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Examining Predictive Accuracy Among Discounting Models

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Abstract

Both descriptive and normative arguments claim that the discount rate to be applied to public projects should be elicited from individual intertemporal preferences. We present a methodology to analyze data from experimental surveys on intertemporal preferences. Focusing on the exponential and the hyperbolic discounting models, we model the experimental data published by Thaler (1981) by means of different specifications. Standard measures of goodness of prediction are then applied to fitted data to select among alternative specifications. We first present our approach by applying it to simulated data. We then present a procedure for statistical estimation of the sample discount rate, testing four specifications.

Keywords: subjective rate of time preference, intertemporal choice

JEL Classification: D91

Evaluation of public projects and policies relies on some criterion of economic efficiency, either in the form of cost-benefit analysis, or, when it seems more suitable, of cost efficiency analysis. Both tests require that the option with higher net value should be selected. Unfortunately, computation of costs and benefits is often problematic: one reason is that many public projects involve costs and benefits that belong to different outcome domains. This problem is typical of projects that deal with risks to the environment or public health: for example, financial benefits may be achieved by incurring environmental costs, or health benefits are achieved by losses in financial terms. In these circumstances the usual strategy is to translate all costs and benefits into a single domain, normally the monetary domain, so that projects can be effectively compared. Several techniques are currently available to implement this "translation" procedure: the most

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widely used are the hedonic pricing and the contingent valuation methods. The former can be used when it is possible to refer to some parallel market prices for the good to be evaluated: for example, insurance prices for outcomes in the health domain, or residential housing prices for outcomes in the environmental domain. When it is not possible to refer to any existing market, analysts apply the contingent valuation method: it is a procedure that requires the direct elicitation of the value that individuals attach to the public good of interest. This procedure is analogous to the elicitation of preferences used in decision analysis and experimental economics to investigate the patterns of behavioral decision making: its reliability rests in large part on the accuracy of the experimental setting (cf. Arrow et al., 1993).

Besides the level of costs and benefits, timing of implementation and duration of the effects of the projects are another important element of the decision. For example, suppose that project A and project B give rise to the same costs and benefits: the only difference is that benefits produced by project A are available before those produced by project B. Then project A would be preferred. Conversely, if, ceteris paribus, costs of project A are to be borne before the (same amount of) costs of project B, then project B would be preferred. Even after all costs and benefits arising from a specific project are expressed in the same monetary terms, it is still necessary to use another conversion procedure to reduce cash flows spanning different time periods. This procedure is called discounting: the way it actually operates depends on behavioral assumptions that will be explored more thoroughly in the next section. Here we only observe that the standard discounting procedure implies application of the same discount rate, usually the official rate of discount, for different variable dimensions, for gains or losses, and for the short or the long run.

The validity of a single discount rate is questioned from a descriptive point of view, see for example Albrecht and Weber (1997). The rationale behind using the official rate of discount is that, assuming perfect capital markets, everyone should behave the same way at the margin. Firms and individuals should borrow and lend until their marginal rate of substitution (MRS) between present and future consumption is equal to the interest rate. If a consumer failed to act as the theory predicts, there would be some way to rearrange his consumption plan to make him better off. For example, if his MRS is higher than the interest rate, the individual would find it attractive to trade some future consumption with present consumption—while the opposite holds if his MRS is lower. So, the market interest rate should reflect perfectly the intertemporal preferences of individuals. Yet, as pointed out by Lind (1990), we can observe that individuals trade at very different interest rates: for example, people may at the same time save at some interest rate, and charge consumption on credit cards at a much higher interest rate. The reason may not just be a matter of transaction costs (easy access to one's own funds is obviously the basic motivation for using credit cards) which invalidate the assumption of perfect capital markets, but it can also depend on the individuals' desire to maintain separate budgets, as a means of control on their spending.[1] A typical example may be the limited amount that people may decide to carry with them when going to the horse races. If "people adopt rules and divide assets into separate budgets to facilitate actions that require self-control, then it also follows that individuals do not necessarily change levels of present and future consumption to equalize their marginal rates of substitution with the marginal rate (i.e., the interest rate) at which they can transform present into future income" (Lind, cit., p. S20). "Therefore, market rates that determine consumers' potential rates of transformation may tell us nothing about people's rates of time preference" (*ibidem*). Lind suggests use of the consumer's rate of time preference, that may be context dependent, rather than use of the official rate of discount in the capital market. This position is also supported by Arrow et al. (1996), who argue that discount rates should be based on how individuals trade off present to future consumption, and admit that discount rates can change with the time horizon to reflect the judgment and behavior of individuals. Given uncertainties in identifying the correct rate of discount, they also suggest that it is appropriate to use a range of discount rates, and that this range should be applied to all analyses on (similar) public projects. Just as present preferences for non-market goods are elicited with experimental methods, the same can be done for intertemporal preferences: so again, experimental methods may help to define a range of discount rates in the relevant setting.

Furthermore, application of the discounting technique to projects that produce effects in non-monetary domains has led some authors to claim even a normative shortcoming of the standard discounting procedure, when public projects have a high impact on health or the environment. According to this view, discounting *should* depend on the problem that is being analyzed: different circumstances would require not only different discount rates, but also different procedures. For example, the standard discounting technique implies that flows in the distant future are so heavily discounted that even huge amounts result in a negligible discounted present value. This would unduly penalize (promote) those projects that present extremely high benefits (costs) that are delayed into the distant future. Keeler and Cretin (1983) discuss policy implications of use of different discount rates for different monetary or non-monetary outcomes. Descriptive experiments have shown differing temporal preferences across monetary and non-monetary domains of health (Chapman, 1996), and air and ocean shore water quality (Guyse, Keller, and Eppel, 2002), and for different time horizons for protective investments (Kunreuther et al. 1998). These descriptive findings and the normative argument against a single discount rate have led to the proposal of an alternative to the traditional ("exponential") discounting model, the hyperbolic discounting model. Both models will be briefly reviewed in Section 1.

In Section 2 we present a methodology to analyze data from experimental surveys on intertemporal preferences. Using a published dataset, we examine exponential and hyperbolic discounting models fit exactly to different certainty equivalent judgments from aggregated data. Section 3 contains two measures for selecting the best performing model, 1) the square root of the mean square error and 2) Theil's U index. In Section 4 we generate a simulated dataset from the discount rates elicited by applying the exponential and hyperbolic models in Section 2. In Section 5 we examine the predictive accuracy of the two models for the simulation dataset. In Section 6 we test different econometric specifications to estimate the sample discount rate, and select the model with the best predictive accuracy. We find the best-fitting model to be the hyperbolic discounting model with an additive normally distributed error term. Section 7 concludes the paper.

1. Discounting models

Most discounting models are based on the behavioral assumptions that people prefer to experience pleasurable experiences as soon as possible, and to delay painful experiences. While the first hypothesis, impatience when faced with pleasurable experiences, seems fairly robust to empirical observation and experimental tests, the second one, procrastination in the face of pain, is more controversial. In fact, it can often be observed that people may prefer to get rid sooner of some unpleasant experience, rather than wait (it may be argued that in so doing, they are avoiding the unpleasant experience of anticipating the future unpleasant experience.)

We will see in the following that different sets of behavioral assumptions generate different types of discounting models. We will refer to the general approach taken by Fishburn and Rubinstein (1982) in examining the effect of the time of realization of an outcome on the relative desirability of the outcome. They study the implications of various axioms for a (weak) preference relation. They start from a simple axiomatic structure: given a non-degenerate real interval X (the set of outcomes); and either a set T of successive non-negative integers, or an interval T of non-negative numbers (the set of time points), they consider the topological space $X \times T$ (the dimensions of outcomes and time). Consider the axioms:

A1. $\geq$ *is a weak order on* $X \times T$;
A2. *If* $x > y$ *then* $(x, t) > (y, t)$;
A3. $\{(x, t)\colon (x, t) \geq (y, s)\}$, *and* $\{(x, t)\colon (x, t) \leq (y, s)\}$ *are closed in the product topology on* $X \times T$.
A4. *If* $s < t$ *then* $x > 0 \rightarrow (x, s) > (x, t)$; $x = 0 \rightarrow (x, s) \sim (x, t)$; $x < 0 \rightarrow (x, s) < (x, t)$.

The first three axioms ensure continuity, monotonicity, and ordering of outcomes in the space $X \times T$; the fourth is the behavioral assumption of impatience for positive outcomes, and procrastination for negative outcomes. Fishburn and Rubinstein show that this axiomatic structure implies the existence of a real valued function u on $X \times T$ that is monotonic in x and t; continuous, and increasing in x; continuous in t if t is continuous; decreasing (constant, increasing) in t if x is greater (equal, less) than zero.

Fishburn and Rubinstein do not present a specific functional form associated with the general set of axioms A1–A4. A representation function is instead provided when an axiom of stationarity is added to the previous set of axioms:

A5. *If* $(x, t)R(y, t + d)$ *then* $(x, s)R(y, s + d)$.

This stationarity axiom states that a delay of time d is treated the same regardless of when it occurs (at time t or s). The model implied by this axiomatic structure assumes the form

E1. $\alpha^t f(x)$,

known as the *exponential discounting model* when f is linear on x. So receiving U.S. \$10 with a t-period delay would be equivalent to $(1/1+\delta)^t$ \$10 today, where δ is the discount rate. The function f need not necessarily be linear, though. Fishburn and Rubinstein show that the representation holds with f concave as well, as when f is a risk averse von Neumann-Morgenstern utility function. While the exponential discounting model (discounting monetary value or utility) is usually presumed to be the best normative discounting model, it has been found to fail to adequately describe people's preference behavior. In particular, people tend to violate the stationarity axiom A5, considering a time delay d more significant if it occurs earlier rather than later (Prelec and Loewenstein, 1991). People may prefer to get \$100 today rather than \$115 in one year, but prefer \$115 in 5 years over \$100 in 4 years. Note that there is a "common" one-year time delay in each case. The dynamic consistency required under the stationarity axiom would require the preference relationship order to remain constant in both the short-term and the long-term case. However, people act as if the one-year delay now would be too long to wait, but the one-year delay in four years would be acceptable. This so-called "common difference effect" prompts various approaches to relax the stationarity axiom to get a more descriptively valid model. Additional anomalies (see Loewenstein and Thaler, 1989 and Loewenstein and Prelec, 1992) showing behavior not consistent with the exponential model include the "magnitude effect" (e.g., Kirby and Marakovic (1996) found discount rates decrease as amounts increased, see also Kirby and Herrnstein (1995)) and "gain-loss asymmetry" in which implicit discount rates for monetary gains are higher than those for losses.

As an alternative to the stationarity axiom, Harvey (1986) proposed the "stretching axiom" to provide a solution to the problem of the excessive discounting of distant future flows implied by the exponential discounting model. It states that the ordering of outcomes in two periods depends on the relative difference (the ratio) between two periods:

A6. *If* $(x, s)R(y, t)$*, then* $(x, d \cdot s)R(y, d \cdot t)$,

where the ratio of the length of time s to t is the same as the ratio of the time periods when each is multiplied or "stretched" by d. The set of axioms A1–A4 plus the axiom A6 supports the following representation, known as the *hyperbolic model*, first axiomatized by Harvey (1986):

E2. $[1/(1+t)^{\gamma}]f(x)$,

where $\gamma > 0$ is a parameter that represents individuals' intertemporal preferences. Albrecht and Weber (1995) discuss hyperbolic discounting models. Herrnstein's (1997) "matching law" refers to the observation of responses (originally for animal's discounting behavior) consistent with hyperbolic discount rates. Loewenstein and Prelec (1992) presented a model to capture such choice patterns that used a hyperbolic discount function. Kirby and Marakovic (1995, 1996) fit hyperbolic and exponential discounting models to human subjects' data. Ainslie (1991) discussed the potential for "rational" economic behavior to result from the hyperbolic discounting model.

2. Test of the models

We now present a methodology to analyze experimental data from surveys on intertemporal preferences. Since this demonstration is to be considered as illustrative of the method, we chose to apply it to the dataset published by Thaler (1981).

The standard approach used in decision theory to analyze this type of data has been to apply some statistical test (usually non-parametric, but also some parametric models have been applied, see Benzion et al. (1989), to test the validity of specific assumptions of different models). Instead, the approach we will use here is more general, in that it considers different models as estimators of the data drawn from the elicitation procedure of the experiment. In the review presented in the preceding section we have examined two main discounting models: the exponential model and the hyperbolic model. In the present application we analyze the performance of these two models, assuming $f(x)$ is linear in the monetary outcome.

There are different procedures to elicit individuals' intertemporal preferences. A choice procedure requires the decision maker to choose between an amount to be received (or paid) now, and another specified amount to be received (paid) some specified time later. Another procedure, called the matching method, requires the individual to assess the amount that would make her indifferent between getting some given amount now, and getting that amount at some specified later time period. Drawing from results in the contingent valuation literature, we can infer that the former procedure is the easiest for the respondent, but it is also less informative. While the matching method produces precise data points, the choice method only generates a dichotomous ordering type of data, and requires more observations for an efficient statistical analysis.

Also, the amounts may be stocks, to be received or paid at a point in time, or flows, to be received or paid along a time stream. The latter setting has been analyzed by Loewenstein and Prelec (1993), who argue that preferences over sequences of outcomes may also be affected by the distribution of the outcomes along the time dimension, in addition to the absolute level of the amounts to be received or paid.[2] A descriptively valid analysis of preferences over sequences of outcomes would require more complex behavioral assumptions than those we considered in the preceding section. To simplify the analysis, in this paper we only consider intertemporal preferences over stocks, rather than flows of outcomes.

The data published by Thaler (1981) are medians of amounts elicited using the matching method: therefore, we have data point observations, that we can use for our illustrative purpose. We have four subsamples, each of them was presented with a given amount now to be matched with some amount in three months, one year, and three years for a total of nine matching points for each subsample. The dataset is represented in Table 1: the left column contains the M_0 present amounts proposed to each subsample. The other columns contain the (median) amounts expressed by respondents when asked the outcome M_t that would have made them indifferent between getting a given M_0 now, or getting M_t later. For example, scenario A required subjects to consider amounts to be gained now and state an indifferent amount to be received in three months (M_3), one year (M_{12}), and three years (M_{36}), respectively. The first three subsamples were presented with gains,

Table 1. Median amounts matching monetary gain/loss M_0 now with delayed amount M_t (from Thaler, 1981)

	Amount now, M_0	Matching amount, M_t, delayed by t months		
Scenario A	Now	3 months	12 months	36 months
Gains	15	30	60	100
	250	300	350	500
	3000	3500	4000	6000
Scenario B	Now	6 months	12 months	60 months
Gains	75	100	200	500
	250	300	500	1000
	1200	1500	2400	5000
Scenario C	Now	1 month	12 months	120 months
Gains	15	20	50	100
	250	300	400	1000
	3000	3100	4000	10000
Scenario D	Now	3 months	12 months	36 months
Losses	−15	−16	−20	−28
	−100	−102	−118	−155
	−250	−251	−270	−310

i.e., amounts to be received, now or later; the last group was instead presented with a loss, i.e., a payment to be sustained now or later.

If the individual is indifferent between M_0 and M_t, the discounted present value of M_t must be equal to M_0. Assuming that a particular model (exponential or hyperbolic) holds, it is then possible to calculate the implicit discount rate. For example, suppose individuals are indifferent between \$15 now and \$30 in three months. We want to discount the \$30 back to the present \$15 (or, equivalently, start with the \$15 and compound the implicit interest "earned" over the next 3 months to end up with \$30.) The implicit monthly rate of discount for the exponential model is the δ that solves the following equation:

$$\$15 = \$30/(1+\delta)^3, \text{ i.e., } \delta = 0.260.$$

The implicit discount rate for the hyperbolic model is instead the γ that solves the following equation:

$$\$15 = \$30/(1+3)^\gamma, \text{ i.e., } \gamma = 0.500.$$

Thaler observed that the implicit monthly discount rates calculated from the exponential discounting model from the elicited matching values present a pattern far from uniform: generally they decrease as the time length and the amount levels increase. When instead the hyperbolic discounting model is applied to the same data, we do not observe a clear pattern. Table 2 shows the implicit (monthly) discount rates inferred from each model.

A quick look at the implicit rates of discount reported in Tables 2(a) and (b) would lead us to think that the hypothesis of a unique discount rate, to be applied to all projects,

Table 2. Implicit (monthly) discount rates from Thaler's data

0 months		1 month	3 months	6 months	12 months	36 months	60 months	120 months
(a) δ (exponential)								
Scenario A								
Gains	\$15		.260		.063	.053		
	\$250		.122		.028	.024		
	\$3000		.054		.019	.019		
Scenario B								
Gains	\$75			.101	.063		.077	
	\$250			.085	.059		.059	
	\$1200			.032	.023		.024	
Scenario C								
Gains	\$15	.101			.063			.011
	\$250	.106			.040			.024
	\$3000	.016			.012			.010
Scenario D								
Losses	−\$15		.022		.007	.001		
	−\$100		.024		.014	.006		
	−\$250		.017		.012	.006		
(b) γ (hyperbolic)								
Scenario A								
Gains	\$15		.500		.132	.111		
	\$250		.540		.131	.112		
	\$3000		.525		.192	.192		
Scenario B								
Gains	\$75			.148	.094		.115	
	\$250			.382	.270		.270	
	\$1200			.461	.337		.347	
Scenario C								
Gains	\$15	.415			.263			.047
	\$250	.469			.183			.112
	\$3000	.396			.289			.251
Scenario D								
Losses	−\$15		.047		.014	.003		
	−\$100		.112		.065	.030		
	−\$250		.173		.121	.060		

independently of their time horizon, and the level of the outcomes involved, should be rejected. This is in fact the conclusion reported by Thaler. A number of techniques for statistical analysis can be used to test the hypothesis in a more rigorous manner. A standard practice is to apply an analysis of variance to the implicit discount rates.

The method we propose here is to investigate if the values obtained from a particular model are a good predictor of the actual values. We want to test the hypothesis that the discount rate is independent of the time horizon and the outcome levels of the project. If the values obtained by applying some constant discount rate to the present values M_0 (by using that rate as if compounding interest over time on the base amount M_0) can be considered an acceptable prediction of the actual values M_t, the hypothesis can be accepted. If a constant discount rate yields an acceptable model, then this parameter value could be used in practical settings for guiding decision making or describing people's preferences.

In the following, we first show an application to simulated data, and then we will apply the method to values obtained through Maximum Likelihood estimation. In the simulation exercise, we multiply the 36 values M_0 by the implicit discount rates obtained from the elicited matching values: we obtain 36 vectors of simulated matching values, "predicted" by a specific discounting model (exponential or hyperbolic) given a specific constant discount rate. The model that gives the best prediction would be selected. Of course, best prediction does not mean good prediction: it may well be that even the best is so bad that we will anyway wish to reject the hypothesis of a constant rate of discount, at least under the exponential or the hyperbolic model, i.e., the two discounting models under analysis. Nevertheless, in practice, it may be necessary to specify a constant discount rate for analysis due to regulatory or administrative requirements, which our method will do. In the next section we will briefly describe the statistical criteria we will apply in our analysis.

3. Model selection

In regression models, goodness of prediction is generally assessed in terms of distance between the actual a_i and the predicted p_i values, and the model is chosen that minimizes the distance, or loss function (see Gourieroux and Monfort, 1995). The most frequently used criterion for goodness of prediction is based on the quadratic loss function $L(p_i - a_i) = (p_i - a_i)^2$, i.e., the square of the Euclidean norm. The criterion involves minimization of the Mean Square Error:

$$\mathrm{MSE} = \frac{1}{n}\sum_{i=1}^{n}(p_i - a_i)^2,$$

i.e., the average of the squared differences between predicted and actual values. Its root, the RMSE, is often preferred because its value level is the same as that of the data. This measure penalizes large departures of the predicted from the actual value; it has optimal statistical properties in that it minimizes the sum of (squared) bias and variance of the predictor.

A problem of a criterion such as the minimization of the MSE (or, equivalently, the RMSE) is that, while it is easy to compare the predictive performance of different models, it may be more difficult to judge if the selected predictor is indeed satisfactory.

To help judgment about the accuracy of prediction, Theil (1966) proposed an index, that is bounded from below at zero, which in the following we will refer to as the U-Theil:

$$\text{U-Theil} = \sqrt{\frac{\text{MSE}}{\sum a_i^2/n}}.$$

Values close to zero indicate a good predictive accuracy, while large values indicate a bad performance, so the interpretation of the results is relatively easier.

We will apply the RMSE and the U-Theil index criteria to the values generated by the two discounting models, to select the discounting model that best predicts the actual experimental data considered in the analysis. We will consider two generating mechanisms: first, we will adopt a simulation approach, that we present in the next section; after, we estimate different statistical models linked to the two alternative discounting functions under analysis.

4. The simulation procedure

We generate a new dataset from the discount rates elicited by applying the exponential and the hyperbolic model to the data. The procedure consists of applying each rate of discount in Table 2(a) and (b) to the whole series of 36 proposed M_t values which match present amounts M_0: this produces a vector of 36 elements of simulated matching values for each of the 36 discount rates. (Thus this simulation uses M_0 monetary values in the present ranging from $-\$250$ to \$3000 in U.S. dollars and time delays t ranging from 3 to 120 months, and applies 36 discount rates generated from 36 distinct M_0 and t combinations.) The set of simulated data for each model is therefore a 36×36 matrix. Tables 3(a) and 3(b) report the mean and standard deviation of the simulated values produced by each model. For example, in the upper left-hand corner of Table 3(a), Thaler's subjects gave an actual median response of \$30 for a three month delay to match \$15 now. When the 36 discount rates δ from Table 2(a) are each used to predict the actual amount, by the exponential formula:

$$\$15 = \left(\frac{1}{1+\delta}\right)^3 \text{ prediction},$$

the mean prediction was \$18 and the standard deviation was \$4. These measures are obviously too rough to provide an indication of the goodness of either model: the statistics refer to the data generated through a wide range of discount rates, and even a perfect prediction with one of the discount rates would be unrevealed by these data. Yet, it is worth observing that the hyperbolic model in Table 3(b) gives on average simulated values closer to the real ones, and especially that it avoids the problem of "explosive" results obtained by the exponential model, especially when the procedure involves long time periods.

Table 3a. Mean and standard deviation of simulated amounts: exponential model

Now	Actual values and mean and standard deviation of predicted values					
Scenario A	3 months		12 months		36 months	
Gains	Actual	Predicted	Actual	Predicted	Actual	Predicted
15	30	18 (4)	60	45 (84)	100	15201 (78975)
250	300	297 (71)	350	755 (1401)	500	253357 (1316250)
300	3500	3559 (855)	4000	9066 (16818)	6000	3040292 (1579499)
Scenario B	6 months		12 months		60 months	
Gains	Actual	Predicted	Actual	Predicted	Actual	Predicted
75	100	111 (69)	200	227 (420)	500	67631271 ($\to +\infty$[a])
250	300	372 (229)	500	755 (1401)	1000	$\to +\infty$ ($\to +\infty$)
1200	1500	1783 (1098)	2400	3626 (6727)	5000	$\to +\infty$ ($\to +\infty$)
Scenario C	1 month		12 months		120 months	
Gains	Actual	Predicted	Actual	Predicted	Actual	Predicted
15	20	16 (1)	50	45 (84)	100	$\to +\infty$ ($\to +\infty$)
250	300	263 (18)	400	755 (1401)	1000	$\to +\infty$ ($\to +\infty$)
3000	3100	3161 (215)	4000	9066 (16818)	10000	$\to +\infty$ ($\to +\infty$)
Scenario D	3 months		12 months		36 months	
Losses	Actual	Predicted	Actual	Predicted	Actual	Predicted
−$15	−16	−18 (4)	−20	−45 (84)	−28	−15201 (78975)
−$100	−102	−118 (28)	−118	−302 (560)	−155	−101343 (526500)
−$250	−251	−297 (71)	−270	−755 (1401)	−310	−253358 (131250)

Note: Values in parentheses are standard deviations.
[a]Values greater than 10^7 are indicated as approaching infinity.

5. Predictive accuracy of models for the simulated dataset

The hypothesis to be checked is that there exists a unique discounting model (i.e., a specific mathematical discounting procedure applied to a specific discount rate) that can predict reasonably well the data. The statistics on the RMSE and the U-Theil index are reported in Table 4.

It can be observed that the exponential model is much riskier than the hyperbolic: due to the exponential functional form, an incorrectly high discount rate may produce "explosively" high predicted values.[3] The hyperbolic model (which does not generate predicted values as high as the exponential model does) in general outperforms the exponential, since all the statistics have lower values. However, one particular exponential specification might turn out to perform well, as shown in Table 5 that reports the best specifications, in terms of RMSE and U-Theil, for each model. Even though the best

Table 3b. Mean and standard deviation of simulated amounts: hyperbolic model

	Actual values and mean and standard deviation of predicted values					
	3 months		12 months		36 months	
Scenario A Gains	Actual	Predicted	Actual	Predicted	Actual	Predicted
15	30	21 (5)	60	29 (13)	100	39 (25)
250	300	347 (81)	350	478 (214)	500	656 (425)
300	3500	4169 (969)	4000	5740 (2571)	6000	7873 (5094)
	6 months		12 months		60 months	
Scenario B Gains	Actual	Predicted	Actual	Predicted	Actual	Predicted
75	100	120 (40)	200	144 (64)	500	231 (172)
250	300	402 (134)	500	478 (214)	1000	771 (573)
1200	1500	1932 (644)	2400	2296 (1029)	5000	3701 (2751)
	1 month		12 months		120 months	
Scenario C Gains	Actual	Predicted	Actual	Predicted	Actual	Predicted
15	20	18 (2)	50	29 (13)	100	58 (51)
250	300	293 (33)	400	478 (214)	1000	972 (851)
3000	3100	3515 (398)	4000	5740 (2571)	10000	11670 (10206)
	3 months		12 months		36 months	
Scenario D Losses	Actual	Predicted	Actual	Predicted	Actual	Predicted
−$15	−16	−21 (5)	−20	−29 (13)	−28	−39 (25)
−$100	−102	−138 (32)	−118	−191 (86)	−155	−262 (170)
−$250	−251	−347 (81)	−270	−478 (214)	−310	−656 (425)

Note: Values in parentheses are standard deviations.

specification is the hyperbolic, with a discount rate $\gamma = 0.251$, also the exponential with discount rate $\delta = 0.010$ predicts the data quite well.

Our method shows how to find the discounting model which best fits the data, without explicitly assuming error in a person's judgments. A stochastic component is required if we want to estimate the discounting model that generates better predictions. We consider this issue in the next section.

Table 4. Summary statistics on measures

		Minimum	Maximum	Median	Mean	Std. dev.
Exponential model	RMSE	610	$\to +\infty$[a]	10043	$\to +\infty$	$\to +\infty$
	U-Theil	0.243	$\to +\infty$	4.004	$\to +\infty$	$\to +\infty$
Hyperbolic model	RMSE	577	6078	1060	1639	1408
	U-Theil	0.230	2.423	0.423	0.653	0.561

[a]Values greater than 10^7 are indicated as approaching infinity.

Table 5. Criteria on selected specifications

Model specification	Discount rates	RMSE	U-Theil
Exponential	$\delta = 0.010$	610	0.243
Hyperbolic	$\gamma = 0.251$	577	0.230

6. Estimation of discount rates assuming an error term

Analogously to the experimental procedures to assess non-market values, that involve the estimation of the sample valuation from elicited individual values (which may be distorted due to judgment errors), a sample discount rate could be estimated from the elicited individual rates. In order to obtain a (parametric) estimate of the discount rate from our sample observations, we should first specify a statistical model with an error term. For each given value M_0, we assume that the matching value M_t is functionally related to M_0 according to some model

$$M_0 = f(M_t; \theta),$$

where θ is a vector of parameters related to the discounting functional form and the error term. Specification of the functional form of both the deterministic and the stochastic component produces an econometric model for individual intertemporal preferences.

When individual socio-economic characteristics are available, the functional form f may be specified in order to include these characteristics as explanatory variables. In our example we do not include any regressors in the model; furthermore, we maintain the usual assumption of linearity of the valuation function that can be easily relaxed just by applying an appropriate transformation to the amounts M. We will make two hypotheses on the functional form of the discounting model, specifying the function in terms of the exponential model or the hyperbolic model. Two alternative hypotheses will be tested also for the error term:

(a) the error term is additive, and is distributed as a *Normal distribution*, with mean zero and variance σ^2;
(b) the error term is multiplicative, and is distributed as a *Lognormal distribution*, so that its logarithm is *normal*, with mean zero and variance ν^2.

These hypotheses give rise to four different specifications: if the discounting model is exponential, and the error term is normal, we have the following:

$$M_0 = \frac{M_t}{(1+\delta_1)^t} + \varepsilon_1, \text{(\textit{exponential normal specification})}. \quad (1)$$

If the discounting model to be applied is the hyperbolic, the specification is:

$$M_0 = \frac{M_t}{(1+t)^{\gamma_2}} + \varepsilon_2, \text{(\textit{hyperbolic normal specification})}. \quad (2)$$

If the error term is multiplicative lognormal, the exponential model has the following form:

$$M_0 = \frac{M_t}{(1+\delta)^t} \cdot \varepsilon$$

which can be transformed as follows:

$$\ln \frac{M_t}{M_0} = t \cdot \ln(1+\delta_3) + \ln \varepsilon_3, \quad (\textit{exponential lognormal specification}). \tag{3}$$

For the hyperbolic model, the corresponding specification form is:

$$M_0 = \frac{M_t}{(1+t)^\gamma} \cdot \varepsilon$$

and by taking logarithms the model can be linearized as follows:

$$\ln \frac{M_t}{M_0} = \gamma_4 \cdot \ln(1+t) + \ln \varepsilon_4 \; (\textit{hyperbolic lognormal specification}). \tag{4}$$

All models are estimated through Maximum Likelihood procedures. The parameter estimates and log-likelihood for the four specifications are shown in Table 6. The terms θ_f and θ_s refer, respectively, to the discount function and the standard deviation parameters relevant to each of the four specifications. In the case of the hyperbolic models, and the normal exponential specification, the estimated parameter corresponds exactly to the parameter of interest $\hat{\gamma}_2$, $\hat{\gamma}_4$ and $\hat{\delta}_1$; while in the lognormal exponential specification, the estimated parameter is $\hat{\theta}_f = \ln(1+\hat{\delta}_3)$, and the parameter $\hat{\delta}_3$ of interest is easily obtained after a transformation.

Even though we are especially interested in the predictive accuracy of the estimates, it may be advisable to check the *approximation* of the alternative specifications to the "true model," in terms of information criteria. A log-likelihood based model selection criterion commonly used in the non-nested models setting is the Akaike criterion, that for models with the same number of parameters simply reduces to a generalized likelihood ratio. Due to the transformation of the dependent variable, the lognormal models cannot be

Table 6. Log-likelihood and parameter estimates

	Exponential normal	Hyperbolic normal	Exponential lognormal	Hyperbolic lognormal
λ	−259.562	−251.725	−24.616	−19.238
θ_f	0.0143	0.2235	0.0169	0.2597
	(0.0013)	(0.0111)	(0.0019)	(0.0238)
θ_s	327.381	263.337	0.4794	0.4129
	(38.582)	(31.035)	(0.0565)	(0.0487)

Note: Values in parentheses are standard errors.

Table 7. RMSE and U-Theil on estimated specifications

Model specification	Discount rates	RMSE	U-Theil
Exponential normal	$\delta = 0.014$	1183	0.472
Hyperbolic normal	$\gamma = 0.223$	545	0.217
Exponential lognormal	$\delta = 0.017$	2166	0.863
Hyperbolic lognormal	$\gamma = 0.260$	2031	0.810

compared in this setting to the normal models, but the two discounting models can be compared under either specification (see Burnham and Anderson, 1998). As it can be observed from the values of the likelihood functions reported in Table 6, the Akaike criterion for goodness of fit chooses the hyperbolic over the exponential discounting function under both the normal and the lognormal specification.

Turning now to the analysis of the accuracy of predictions, we report in Table 7 the RMSE and U-Theil values for the estimated models. It is quite clear from the values of the RMSE that the Normal specifications dominate the lognormal, and this leads us to discard the hypothesis of a lognormal error term. The hyperbolic normal model shows the best predictive accuracy, with a low RMSE and a U-Theil close to zero. However, also the exponential Normal model performs relatively well, and it may be advisable to analyze further the predictive accuracy of the two Normal models.

So far, we have applied our selection criteria to the original data set. This means that the same data are used to fit and to assess the models: this may create a problem of "overfitting," i.e. the parameter estimates of the selected model may be strictly dependent on the particular data set employed, and the measure of the predictive error may be biased. In order to control for overfitting, we apply a "leave-one-out" cross-validation procedure, which is suitable for small data sets (see Efron and Tibshirani, 1993, p. 240). The procedure consists of estimating the model leaving one observation out, and using the parameter estimates to predict the value of the missing observation. The process is repeated n times (the number of the observations in the sample), and the predicted values are used to produce an estimate of the Mean Square Error. The cross-validated RMSE and U-Theil for the Exponential and Hyperbolic Normal models are reported in Table 8. As seen in the table, the overfitting problem is quite serious for the exponential model, while the hyperbolic estimates seem fairly stable. All these results lead us to accept, for Thaler's data, the hyperbolic model with a normally distributed additive error term and the (unique) discount rate $\gamma = 0.223$.

Table 8. Cross-validated RMSE and U-Theil on normal specifications

Model specification	Cross-validated RMSE	Cross-validated U-Theil
Exponential normal	5627	2.243
Hyperbolic normal	656	0.261

7. Conclusions

A method for characterizing intertemporal preferences by selecting the discounting model which best fits data on people's preferences is presented. Such an approach can be useful when analyzing experimental data or in policy making when a discounting model is needed to characterize residents' temporal discounting preferences. In practice, it may be necessary to specify a constant discount rate in economic analyses which must consider people's intertemporal preferences due to regulatory or administrative requirements. Standard measures of goodness of prediction are applied to fitted data to select among alternative specifications of discounting models. We limited our analysis to the exponential and the hyperbolic discounting models. We first presented our approach by applying it to simulated data, that we obtain by manipulating a matrix of experimental data on intertemporal preferences published by Thaler (1981). We then proceeded to estimate the sample discount rate, testing four different specifications: exponential or hyperbolic discount models, modeled with a normal or lognormal distribution of the error term. We found that the hyperbolic discounting model with an additive normal error term provided the best fit. Furthermore, in contrast to Thaler's conclusions, we found that its predictive accuracy is good enough to warrant acceptance of the hypothesis that the data are expressed by a unique discount rate: i.e., the hyperbolic discount rate $\gamma = 0.223$. As Camerer (1998) notes, the economics profession has been slow to accept the hyperbolic discounting model. Our method provides an approach for determining when it is most appropriate.

The estimation procedure we adopted can be easily extended to other functional forms, allowing for non-linearity of the valuation function, socio-economic individual characteristics included as regressors, and different specifications for the discounting model.

Our aim in this paper was to present a method to determine a single best-fitting discount rate, using either an exponential or hyperbolic model, which can best capture multiple preference judgments. In policy decision making, using a model (such as exponential discounting) which does not match people's preferences can lead to sub-optimal decisions. However, use of more flexible models (such as hyperbolic discounting or different discount rates for different attributes, amounts, or delays) can lead to "irrational" outcomes. Care must be taken to weigh the benefits of more accurately representing people's preferences with potential disadvantages of logical violations of desirable properties, such as a constant discount rate or the stationarity axiom of exponential discounting models.

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Notes

1. A person's willingness to accept different interest rates for saving (a gain) and for credit card charges (a loss) may reflect the gain/loss asymmetry effect in time discounting.
2. In a related line of research involving sequences of choices, Rachlin and Siegel (1994) found subjects making repeated choices between probabilistic and near-certain monetary rewards tended to be less risk-averse the shorter the intertrial interval. Subjects were also less risk-averse when the choice-outcome sequences were clustered in threes than when each choice-outcome sequence was separated from its neighbors by equal intertrial intervals.
3. In fact, people often fail to realize the power of compound interest to make them rich, so they fail to save while young and let their interest compound exponentially. This is another example of people's preferences and perceptions not being consistent with the exponential discounting model.

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