulation setting:

$$X = \prod_{p \in P} X^p = \{x = (x^1, ..., x^p) : x^p \in X^p \}.$$
Once again, we have the result:

**Theorem 9.5.** Suppose that $F$ is Lipschitz continuous. Then $x$ is a Taylor ESS if and only if:

- $x$ is a Nash equilibrium: $(y - x)'F(x) \leq 0$ for all $y \in X$, and
- There is a neighborhood $O$ of $x$ such that for all $y \in O - \{x\}$, $(y - x)'F(x) = 0$ implies that $(y - x)'F(y) < 0$. 

For some local stability results we require a strengthening of the Nash equilibrium condition.

In a **quasistrict equilibrium** $x$, all strategies in use earn the same payoff, a payoff that is strictly greater than that of each unused strategy.

This is a generalization of strict equilibrium, which in addition requires $x$ to be a pure state.

The second part of the Taylor ESS condition is also strengthened, replacing the inequality with a differential version.
Definition. We call $x$ a regular Taylor ESS if and only if:

$x$ is a quasistrict Nash equilibrium: $F^p_i(x) = \bar{F}^p(x) > F^p_j(x)$
when $x^p_i > 0$, $x^p_j = 0$, and

For all $y \in X - \{x\}$, $(y - x)'F(x) = 0$ implies that $(y - x)'DF(x)(y - x) < 0$.

Note: every regular Taylor ESS is a Taylor ESS.
Local Stability via Lyapunov Functions

- Let $Y \subseteq X$. Recall that the function $L : Y \rightarrow \mathbb{R}$ is a Lyapunov function for a differential equation (D) or differential inclusion (DI) if its value changes monotonically along every solution trajectory.

- In Lecture 7, we showed that if one can find a Lyapunov function whose domain is $X$ (i.e. $Y = X$), then one can prove global convergence for various evolutionary dynamics.

- For our purpose here, all we need to prove local stability of state $x$ is to construct a Lyapunov function whose domain contains $x$ (i.e. $Y \subset X$).
Local Stability via Lyapunov Functions

Once again:

**Theorem 9.6. (Lyapunov Stability)** Let \( A \subseteq X \) be closed and let \( Y \subseteq X \) be a neighborhood of \( A \). Let \( L : Y \to \mathbb{R}_+ \) be Lipschitz continuous with \( L^{-1}(0) = A \). If each solution \( \{x_t\} \) of (D) [or (DI)] satisfies \( \dot{L}(x_t) \leq 0 \) for almost all \( t \geq 0 \), then \( A \) is Lyapunov stable under (D) [or (DI)].

**Theorem 9.7. (Asymptotic Stability)** Let \( A \subseteq X \) be closed and let \( Y \subseteq X \) be a neighborhood of \( A \). Let \( L : Y \to \mathbb{R}_+ \) be \( C^1 \) with \( L^{-1}(0) = A \). If each solution \( \{x_t\} \) of (D) [or (DI)] satisfies \( \dot{L}(x_t) < 0 \) for all \( x \in Y - A \), then \( A \) is asymptotically stable under (D). If in addition, \( Y = X \), then \( A \) is globally asymptotically stable under (D).

These results are used to prove the following theorems which establish the connection between ESS and local stability.
Local Stability via Lyapunov Functions

**Theorem 9.8.** Let $x^*$ be a Taylor ESS of $F$. Then $x^*$ is asymptotically stable under the replicator dynamic for $F$.

**Theorem 9.9.** Let $x^*$ be a regular Taylor ESS of $F$. Then $x^*$ is asymptotically stable under the best response dynamic for $F$. 
Theorem 9.10. Let $x^*$ be a regular Taylor ESS of $F$. Then for some neighborhood $O$ of $x^*$ and each small enough $\eta > 0$, there is a unique $\text{logit}(\eta)$ equilibrium $\tilde{x}^{\eta}$ in $O$, and this equilibrium is asymptotically stable under the $\text{logit}(\eta)$ dynamic. Finally, $\tilde{x}^{\eta}$ varies continuously in $\eta$, and $\lim_{\eta \to 0} \tilde{x}^{\eta} = x^*$. 
Linearization of Dynamics

- Another technique for establishing local stability of a rest point is to linearize the dynamic around the rest point.

- This requires the dynamic to be smooth around the rest point, but does not require the guesswork of finding a Lyapunov function.

- If a rest point is found to be stable under the linearized dynamic, then it is **linearly stable**.

- Linearization will also be used to prove that a rest point is unstable, and we shall use this in the next lecture to study nonconvergence of evolutionary dynamics.
Linear Approximation

The linear (first-order Taylor) approximation to a function $F$ around point $a$ is:

$$F(a + h) \approx F(a) + DF(a)h.$$ 

Let $o(|h|)$ be the remainder, the difference between the two sides:

$$o(|h|) \equiv F(a + h) - F(a) - DF(a)h.$$
Suppose $F$ is a function of one variable. Then:

$$\frac{o(|h|)}{h} = \frac{F(a + h) - F(a)}{h} - F'(a) \rightarrow 0 \text{ as } h \rightarrow 0,$$

by the definition of the derivative $F'(a)$.

The approximation gets better as $h$ gets smaller, but it gets better at an order of magnitude smaller than $h$. 
Let $A$ be an $n \times n$ matrix. A non-zero vector $v$ is an eigenvector of $A$ if it satisfies:

$$Av = \lambda v,$$

for some scalar $\lambda$ called an eigenvalue of $A$.

Note that:

$$Av = \lambda v \implies (A - \lambda I)v = 0 \implies |A - \lambda I| = 0.$$

Therefore, an eigenvalue of $A$ is a number $\lambda$ which when subtracted from each of the diagonal entries of $A$ converts $A$ into a singular matrix.
EXAMPLE: $A = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$

$|A - \lambda I| = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$. 

Therefore, $A$ has two eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$. 


Linearization of Dynamics

- A single-population dynamic \( \dot{x} = V(x) \), which we shall refer to as (D), describes the evolution of the population state through the simplex \( X \).

- Near \( x^* \), the dynamic (D) can typically be well approximated by the linear dynamic:

\[
\dot{z} = DV(x^*)z, \quad (L)
\]

where (L) is a dynamic on the tangent space \( TX \).

- (L) approximates the motion of deviations from \( x^* \) following a small displacement \( z \).
★ Consider the linear mapping which maps each displacement vector \( z \in TX \) into a new tangent vector \( DV(x^*)z \in TX \).

★ The scalar \( \lambda = a + ib \) is an eigenvalue of this map if \( DV(x^*)z = \lambda z \).

★ If all eigenvalues of this map have negative real part, then the rest point \( x^* \) is **linearly stable** under (D).
Linearization of Dynamics

**Theorem 9.11.** Let $x^*$ be a regular Taylor ESS of $F$. Then $x^*$ is linearly stable under the replicator dynamic.

**Theorem 9.12.** Let $x^* \in \text{int}(X)$ be a regular Taylor ESS of $F$. Then for some neighborhood $O$ of $x^*$ and all $\eta > 0$ less than some threshold $\hat{\eta}$, there is a unique and linearly stable $\logit(\eta)$ equilibrium $\tilde{x}^\eta$ in $O$. 