NUMERICAL STUDY

OF

DROPLET-STREAM VAPORIZATION

AT ZERO REYNOLDS NUMBER

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Abstract

This paper presents a numerical simulation of the vaporization of an infinite stream of droplets in the limit of zero Reynolds number. Analytical transformations are developed so that the solutions can be sought in a regular domain, suitable for the finite difference method and points can be clustered in high-gradient regions to guarantee precision for the numerical approximation without corresponding increase in the computational costs. Once obtained, the transformed equations are discretized, leading to an algebraic system of equations for the velocity potential and the temperature. This system is solved using subroutines available in scientific routines libraries, with interactive refinement of the solution. The transformations are then inverted and the profiles within the physical domain obtained for the quantities of interest. The results are compared to analytical solutions available to limiting cases, for validation purposes. Converged numerical solutions for stream with different droplet spacings are presented, showing the effect of droplet interaction. The results indicate that the vaporization rate of a droplet in a stream can be obtained from the isolated droplet vaporization case times a function depending on the interdroplet distance.

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Nomenclature

\(a\) \hspace{1em} \text{Truncation Distance} \\
\(b\) \hspace{1em} \text{Half Interdroplet Distance} \\
\(c\) \hspace{1em} \text{Clustering Parameter} \\
\(h\) \hspace{1em} \text{Enthalpy} \\
\(\mathcal{J}_a\) \hspace{1em} \text{Jacob Number} \\
\(k\) \hspace{1em} \text{Thermal Conductivity} \\
\(\dot{m}\) \hspace{1em} \text{Mass Vaporization Rate} \\
\(N\dot{u}\) \hspace{1em} \text{Nusselt Number} \\
\(r\) \hspace{1em} \text{Radial Coordinate} \\
\(R\) \hspace{1em} \text{Droplet Radius} \\
\(T\) \hspace{1em} \text{Temperature} \\
\(u\) \hspace{1em} \text{Velocity Component} \\
\(U\) \hspace{1em} \text{Tangential Velocity} \\
\(V\) \hspace{1em} \text{Normal Velocity} \\
\(\alpha\) \hspace{1em} \text{Thermal diffusivity} \\
\(\epsilon\) \hspace{1em} \(\eta\) - transformed coordinate \\
\(\zeta\) \hspace{1em} \(\theta\) - transformed coordinate \\
\(\eta\) \hspace{1em} \(r\) - transformed coordinate \\
\(\theta\) \hspace{1em} \text{Angular coordinate} \\
\(\rho\) \hspace{1em} \text{Density} \\
\(\Phi\) \hspace{1em} \text{Velocity Potential}

Subscripts

\(c\) \hspace{1em} \text{Physical Domain Diagonal} \\
\(D\) \hspace{1em} \text{Diameter} \\
\(f\) \hspace{1em} \text{Vaporization} \\
\(r\) \hspace{1em} \text{Radial Component} \\
\(nf\) \hspace{1em} \text{Reference} \\
\(s\) \hspace{1em} \text{Droplet Surface} \\
\(\theta\) \hspace{1em} \text{Tangential Component}

Superscripts

\(*\) \hspace{1em} \text{Nondimensional} \\
\(---\) \hspace{1em} \text{Average}
Introduction

Droplet vaporization occurs in a wide range of technological applications such as Diesel and liquid rocket engines. For an isolated droplet, the velocity, temperature and concentration profiles can be found under simplifying assumptions by solving the governing equations analytically [1, 2]. In most practical cases, when a group of droplets is present and the interdroplet distances are small enough to make interaction between them significant, analytical solutions are not possible. This is the case of most practical devices, where dense sprays prevail and droplets are subjected to conditions different from the ones experienced by isolated ones.

Theoretical studies of droplet-array vaporization have been carried out by neglecting the convective terms in the governing equations, assuming slow vaporization process, and then using the linearity of the obtained equations to build solutions by the method of images [3, 4]. Later, these ideas were extended to rapidly vaporizing droplets [5] by using transformations to reduce the equations to a linear form where the method of images could be applied. These solutions confirmed the importance of droplet interaction for the vaporization of arrays of droplets. Numerical solutions [6] – [13] have been limited to the study of a few droplets due to the need for a fine enough computational grid to resolve the small scale phenomena near the droplets. In convective cases, numerical simulations show that the drag exerted on a droplet, as well as, heat transfer parameters are significantly different from the ones obtained for an isolated droplet due to interaction effects [8, 11, 12, 13]. Results show an inner region, strongly influenced by the individual droplets and an outer region dominated by the external flow [9]. Another approach has made use of droplet submodels to predict group behavior. The accuracy of these so called global methods [14] is dependent on the accuracy of the submodels and their capabilities in describing the small scales of the problem. The vast majority of numerical models have been limited to high and moderate Reynolds numbers since these prevail in the initial portion of the lifetime of most droplets in a real combustor. However, the zero Reynolds number limit is the logical next step beyond the classical isolated droplet theory resulting in the $d^3$ law and, consequently, it is the subject of the present study.

Droplet stream vaporization has been also the subject of experimental work both in convective and nonconvective situations. As in the numerical approaches, experiments are limited by the small spatial scale of the physical phenomena [14]. The classical $d^3$ law for an isolated droplet was shown not to be followed by the droplets in a stream and
buoyancy effects were substantially affected by the droplet interaction leading to a decrease of the droplet vaporization rate, when compared to the isolated droplet case [15]. Measurements for combusting droplet streams [16] show a cylindrical flame structure, after a distance of 15 droplet diameters from the stream axis, indicating that, at such distances, effects from individual droplets are not significant. This behavior is expected to exist in the vaporizing stream case. However, for the near field temperature profile, different behavior is observed as the angular position from the stream axis varies, stressing the multi-dimensional character of the problem.

The present work addresses the problem of droplet stream vaporization by resolving the small scale phenomena near and between droplets of a linear array at zero Reynolds number. A computational grid is developed such that this requirement can be satisfied. Results for the velocity and temperature fields are presented for different droplet spacing, so that the interaction between droplets can be analyzed. Then, the vaporization rate is correlated to the droplet spacing.

**Analysis**

A schematic of the problem is shown in Fig. 1. A section of a long stream of droplets, where the end effects can be neglected, is considered. As a first approach, the flow generated by the vaporizing droplets can be studied under the assumption of the potential flow theory. It should be noted that for a single isolated droplet, the potential flow solution is indeed the exact solution and that the superposition of several droplets should not alter this significantly at zero or low Reynolds number. The results obtained can provide some important information about the flow field and the applicability of the constructed grid. Moreover, analytical results are available in the literature for the far-field region, which makes possible the validation of the developed numerical code. After symmetry considerations are made, the domain where the solution is sought is obtained (Fig. 1) and, using a spherical coordinate system centered on one droplet, the equations governing the vaporization of a stream of droplets can be written in their dimensional quasi-steady form as:

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) = 0
\]

\[
u_r \frac{\partial T}{\partial r} + \frac{u_\theta}{r} \frac{\partial T}{\partial \theta} = \alpha \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) \right)
\]
in

\[ R < r < \frac{b}{\cos \theta}, \quad 0 < \theta < \frac{\pi}{2} \]

where the velocity components are

\[ u_r = \frac{\partial \Phi}{\partial r} \quad \text{and} \quad u_\theta = \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \]

with boundary conditions:

\[ \frac{\partial \Phi}{\partial r} = V_s(\theta) \quad T = T_s \quad ; \ r = R \quad 0 < \theta < \frac{\pi}{2} \]

(3)

\[ \Phi \to 0 \quad T \to T_\infty \quad ; \ r \to \infty \]

(4)

\[ \frac{\partial \Phi}{\partial \theta} = 0 \quad \frac{\partial T}{\partial \theta} = 0 \quad ; \ \theta = 0 \quad R < r < b \]

(5)

\[ \frac{\partial \Phi}{\partial \theta} = 0 \quad \frac{\partial T}{\partial \theta} = 0 \quad ; \ \theta = \frac{\pi}{2} \quad R < r < \infty \]

(6)

\[ -\frac{\partial \Phi}{\partial \theta} \sin \theta + \frac{\partial \Phi}{\partial r} b = 0 \quad -\frac{\partial T}{\partial \theta} \sin \theta + \frac{\partial T}{\partial r} b = 0 \quad ; \ r \cos \theta = b \quad 0 < \theta < \theta_c \]

(7)

Equations (1) and (2) are coupled by the energy requirement:

\[ k \left. \frac{\partial T}{\partial r} \right|_{r=R} = \rho h_f g V_s = \rho h_f g \left. \frac{\partial \Phi}{\partial r} \right|_{r=R} \]

(8)

at the droplet surface. The droplet surface is assumed at a constant temperature equal to the vaporizing temperature of the liquid at the prevailing pressure. We plan to relax this assumption in a subsequent study.

The equations can be nondimensionalized using new variables defined by:

\[ r^* = \frac{r}{R} \quad \Phi^* = \frac{\Phi}{V_{ref} R} \quad T^* = \frac{T - T_s}{T_\infty - T_s} \]

(9)

where

\[ V_{ref} = \frac{k}{\rho h_f g} \frac{T_\infty - T_s}{R} \]

(10)

The nondimensionalized governing equation can be written as:

\[ \frac{1}{r^* \Phi^*} \frac{\partial}{\partial r^*} \left( r^* \Phi^* \right) + \frac{1}{r^* \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \Phi^* \right) = 0 \]

(11)
\[ u_r \frac{\partial T^*}{\partial r^*} + u_\theta \frac{\partial T^*}{\partial \theta} = \frac{1}{\lambda} \left( \frac{1}{r^{2} \partial r^*} \left( r^{*2} \frac{\partial T^*}{\partial r^*} \right) + \frac{1}{r \sin \theta \partial \theta} \left( \sin \theta \frac{\partial T^*}{\partial \theta} \right) \right) \]  \hspace{1cm} (12)

in

\[ 1 < r^* < \frac{b^*}{\cos \theta}, \quad 0 < \theta < \frac{\pi}{2} \]

where the velocity components are

\[ u_r = \frac{u_r}{V_{ref}} = \frac{\partial \Phi^*}{\partial r^*} \quad \text{and} \quad u_\theta = \frac{u_\theta}{V_{ref}} = \frac{1}{r^*} \frac{\partial \Phi^*}{\partial \theta} \]

with boundary conditions:

\[ \frac{\partial \Phi^*}{\partial r^*} \bigg|_{r^*=0} = V_s(\theta) = V_{ref}(\theta) \quad T^* = 0 \quad ; \quad r^* = 1 \quad 0 < \theta < \frac{\pi}{2} \]  \hspace{1cm} (13)

\[ \Phi^* \rightarrow 0 \quad T^* \rightarrow 1 \quad ; \quad r^* \rightarrow \infty \]  \hspace{1cm} (14)

\[ \frac{\partial \Phi^*}{\partial \theta} = 0 \quad \frac{\partial T^*}{\partial \theta} = 0 \quad ; \quad \theta = 0 \quad 1 < r^* < b^* \]  \hspace{1cm} (15)

\[ \frac{\partial \Phi^*}{\partial \theta} = 0 \quad \frac{\partial T^*}{\partial \theta} = 0 \quad ; \quad \theta = \frac{\pi}{2} \quad 1 < r^* < \infty \]  \hspace{1cm} (16)

\[- \frac{\partial \Phi^*}{\partial \theta} \sin \theta + \frac{\partial \Phi^*}{\partial r^*} b^* = 0 \quad - \frac{\partial T^*}{\partial \theta} \sin \theta + \frac{\partial T^*}{\partial r^*} b^* = 0 \quad ; \quad r^* \cos \theta = b^* \quad 0 < \theta < \theta_c \]  \hspace{1cm} (17)

The Jacob number, appearing in the nondimensional energy equation, is defined as

\[ \text{Ja} = \frac{c_p(T_{\infty} - T_s)}{k_{fg} R} = \frac{V_{ref} R}{\alpha} \]  \hspace{1cm} (18)

The coupling condition can be written in nondimensional form as

\[ \left. \frac{\partial T^*}{\partial r^*} \right|_{r^*=1} = V_s = \left. \frac{\partial \Phi^*}{\partial r^*} \right|_{r^*=1} \]  \hspace{1cm} (19)

### Numerical Considerations

The numerical simulation of this vaporization problem is carried out using the finite difference method. This choice can be explained by the robustness of this method when applied to convection-diffusion problems within sufficiently simple domains [17, 18]. Also, it allows the analysis of more evolving physical situations of interest, such as the change in shape and in size of the droplet, with less amount of work than other numerical methods available for the solution of partial differential equations. For the
current numerical approach, the clustering of points in regions where steep gradients are expected and the domain regularization are some features that can be achieved by simple algebraic transformations when the discretizing grid is generated [20, 21].

For the numerical integration of the equations, boundary conditions at infinity (Eq. 14) are applied at a finite distance $a$ (Fig.2). Due to this approximation, the obtained results must be tested for increasing $a$ in order to satisfy a established precision requirement.

The proposed transformation will map the symmetry plane between two droplets ($r^s \cos \theta = b^s$) and the outer boundary ($r^s \sin \theta = a^s$) onto a single side in the transformed domain. By regularizing the domain and making the discretization simpler, the use of nonuniform grids is avoided, specially near the droplet where precision is necessary, due to the gradients existing in this region of the physical domain.

We begin by defining:

$$f(\theta) = \begin{cases} \frac{b^s}{\cos \theta}, & \text{for } 0 \leq \theta \leq \theta_c \\ \frac{a^s}{\sin \theta}, & \text{for } \theta_c < \theta \leq \frac{\pi}{2} \end{cases}$$

(20)

where $\theta_c$ corresponds to the angle of the domain diagonal.

The function $f(\theta)$ defined above is continuous but its derivative can not be defined along the $\theta = \theta_c$ line. Looking ahead in the analysis, one can notice that the derivatives of this function will be present in the transformed equations, which would lead to discontinuous coefficients for these equations. In order to avoid this shortcoming, which can cause complications in the numerical solution, the transformation developed for the polar angle places a grid line along $\theta = \theta_c$ and appropriate matching conditions are applied to the mesh points along that line. These conditions guarantee that fluxes of mass and energy are conserved across the referred line. This corresponds to splitting the physical domain into two subdomains divided by the $\theta = \theta_c$ line with boundary condition specified so that both solutions match smoothly, despite the discontinuity introduced by the transformation.

The matching conditions can be written, in nondimensional form as:

$$\left. \frac{\partial \Phi^*}{\partial r^s} \right|_+ = \left. \frac{\partial \Phi^*}{\partial r^s} \right|_- \quad \left. \frac{\partial T^*}{\partial r^s} \right|_+ = \left. \frac{\partial T^*}{\partial r^s} \right|_- ; \quad 1 < r^s < f(\theta_c) \quad \theta = \theta_c$$

(21)
\[
\frac{\partial \Phi^a}{\partial \theta} \bigg|_+ = \frac{\partial \Phi^a}{\partial \theta} \bigg|_- \quad \frac{\partial T^a}{\partial \theta} \bigg|_+ = \frac{\partial T^a}{\partial \theta} \bigg|_- \quad ; 1 < r^a < f(\theta_c) \quad \theta = \theta_c
\]  

Since both potential and temperature are specified at \(r^a = f(\theta_c)\) for the diagonal, the conditions above also specify that these variables have the same value for each grid point, if evaluated using results from either side of the diagonal.

\[
\Phi_+ = \Phi_- \quad T_+ = T_- \quad ; R < r < f(\theta_c) \quad \theta = \theta_c
\]  

and

\[
\Phi^a_+ = \Phi^a_- \quad T^a_+ = T^a_- \quad ; 1 < r^a < f(\theta_c) \quad \theta = \theta_c
\]  

Using these ideas, the first transformation can be defined and the new variables can be written as:

\[
\eta = \frac{r^a - R}{f(\theta) - R}
\]

and

\[
\zeta = \begin{cases} 
\frac{\zeta}{\theta_c} \cdot \theta & \text{, for } 0 \leq \theta \leq \theta_c \\
\frac{1 - \zeta}{\pi/2 - \theta_c} (\theta - \theta_c) + 1 & \text{, for } \theta_c < \theta \leq \frac{\pi}{2}
\end{cases}
\]

Values of \(\zeta_c\) are chosen such that the ratio between the number of points placed below and above the diagonal is proportional to the aspect ratio of the truncated physical domain. In doing so, a better distribution of grid points can be achieved, since, as the domain is stretched, fewer points are required above the domain diagonal due to the smaller gradients present in this region as the far-field conditions are approached.

It is desirable to cluster points in regions where steep gradients in the solution are expected, as is the case near the droplet surface. In doing so, less points need to be used for the same precision, which translates into less expensive computations.

For this purposes we can define [18]

\[
\epsilon = 1 - \frac{\ln\frac{(c+1)-\eta}{(c-1)+\eta}}{\ln\frac{(c+1)}{(c-1)}}
\]  

\[8\]
to be used, instead of \( \eta \). The clustering parameter \( c \) assumes values between 1 and \( \infty \), and its influence on the results must be evaluated. A change in the rate of convergence is expected, as the grid is refined, for different values of this parameter and care must be exercised to avoid numerical errors as the distance between points become very small.

The governing equations must be rewritten using the new variables. Since the transformations developed above are all analytical, their metrics can be promptly obtained. The transformed equations present new terms, such as cross derivatives, and coefficients dependent on the position in the transformed domain. The boundary conditions must be subjected to the same transformation as the differential equations. The transformed equations and their corresponding boundary conditions, as well as the transformation metrics, are presented in appendix A.

Once obtained, the transformed equations can be discretized using the finite – difference method. A second-order accurate scheme is used throughout the domain, including the boundary points, with exception of the cross derivatives terms in the corner points, where a backwards first order approximation is used. The discretization leads to a system of algebraic equations whose unknowns are the values of the potential and temperature at each grid point. A typical equation in this system presents nine non zero coefficients. The system of equations is solved first for the potential, using a guessed value for the droplet surface velocity. With the potential obtained for each grid point, the velocity components can be found and the temperature field obtained by solving the corresponding discretized equations. Using the temperature gradient at the droplet surface, a new value for the surface velocity can be found (Eq. 19) and used to improve the potential. This procedure is repeated until an established tolerance is reached for the surface velocity. Specifying a constant value for the potential at the outer boundary, which corresponds to a parallel flow, has the effect of making the discrete system of equations for the potential to have a unique solution.

The iterative scheme for the solution employs the analytically obtained isolated droplet surface velocity as the initial guess. Under-relaxation is used between iterations to guarantee convergence, as higher values of \( Ja \) are considered. This fact can be explained by the increasing importance of the convective terms in the energy equation as \( Ja \) increases. Upper-limit values for the under-relaxation parameter were found to be 1.00, 0.75, 0.40 for \( Ja \) values of 1, 10, 100 respectively. For values of \( Ja \) lower than unity, no under-relaxation is necessary.
The system obtained from the procedure described above is solved using the IMSL routine DLSARB [22], which incorporates an iterative refinement of the solution [23] and makes use of the sparseness of the system matrix by means of band storing its coefficients [19]. Once convergence is achieved, the transformations must be inverted, so that, the values obtained in the transformed plane can be allocated to the corresponding points in the physical domain. Since all transformations are analytic, the inversion relations can be written as

\[ r^* = \eta \{ f(\theta) - 1 \} + 1 \]  

(28)

where

\[ \eta = \frac{(c+1) - (c - 1)(\frac{c+1}{c-1})^{(1-\epsilon)}}{(\frac{c+1}{c-1})^{(1-\epsilon)} + 1} \]  

(29)

and

\[ \theta = \begin{cases} \frac{\theta_c \cdot \zeta}{\zeta_c}, & \text{for } 0 \leq \theta \leq \theta_c \\ \frac{\pi/2 - \theta_c}{1 - \zeta_c} (\zeta - 1) + \frac{\pi}{2}, & \text{for } \theta_c < \theta \leq \frac{\pi}{2} \end{cases} \]  

(30)

Some grids generated using this method are shown in Fig. 2. The developed transformations place a coordinate line along the droplet surface, avoiding the use of nonuniform grids [24]. Normal gradients to the droplet, present in boundary (Eq. 13) and coupling (Eq. 19) conditions can be readily obtained, since the grid lines are also orthogonal to the droplet surface.

Results

In order to validate the numerical scheme, we compare the calculations with analytical results for a number of limiting cases.

The first of these modified problems is the one related to a single droplet in an infinite medium. Equations and boundary conditions for this problem [25] are the same as Eq.(11)-(17), when the half distance between droplets \( l^* \) approaches infinity. For numerical simulation purposes, this conditions will be obtained by considering \( l^* \) large enough, such that the interaction between droplets can be neglected. Analytical solutions for the nondimensional potential \( (\Phi^*) \) and temperature \( (T^*) \) fields are available [25], as well as
values for the surface velocity, that couples both equations. Converged numerical solutions are presented in Fig. 3 for comparison, for the region near the droplet. Necessary values for the interdroplet distance to guarantee droplet-isolation proved to be function of the Jacob number \((J_d)\). Typical values used for \(b^e\) were 200 and 400 for \(J_d\) values of 1 and 10, respectively. The analytical solutions are presented in appendix B for completeness of discussion. Although the transformations were not developed for this specific case, the numerical results show very good agreement with the analytical one, validating the numerical procedure. The nondimensional droplet surface velocity obtained numerically agrees with the corresponding analytical value to 4 significant digits. Besides, by comparing the numerically obtained value for the droplet surface velocity against the analytical one, a clustering parameter \(c\) lower bound value can be obtained, for a given precision established for the calculations. This lower bound value can be used when the stream of droplets is considered, since for finite droplet distances, large values of \(c\) can be used, due to the natural concentration of grid points near the droplet surface when finite inter-droplet distances are considered.

Substituting the droplet in the previous case by a porous sphere, where the mass flux is prescribed, also permits an analytical solution to be found (appendix B) and is used as another example to validate the numerical procedure. Results are presented in Fig. 4 showing the good agreement between both solutions. For this case, the full system of equations has to be modified by relaxing the coupling condition (Eq. 19) between the potential and energy equation. If the inter-droplet distance is made finite, interaction becomes important and an analytical solution can not be found, due to the nonregularity of the domain. Numerical simulations of the porous sphere problem were performed and the results are presented in Fig. 5 for the quantities of interest. One important feature to be mentioned is the fact that the sphere boundary is not an equipotential surface, as in the isolated case, thus indicating the existence of a tangential velocity component at the sphere surface for this nonconvective case.

The full system of equations can be now analyzed, and solutions are presented in Figs. 6 to 8 for different inter-droplet distances \(2b^e\), showing the equipotential contours, velocity vectors and temperature distribution, within the region near the droplet. Aspect ratios values up to 7:1 were used in the simulations, without loss of precision in the near field, leading to to variations in the total droplet vaporization rate of less than \(2\%\). It is important to point out that the droplet surface nearly becomes an equipotential surface for
this case, despite the presence of the other droplets. The tendency for a tangential velocity to exist is countered by the angular dependence of the vaporization velocity, as shown in Fig. 9 for different droplet spacings. Therefore, within the precision of these calculations, the velocity field happens to be identical to the viscous flow solution, as the absence of a tangential velocity at the droplet surface is equivalent to the nonslip requirement at the surface. This fact is demonstrated by Fig. 9 which shows the negligible values of the tangential velocity at the droplet surface. Away from an individual droplet, the influence from the other droplets in the stream becomes important, and the profiles are similar to those obtained if the stream of droplets is approximated as a vaporizing liquid cylinder.

The nondimensional vaporization rate can be obtained from

\[
\dot{m}^* = \frac{\dot{m}}{4\pi\rho R^2 V_{mf}} = \int_0^{\pi/2} V_s^*(\theta) \sin(\theta) d\theta
\]

(31)

The average droplet Nusselt number, based on the diameter, is related to the nondimensional mass vaporization rate by

\[
\overline{Nu}_D = 2 \dot{m}^*
\]

(32)

Once the converged solution is obtained for each droplet spacing, the vaporization rate is calculated using Eq.(31). The surface velocity profiles are interpolated using cubic splines and the integral in Eq.(31) is evaluated using the subroutines DCSAKM and DCSITG from IMSL [22], respectively.

It would be desirable to correlate the results for the stream vaporization with an equation of the form:

\[
\frac{\dot{m}^*}{\dot{m}^*_{iso}} = G(b^*)
\]

(33)

The ratio of the nondimensional droplet vaporization rate in a stream to that of an isolated droplet, is presented in Fig. 10 as a function of the droplet spacing \(b^*\) for several values of the Jacob number. The results in Fig. 10 show a negligible Jacob number dependence of the ratio of mass vaporization rates. This fact indicates that droplets in a stream and under isolated conditions experience a similar behavior for given fluid thermodynamics properties, making the functional form postulated in Eq.(33) valid. A
power function approximation of the results presented in Fig. 10 leads to the following expression for \( G(b^*) \):

\[
G(b^*) = 1 - g_0 \left( b^* \right)^{-g_1}
\]  

(34)

where \( g_0 = 0.004078 \), \( g_1 = 0.808746 \).

Conclusions

The vaporization of an infinitely-long droplet stream with finite interdroplet spacing has been numerically investigated for the case of zero Reynolds number (stagnant conditions). The numerical procedure has been validated by comparison with classical analytical results for an isolated droplet. The results indicate that the vaporization of a droplet stream can be calculated as the product of the isolated vaporization rate times a function of the interdroplet distance. Furthermore, under the conditions of no free or forced convection, these calculations show that the potential flow solution is indeed also the solution of the constant property Navier-Stokes equations, within the precision of the calculations.
Appendix A

Transformation metrics

\[ \eta_r^* = \frac{1}{f(\theta) - 1} \]  
\[ \eta_\theta = -\frac{r^* - 1}{[f(\theta) - 1]^2} f'(\theta) \]  
(A.1)

\[ \eta_r^* = 0 \]  
\[ \eta_{\theta\theta} = -(r^* - 1)[f(\theta) - 1]f''(\theta) - 2f'^2(\theta) \]  
\[ \frac{[f(\theta) - 1]^3}{f(\theta) - 1} \]  
(A.2)

\[ \zeta^*_r = 0 \]  
(A.3)

\[ \zeta_\theta = \frac{\zeta_c}{\theta_c} \quad 0 \leq \theta \leq \theta_c \]  
(A.4)

\[ \zeta_\theta = \frac{1 - \zeta_c}{\pi f^2 - \theta_c} \quad \frac{\pi f^2}{2} \leq \theta \leq \frac{\pi}{2} \]  
(A.5)

\[ \zeta_{r^*r^*} = 0 \]  
\[ \zeta_{\theta\theta} = 0 \]  
(A.6)

\[ \epsilon_\eta = \frac{2c}{ln \frac{c^2 + 1}{\eta - 1}} \]  
\[ \frac{1}{c^2 - (\eta - 1)^2} \]  
(A.7)

\[ \epsilon_m = \frac{4c}{ln \frac{c^2 + 1}{\eta - 1}} \]  
\[ \frac{\eta - 1}{c^2 - (\eta - 1)^2} \]  
(A.8)

Transformed Potential Equation

\[ \nabla^2 \Phi^* = \{ F_a \} \frac{\partial^2 \Phi^*}{\partial \zeta^2} + \{ F_b \} \frac{\partial^2 \Phi^*}{\partial \zeta^2} + \{ F_c \} \frac{\partial^2 \Phi^*}{\partial \zeta^2} + \{ F_d \} \frac{\partial \Phi^*}{\partial \zeta} + \{ F_e \} \frac{\partial \Phi^*}{\partial \zeta} = 0 \]  
(A.9)

where

\[ F_a = \{ F_2 + F_4 \} F_8^2 \]  
(A.10)

\[ F_b = \zeta_\theta^2 \]  
(A.11)

\[ F_c = \{ F_5 + F_6 \} F_8 \]  
(A.12)
\[ F_d = \{ F_1 + F_3 \} F_8 + \{ F_2 + F_4 \} F_7 \]

\[ F_8 = \frac{Q \phi}{r^2 \sin \theta} \cos \phi \]  

(A.13)

\[ F_9 = \frac{2}{r^2 [f(\theta) - 1]^2} \]

(A.14)

\[ F_1 = \frac{2}{r^2 [f(\theta) - 1]^2} \]

(A.15)

\[ F_2 = \frac{1}{[f(\theta) - 1]^2} \]

(A.16)

\[ F_3 = \frac{r^* - 1}{r^2 \sin \phi} \left[ \frac{\cos \phi f'(\theta) + \sin \phi f''(\theta) - 2 \sin \phi \frac{a^*}{\sin \phi} f''(\theta)}{[f(\theta) - 1]^3} \right] \]

(A.17)

\[ F_4 = \frac{1}{r^2} \left[ \frac{r^* - 1}{[f(\theta) - 1]^2} f'(\theta) \right]^2 \]

(A.18)

\[ F_5 = F_6 = -\frac{1}{r^2} \left[ \frac{r^* - 1}{[f(\theta) - 1]^2} \phi \right] \]

(A.19)

\[ F_7 = \frac{4c}{\ln \left( \frac{c^2}{(\eta - 1)^2} \right)} \frac{\eta - 1}{c^2 - (\eta - 1)^2} \]

(A.20)

\[ F_8 = \frac{2c}{\ln \left( \frac{c^2}{(\eta - 1)^2} \right)} \frac{1}{c^2 - (\eta - 1)^2} \]

(A.21)

\[ f'(\theta) = \begin{cases} \frac{b^* \sin \theta}{\cos^2 \theta}, & \text{for } 0 \leq \theta \leq \theta_c \\ -\frac{a^* \cos \theta}{\sin \theta}, & \text{for } \theta_c < \theta \leq \frac{\pi}{2} \end{cases} \]

(A.22)

\[ f''(\theta) = \begin{cases} \frac{b^* (1 + \sin^2 \theta)}{\cos^2 \theta}, & \text{for } 0 \leq \theta \leq \theta_c \\ -\frac{a^* (1 + \cos^2 \theta)}{\sin^2 \theta}, & \text{for } \theta_c < \theta \leq \frac{\pi}{2} \end{cases} \]

(A.23)

\[ u^* \left( \frac{\partial T^*}{\partial r^*} \right) + u^* \left( \frac{\partial T^*}{\partial \theta^*} \right) = \frac{1}{Ja} \nabla^2 T^* \]

Transformed Energy Equation

(A.24)
\{F_9 \cdot F_3\} \frac{\partial T^*}{\partial \epsilon} + \{F_{10}\} \frac{\partial T^*}{\partial \zeta} = \frac{1}{\kappa} \nabla^2 T^* \tag{A.25}

where

\[ F_9 = u_r^* \eta_r^* + \frac{v_0^*}{r^*} \eta_\theta \tag{A.26} \]

\[ F_{10} = \frac{v_r^*}{r^*} \zeta \theta \tag{A.27} \]

and the Laplacian operator was already defined for the Potential Equation.

**Boundary Conditions**

\[ \frac{\partial \Phi^*}{\partial \epsilon} = \frac{1}{\eta_r^* \epsilon_\eta} V_s^*(\theta) \quad T^* = 0 \quad \epsilon = 0 \quad 0 < \zeta < 1 \tag{A.28} \]

\[ \Phi^* = 0 \quad T^* = 1 \quad \epsilon = 1 \quad \zeta_c < \zeta < 1 \tag{A.29} \]

\[ \frac{\partial \Phi^*}{\partial \zeta} + \frac{\partial T^*}{\partial \epsilon} = 0 \quad \zeta = 0 \quad 0 < \epsilon < 1 \tag{A.30} \]

\[ \frac{\partial \Phi^*}{\partial \zeta} + \frac{\partial T^*}{\partial \epsilon} = 0 \quad \zeta = 1 \quad 0 < \epsilon < 1 \tag{A.31} \]

\[ -\{\zeta \sin \theta\} \frac{\partial \Phi^*}{\partial \zeta} + \{b^* \eta_r^* - \eta_\theta \sin \theta\} \epsilon_\eta \frac{\partial \Phi^*}{\partial \epsilon} = 0 \quad \tag{A.32} \]

**Diagonal Conditions**

\[ \frac{\partial \Phi^*}{\partial \epsilon} \bigg|_+ = \frac{\partial \Phi^*}{\partial \epsilon} \bigg|_- \quad \frac{\partial T^*}{\partial \epsilon} \bigg|_+ = \frac{\partial T^*}{\partial \epsilon} \bigg|_- \quad 0 < \epsilon < 1 \quad \zeta = \zeta_c \tag{A.33} \]

\[ \{\eta_\theta \epsilon_\eta\} \frac{\partial \Phi^*}{\partial \zeta} \bigg|_+ + \{\zeta_\theta\} \frac{\partial \Phi^*}{\partial \zeta} \bigg|_+ = \{\eta_\theta \epsilon_\eta\} \frac{\partial \Phi^*}{\partial \zeta} \bigg|_- + \{\zeta_\theta\} \frac{\partial \Phi^*}{\partial \zeta} \bigg|_- \quad \tag{A.34} \]

\[ \{\eta_\theta \epsilon_\eta\} \frac{\partial T^*}{\partial \zeta} \bigg|_+ + \{\zeta_\theta\} \frac{\partial T^*}{\partial \zeta} \bigg|_+ = \{\eta_\theta \epsilon_\eta\} \frac{\partial T^*}{\partial \zeta} \bigg|_- + \{\zeta_\theta\} \frac{\partial T^*}{\partial \zeta} \bigg|_- \quad 1 < \epsilon < 1 \quad \zeta = \zeta_c \tag{A.34} \]
Appendix B

Analytical Solutions

1 - Vaporizing droplet in a infinite domain [25]

\[
\dot{m} = \frac{1}{Ja} \ln(1 + Ja) \tag{B.1}
\]

\[
\Phi = \dot{m} \left( 1 - \frac{1}{r^2} \right) \tag{B.2}
\]

\[
T^* = \frac{1}{Ja} \left\{ (1 + Ja)^{1/r^*} - 1 \right\} \tag{B.3}
\]

2 - Isolated porous sphere (\(\dot{m}^*\) given)

\[
\Phi^* = \dot{m}^* \left( 1 - \frac{1}{r^2} \right) \tag{B.4}
\]

\[
T^* = \left( 1 - e^{-\dot{m}^* \cdot \lambda_t} \right)^{-1} \left( e^{\dot{m}^* \cdot \lambda_t / r^*} - e^{-\dot{m}^* \cdot \lambda_t} \right) \tag{B.5}
\]

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