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SIMPLICIAL TRIANGULATIONS OF TOPOLOGICAL MANIFOLDS

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In this lecture we will motivate and outline our work concerning simplicial triangulations of topological manifolds. Details of these and related results will appear in [8], [9], and [10].

The primary question we are concerned with is when can a given topological manifold M be triangulated as a simplicial complex, and if so, in how many "different" ways can it be triangulated? The work of R. Kirby and L. Siebenmann ([11], [12]) shows that in each dimension greater than four there exist closed topological manifolds which admit no piecewise linear manifold structure and hence cannot be triangulated as a combinatorial manifold. However, R. D. Edwards [5] has recently demonstrated the existence of noncombinatorial triangulations of S^n, $n \geqq 5$. It is still unknown whether or not every topological manifold can be triangulated as a simplicial complex.

Let us first determine what restrictions are put on a triangulation of a topological manifold. Note that if X is a compact space, then the $(n-k)$-suspension of X, denoted $\Sigma^{n-k} X$, is homeomorphic ($\approx$) to the n-sphere S^n if and only if $c'X \times R^{n-k}$ is an open topological n-manifold, where $c'X$ denotes the open cone over X. Thus K is a triangulation of a topological n-manifold M without boundary if and only if the link L^k of an $(n-k-1)$-simplex in the first barycentric subdivision K' has the homology of S^k and $\Sigma^{n-k} L^k \approx S^n$. We improve this as follows.

Recall that a (polyhedral) *closed homology manifold* is a compact polyhedron with the property that the links of $(n-k)$-simplices have the homology of S^{k-1}.

THEOREM 1. *A closed homology n-manifold M is a topological n-manifold if and only if*

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(1) *for every 3-dimensional link L^3 of M, $\Sigma^{n-3} L^3 \approx S^n$, and*
(2) *every $(n-1)$-dimensional link of M is 1-connected.*

OUTLINE OF THE PROOF. By our observation above we need only check that $\Sigma^{n-k} L^k \approx S^n$ for every k-dimensional link L^k of M for $4 \leq k \leq n-1$.

Case 1. $k = 4$. We show that $c'L^4 \times R^{n-5}$ is an open topological n-manifold. Now L^4 is a closed homology 4-manifold, so the only non PL sphere links are the links of a finite number of vertices. For simplicity suppose there is only one such bad link and call it $\bar{L}$. Then $L^4 = P^4 \cup c\bar{L}$, where the union is taken along $\partial P^4 = \partial(c\bar{L}) = \bar{L}$. The double Q^4 of P^4 is a PL homology 4-sphere so that a recent result of R. D. Edwards [6] implies that $\Sigma^{n-4}Q^4 \approx S^n$; hence $c'Q^4 \times R^{n-5}$ is an open topological n-manifold. By the codimension one approximation theorem of Bryant-Edwards-Seebeck or of Ancel-Cannon [1], we can re-embed $c'P^4 \times R^{n-5} \subset c'Q^4 \times R^{n-5}$ via an embedding h so that its complement in $c'Q^4 \times R^{n-5}$ is 1-ULC. Since $c'\bar{L} \times R^{n-5}$ is an open topological manifold by (1), the taming theorem of J. Cannon [4] implies that $h(c'\bar{L} \times R^{n-5})$ is collared in the closure of the complement of $h(c'P^4 \times R^{n-5})$ in $c'Q^4 \times R^{n-5}$. This then implies that $h(c'P^4 \times R^{n-5}) \cup (h(c'\bar{L} \times R^{n-5}) \times [0, 1]) \approx c'L^4 \times R^{n-5}$ is an open topological n-manifold as required.

Case 2. $k \geq 5$. Suppose inductively that $\Sigma^{n-k+1}L^{k-1} \approx S^n$. This then implies that $L^k \times R^{n-k}$ and hence $L^k \times T^{n-k}$ is a topological manifold for every k-dimensional link L^k of M. By the results of [7] or [14], there exist a topological homology k-sphere H^k and a simple homotopy equivalence $f: H^k \times T^{n-k} \to L^k \times T^{n-k}$ which is homotopic to a homeomorphism h. As $k \geq 5$, the Kirby-Siebenmann obstruction to putting a PL manifold structure on H^k is zero, so that we can assume that H^k is a PL homology k-sphere. Now lift h to a bounded homeomorphism $h': H^k \times R^{n-k} \to L^k \times R^{n-k}$ which therefore extends to a homeomorphism $h': H^k * S^{n-k-1} \to L^k * S^{n-k-1}$ (cf. [16]). By a recent result of R. D. Edwards [6] $H^k * S^{n-k-1} \approx S^n$, so that $\Sigma^{n-k}L^k \approx S^n$ as required. □

So we now know how to identify a simplicial triangulation of a topological manifold. What "nice" properties of a simplicially triangulated topological manifold would one like? Note that if K is a polyhedron, then $K \times R$ is a PL $(n+1)$-manifold if and only if K is a PL n-manifold. This is a fundamental transversality property for PL manifolds. However, if $K \times R$ is a topological $(n+1)$-manifold it is not necessarily the case that K is a topological n-manifold. But observe that the links of K have all the suspension properties of the n-skeleton of a simplicially triangulated topological $(n+1)$-manifold. This then motivates the following definition.

A TRI_n *m-manifold* is a homology m-manifold M such that if $k < n$ and L is a k-dimensional link of M, then $\Sigma^{n-k}L^k \approx S^n$ or $\Sigma^{n-k+1}L^k \approx D^{n+1}$, where D^{n+1} is the $(n+1)$-disk. We now list some facts about TRI_n manifolds. Let K be a polyhedron.

(1) $K \times R$ is a TRI_n manifold if and only if K is a TRI_n manifold.

(2) If K is a TRI_n m-manifold without boundary and with $n \geq m$, then $K \times R^{n-m}$ is a topological n-manifold without boundary.

(3) If K is a TRI_n m-manifold with $m > n \geq 6$, then there exists a TRI_n m-manifold $\bar{K}$ which is also a topological manifold and a PL contractible map $f: \bar{K} \to K$. (By Theorem 1, K is a topological manifold except that the $(m-1)$-dimensional

links of K need to be 1-connected and we blow up these links via a PL contractible map to be 1-connected.)

We now wish to construct a "normal" bündle theory for TRI_n manifolds similar to PL block bundles. A TRI_n q-sphere is a TRI_n q-manifold H^q having the homology of S^q and if $q < n$ we further require that $\Sigma^{n-q}H^q \approx S^n$. A TRI_n *cell complex* is then a cone complex whose cones are cones on TRI_n spheres. A TRI_n cone q-bundle ξ^q/K over a TRI_n cell complex K assigns to each p-cell α of K a block B_α which is the cone on a $\mathrm{TRI}_n(p + q - 1)$-sphere and these cones fit together like the eclls in a cell complex. Using the mock bundle recipe of Buoncristiano, Rourke and Sanderson [3] for representing homotopy functors, Theorem 1 and facts (1)—(3) above, there exists a classifying space $B\,\mathrm{TRI}_n(q)$ for TRI_n cone q-bundles, $q \geqq n \geqq 6$. Let $B\,\mathrm{TRI}_n = \lim_{q\to\infty} B\,\mathrm{TRI}_n(q)$. Using fact (3) above, one shows that every TRI_n cone q-bundle, $q \geqq n \geqq 6$, is concordant to a topological block bundle, so that there is a natural map $j\colon B\,\mathrm{TRI}_n \to B\,\mathrm{TOP}$, where $B\,\mathrm{TOP}$ classifies stable topological block bundles.

We now return to our primary question, when can a given topological m-manifold M be triangulated as a simplicial complex, and if so, in how many "different" ways? Let N be a codimension zero submanifold of ∂M and let Σ_0 be a TRI_n manifold structure on N which extends to a neighborhood of N in M. Let $\mathscr{S}_{\mathrm{TRI}_n}(M \text{ rel } N, \Sigma_0)$ denote the set of TRI_n manifold structures on M agreeing with Σ_0 near N modulo the equivalence relation (called TRI_n *concordance*) that two such structures Γ_0 and Γ_1 on M are TRI_n concordant if there exists a TRI_n manifold structure Γ on $M \times I$ agreeing with $\Sigma_0 \times I$ near $N \times I$ and $\Gamma | M \times \{i\} = \Gamma_i$ for $i = 0, 1$.

Similarily let $\mathrm{Lift}(\tau \text{ rel } N, F_0)$ denote the set of lifts of the map $\tau\colon M \to B\,\mathrm{TOP}$, which classifies the stable topological tangent bundle of M, to $B\,\mathrm{TRI}_n$ through $j\colon B\,\mathrm{TRI}_n \to B\,\mathrm{TOP}$ that agree near N with a fixed lift F_0 of τ near N induced by Σ_0, modulo the equivalence relation of vertical homotopy rel N.

THEOREM 2 (CLASSIFICATION THEOREM). *Let M, Σ_0, and F_0 be as above. If $m > n \geqq 6$ ($m \geqq n \geqq 6$ if $N = \partial M$), then M admits a* TRI_n *manifold structure agreeing with Σ_0 near N if and only if τ has lift $M \to B\,\mathrm{TRI}_n$ equaling F_0 near N. In fact there is a bijection* $\mathscr{S}_{\mathrm{TRI}_n}(M \text{ rel } N, \Sigma_0) \to$ *Lift*$(\tau \text{ rel } N, F_0)$.

TOWARDS A PROOF OF THEOREM 2. Assume $\partial M = \emptyset$ and suppose $\tau\colon M \to B\,\mathrm{TOP}$ lifts to $B\,\mathrm{TRI}_n$. Then embed M in R^s for some large s and let Q be a PL manifold neighborhood of N equipped with a deformation retraction $r\colon Q \to M$. Then τr classifies a topological bundle over Q whose total space is homeomorphic to $M \times R^k$, for some k. As τr lifts to $B\,\mathrm{TRI}_n$, $M \times R^k$ is a TRI_n manifold. We now wish to show that this implies that M has a TRI_n manifold structure. It clearly suffices to show that if $M \times R$ is a TRI_n manifold, then so is M. This is accomplished via

THEOREM 3 (PRODUCT STRUCTURE THEOREM). *Let M^m be a connected topological m-manifold and let Θ be a* TRI_n *manifold structure on $M \times R$. Let N be a codimension zero submanifold of ∂M and Σ_0 a* TRI_n *manifold structure on N which extends to a neighborhood of N in M such that $\Sigma_0 \times R$ agrees with Θ near $N \times R$. If $m > n \geqq 6$ ($m \geqq n \geqq 6$ if $N = \partial M$), then there exists a* TRI_n *manifold structure Γ*

on M agreeing with Σ_0 near N, unique up to concordance rel Σ_0, *such that $\Gamma \times R$ is concordant* rel $\Sigma_0 \times R$ *to Θ.*

TOWARDS A PROOF OF THEOREM 3. Our proof is modeled on W. Browder's *Structures on $M \times R$* [2]. Assume M is closed. Triangulate $M \times R$ and R so that there is a simplicial map $\pi: M \times R \to R$ homotopic to the projection of $M \times R$ onto R. Let $*$ be a point interior to a simplex of R. Then $\pi^{-1}(*) \times R$ is a codimension zero TRI_n submanifold of $M \times R$, so that by fact (1) above $K = \pi^{-1}(*)$ is a TRI_n manifold. We can assume K is connected, so let W be the cobordism between K and M. By doing a series of handle exchanges we wish to make W into a topological manifold and the inclusion of K into W a simple homotopy equivalence. Then the topological s-cobordism theorem would yield a TRI_n manifold structure on M.

Step 1. We first do the handle exchanges in the homology manifold category. To do this we need surgery below the middle dimension for homology manifolds, a Whitney type trick, and some algebra. The first requirement is accomplished by Matsui [13]; the second is accomplished by using the topological Whitney trick in $M \times R$ and then making it polyhedral by using the homology transversality theorem of [7] and the established surgery below the middle dimension; and the last requirement is purely formal. Thus by doing a series of homology handle exchanges we arrive at a homology m-manifold K' and a cobordism W between K' and M with $K' \subset W$ a simple homotopy equivalence. Also $W = W' \cup W''$ where W' union a collar is a topological manifold and W'' is a homology manifold cobordism from K to K'.

Step 2. We observe that as K is a TRI_n manifold, by using Theorem 1 and fact (3) we can resolve the singularities of W'' via a simple homotopy equivalence so that W is in fact a TRI_n manifold which is a topological manifold. Thus W is our desired topological s-cobordism. □

We now discuss the (homotopic) fiber $\mathrm{TOP}/\mathrm{TRI}_n$ of $j: B\,\mathrm{TRI}_n \to B\,\mathrm{TOP}$. Let $\theta_3^{\mathrm{TRI}_n}$ denote the group of oriented PL homology 3-spheres modulo those which bound acyclic TRI_n 4-manifolds; let $\Theta_3^{\mathrm{TRI}_n/\mathrm{PL}}$ denote the group of oriented PL homology 3-spheres which bound acyclic TRI_n 4-manifolds modulo those which bound acyclic PL 4-manifolds; and let θ_3^H denote the group of PL homology 3-spheres modulo those which bound acyclic PL 4-manifolds. The only concrete theorem known about θ_3^H is the existence of the Kervaire-Milnor-Rochlin surjection $\alpha: \theta_3^H \to Z_2$. From the definitions we have the short exact sequence $0 \to \theta_3^{\mathrm{TRI}_n/\mathrm{PL}} \to \theta_3^H \to \theta_3^{\mathrm{TRI}_n} \to 0$.

THEOREM 4. *If $n \geq 6$, the homotopy groups of* $\mathrm{TOP}/\mathrm{TRI}_n$ *are zero except possibly for π_3 and π_4. Furthermore there are two exact sequences*

(1) $0 \to \pi_4 \to$ *kernel*$(\alpha: \theta_3^H \to Z_2) \to \theta_3^{\mathrm{TRI}_n} \to \pi_3 \to 0$,

(2) $0 \to \pi_4 \to \theta_3^{\mathrm{TRI}_n/\mathrm{PL}} \to^{\alpha} Z_2 \to \pi_3 \to 0$,

where α is the Kervaire-Milnor-Rochlin map.

COROLLARY 5. (1) $\pi_3(\mathrm{TOP}/\mathrm{TRI}_n)$ *has at most* 2 *elements.*

(2) $\pi_3(\mathrm{TOP}/\mathrm{TRI}_n) = 0$ *if and only if there exists a* PL *homology* 3*-sphere with* $\alpha(H^3) = 1$ *and* $\Sigma^{n-3} H^3 \approx S^n$.

(3) $\pi_4(\mathrm{TOP}/\mathrm{TRI}_n) = 0$ *if and only if any* PL *homology* 3*-sphere with* $\alpha(H^3) = 0$ *and* $\Sigma^{n-3} H^3 \approx S^n$ *bounds an acyclic* PL *4-manifold.*

We also have the following existence theorem.

THEOREM 6. *Every topological m-manifold has a* TRI_n *manifold structure for* $m > n \geqq 6$ ($m \geqq n \geqq 6$ *if* $\partial M = \emptyset$) *if and only if there exists a* PL *homology 3-sphere* H^3 *satisfying the following* 3 *properties.*

(1) $\alpha(H^3) = 1$,

(2) $\Sigma^{n-3}H^3 \approx S^n$,

(3) $H^3 \# H^3$ *bounds a* PL *acyclic 4-manifold.*

REMARK. When $m = 5$, L. Siebenmann demonstrated in [16] that the existence of a PL homology 3-sphere H^3 satisfying (1) and (2) above implied that all closed oriented 5-manifolds can be triangulated as simplicial complexes. If $H^3 \# H^3$ bounds a contractible PL 4-manifold, then he also shows that all 5-manifolds can be triangulated. For closed topological m-manifolds M with $6 \leqq n \leqq 8$ and with the integral Bockstein of the Kirby-Siebenmann obstruction to putting a PL manifold structure on M being zero, M. Scharlemann [15] has shown that (1) and (2) above imply that M is triangulable as a simplicial complex. T. Matumoto [14] proves a version of the sufficiency of Theorem 6 with (2) replaced by the condition that $\Sigma^{n-4}H^3 \approx S^{n-1}$.

REMARK. Our proof of Theorem 6 actually shows that if there exists a PL homology 3-sphere satisfying (1)—(3) above, then every topological m-manifold has a TRI_n manifold structure in which the 3-sphere links are PL homeomorphic to connected sums of H^3, $-H^3$, and S^3.

TOWARDS A PROOF OF THE SUFFICIENCY OF THEOREM 6. Let H^3 be a PL homology 3-sphere satisfying (1)—(3) of Theorem 6. One can consider TRI_n manifolds M whose 3-dimensional sphere links in M and ∂M are PL homeomorphic to connected sums of H^3, $-H^3$, and S^3. Call such manifolds H^3 manifolds. One can construct a classifying space BH^3 for stable TRI_n cone bundles based on H^3 manifolds. There are natural maps $i_0: BH^3 \to B\,\mathrm{TOP}$, $i_1: B\,\mathrm{PL} \to BH^3$ and $i_2: BH^3 \to B\,\mathrm{TRI}_n$. The fiber of i_1 is a $K(Z_2, 3)$ so that by considering the homotopy exact sequence of the triple ($B\,\mathrm{TOP}$, BH^3, $B\,\mathrm{PL}$) we have that i_0 is a homotopy equivalence. The result now follows from Theorem 2. □

More generally we have the following existence theorem. Let $Sq_k: H^4(\ ; Z_2) \to H^5(\ ; Z_k)$ denote the Bockstein associated with the short exact coefficient sequence $0 \to Z_k \to^{\times 2} Z_{2k} \to Z_2 \to 0$. Also let $\Delta(M) \in H^4(M; Z_2)$ denote the Kirby-Siebenmann obstruction to the existence of a PL manifold structure on M.

COROLLARY 7. *If there exists a closed topological m-manifold M with a* TRI_n *manifold structure,* $m \geqq n \geqq 6$, *and if* $Sq_{2k}\Delta(M) \neq 0$, *then there exists a* PL *homology 3-sphere* H^3 *such that*

(1) $\alpha(H^3) = 1$,

(2) $\Sigma^{n-3}H^3 \approx S^n$, *and*

(3) *the 2k-fold connected sum of* H^3 *bounds a* PL *acyclic 4-manifold.*

Also, if there exists a PL *homology 3-sphere* H^3 *satisfying* (1)—(3), *then every topological m-manifold M with* $Sq_k\Delta(M) = 0$ *has a* TRI_n *manifold structure if* $m > n \geqq 6$ ($m \geqq n \geqq 6$ *if* $\partial M = \emptyset$).

We also remark that there is a surgery theory for TRI_n manifolds completely analogous to topological surgery theory. This is given in [9].

We also investigate the question of whether a given topological n-manifold, $n \geqq 5$, can be triangulated as a simplicial homotopy manifold. For example,

PROPOSITION 5. *Suppose that every* PL *homotopy* 3-*sphere bounds a contractible* PL 4-*manifold. Then there is a one-to-one correspondence between the set of concordance classes of simplicial homotopy manifold triangulations of a topological n-manifold* M, $n \geqq 5$, *and concordance classes of* PL *manifold structures on* M.

PROPOSITION 6. *Suppose there exists a bad counterexample to the* 3-*dimensional Poincaré conjecture; namely suppose there exists a* PL *homotopy* 3-*sphere* H^3, *with*

(i) $\alpha(H^3) = 1$, *and*

(ii) $H^3 \# H^3$ *bounds a contractible* PL 4-*manifold.*

Then every topological n-manifold, $n \geqq 5$, *can be triangulated as a simplicial homotopy manifold.*

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