ANOTHER CONSTRUCTION OF AN EXOTIC $S^1 \times S^3 \# S^2 \times S^2$

Ronald Fintushel$^1$ and Ronald J. Stern$^2$

This note was motivated by Selman Akbulut's talk at this conference. (See [A].) As Akbulut pointed out, if one could construct an exotic twisted $S^3$-bundle over $S^1$, with a homotopy equivalence $g: N^4 \to S^1 \times S^3$, then if a transverse preimage of an $S^3$-fiber is a homology sphere $H^3$, we must have $\mu(H^3) \neq 0$. But splitting $N^4$ along $H^3$ yields an acyclic 4-manifold whose boundary is $H^3 \# H^3$. Thus searching for an exotic $S^1 \times S^3$ is an approach toward finding the long sought after element of order 2 in $\Theta^3_H$.

Akbulut's construction is suggested by the fact that the complement of a tubular neighborhood $E(RP^2)$ of $RP^2$ in $R^4$ is $S^1 \times S^3$. His idea was to look for an $RP^2$ in $Q^4$, Cappell and Shaneson's exotic $RP^4([CS])$, such that $\pi_1(Q^4 - RP^2) = Z$, and then form $Q^4 - E(RP^2) \cup S^1 \times B^3$. Unable to find such an $RP^2$ embedded in $Q^4$, Akbulut was nonetheless able to find an $RP^2$ in $Q^4 \# S^2 \times S^2$ with $\pi_1(Q^4 \# S^2 \times S^2 - RP^2) = Z$ and he was then able to form $Q^4 \# S^2 \times S^2 - E(RP^2) \cup S^1 \times B^3$ an exotic $S^1 \times S^3 \# S^2 \times S^2$.

After seeing Akbulut's talk we decided to see if one could construct an exotic $S^1 \times S^3 \# S^2 \times S^2$ using the techniques we promoted in [FS$_1$] and [FS$_2$]. As we show this is quite simple to do and the invariant $\rho$ of these papers can be used to detect the fact that the construction is exotic. Instead of viewing $S^1 \times S^3$ as $S^1 \times B^3 \cup S^1 \times B^3$, it is more convenient from our point of view to think of $S^1 \times S^3$ as $S^2 \times MB \cup S^1 \times B^3$ (MB = Mbius band). For our construction we start with $K^3$ a Seifert-fibered homology $S^2 \times S^1$ obtained by surgering an exceptional fiber of $\Sigma(3,5,19)$ and form $X^4$, the mapping cylinder of the free involution contained in the $S^1$-action on $K^3$. If we could show that $K^3$ bounded a homotopy $B^3 \times S^1$ with $\pi_1$ mapping onto, we could take its union with $X^4$ and thus construct a fake $S^1 \times S^3$. We cannot do this, but we are able to show that $K^3$ bounds a homotopy $B^3 \times S^1 \# S^2 \times S^2$ and thus we are able to form $M^4$, a homotopy $S^1 \times S^3 \# S^2 \times S^2$. As in [FS$_1$] we can show that if $M^4$ were s-cobordant to $S^1 \times S^3 \# S^2 \times S^2$ then

$^1$Supported in part by NSF grant MCS 7900244A01.
$^2$Supported in part by NSF grant MCS 8002843A01.
\[
\mu(K/\mathbb{Z}_2) - \frac{1}{2} \alpha(K,\mathbb{Z}_2) = \rho(M^4) = \rho(S^1 \times S^3 \# S^2 \times S^2) \\
= \mu(S^2 \times S^1) - \frac{1}{2} \alpha(S^2 \times S^1, \mathbb{Z}_2) = 0 \text{ (mod 16)}
\]

for some almost framing of \( K/\mathbb{Z}_2 \). However \( \alpha(K,\mathbb{Z}_2) = 0 \) and the two \( \mu \)-invariants of \( K/\mathbb{Z}_2 \) are both 8 (mod 16); so \( M^4 \) is exotic. Finally, we are able to show that the double cover \( \tilde{M} \) is standard, i.e. \( \tilde{M} \) is diffeomorphic to \( S^1 \times S^3 \# S^2 \times S^2 \# S^2 \times S^2 \).

We now proceed with the construction of \( M^4 \). Let \( K^3 \) be the homology \( S^2 \times S^1 \) which is the boundary of the plumbing manifold

\[
\begin{array}{cccc}
-3 & -1 & -4 & -4 \\
-3 & & & \\
-2 & & & \\
\end{array}
\]

Then \( K \) is Seifert fibered with Seifert invariants \(((1,1),(3,-1),(5,-2),(15,-4))\); so the involution contained in the \( S^1 \)-action on \( K \) is free. Let \( X^4 \) be the mapping cylinder of the orbit map \( K \to K/\mathbb{Z}_2 \). As was shown in our earlier paper [FS' Lemma 3.1] there is a \( \mathbb{Z}_2 \)-equivariant map \( K \to S^2 \times S^1 \) which induces isomorphisms on homology. (The involution on \( S^2 \times S^1 \) is identity \times antipodal.) Taking mapping cylinders there is an induced map \( f: X \to S^2 \times M \) which induces isomorphisms on homology.

We have the following Kirby calculus picture for \( K \):

(\text{cf [FS; p. 362].})

Now construct a cobordism \( Y^4 \) from \( K \) to \( 3 \times Y = \hat{K} \) by attaching the following 2-handles to \( K \times I \):

\[
4
\]

\[
1
\]

\[
1
\]

\[
-4
\]
We claim that $f$ extends over these 2-handles to a map:

$$f : X \cup Y \times S^2 \times MB \cup (S^2 \times S^1 \times I \# S^2 \times S^2)$$.

To see this follow the 2-handles back through the Kirby calculus argument in [FS$_2$; p. 361-362]. The attaching circles are $k_0$ with 0-framing and $k_2$ with 2-framing:

On $K$-(exceptional fibers), $f$ preserves $S^1$-fibers and is a 15-fold covering. The image of $f$ (see [FS$_2$; Lemma 3.1]) is $S^2 \times S^1$:

In $S^2 \times S^1$, $f(k_0)$ is nullhomologous (in the above diagram we see that $f(k_0)$ bounds a genus 1 surface) therefore $f(k_0)$ is nullhomotopic in $S^2 \times S^1$. So there is a homotopy in $S^2 \times S^1$ of $f(k_0)$ to a trivial knot. By the homotopy
extension property this extends to a homotopy from the identity of \( S^2 \times S^1 \) to a map \( g \) of \( S^2 \times S^1 \) to itself which takes \( f(k_0) \) to a trivial knot. We can also easily arrange that \( g(f(k_1)) \) be a meridian of \( g(f(k_0)) \). Composing \( f \) with the above ambient homotopy, we extend \( f: X \cup K \times I \cup S^2 \times S^1 \times I \) so that \( f|K \times \{1\} : S^2 \times S^1 \times \{1\} \) maps tubular neighborhoods of \( k_1 \) and \( k_2 \) onto tubular neighborhoods of the components of a trivial Hopf link in \( S^2 \times S^1 \times \{1\} \).

For some framings \( a_1 \) on \( f(k_1 \times 1) \) and \( a_2 \) on \( f(k_2 \times 1) \), \( f \) will extend over \( Y = K \times I \cup h^2(k_1) \cup h^2(k_2) \cup S^2 \times S^1 \times I \cup h^2(f(k_1)) \cup h^2(f(k_2)) \). Because \( f|K \) induces isomorphisms on homology the naturality of the Mayer-Vietoris sequence and the 5-lemma imply that the intersection form of these two manifolds is the same. The intersection form of \( Y \) has matrix

\[
\begin{pmatrix}
0 & 1 \\
1 & 2 \\
\end{pmatrix}
\]

and therefore is even unimodular with signature 0. Hence the same is true for the intersection form

\[
\begin{pmatrix}
a_1 & 1 \\
1 & a_2 \\
\end{pmatrix}
\]

of \( S^2 \times S^1 \times I \cup h^2(f(k_1)) \cup h^2(f(k_2)) \). This means that this intersection form is the same as the intersection form of \( S^2 \times S^2 \). Hence \( S^2 \times S^1 \times I \cup h^2(f(k_1)) \cup h^2(f(k_2)) \cong S^2 \times S^1 \times I \# S^2 \times S^2 \).

Another 5-lemma argument shows that \( f|K: S^2 \times S^1 \) induces isomorphisms on homology. \( \hat{K} \) is:

```
\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]
ANOTHER CONSTRUCTION OF AN EXOTIC $S^1 \times S^3 \# S^2 \times S^2$

But the link

ribbon
in $S^3$ is concordant by the ribbon move shown to

![Diagram](image)

Hence there is a homology cobordism $Z$ from $\hat{K}$ to $S^2 \times S^1$=

![Diagram](image)

with $\pi_1(\hat{K}) \to \pi_1(Z)$ and $\pi_1(S^2 \times S^1) \to \pi_1(Z)$ onto. Let $\tilde{f}: S^2 \times S^1 \to S^2 \times S^1$ be a diffeomorphism inducing on homology the same homomorphism as $(f|\hat{K})_*$. (Here we identify $H_*(\hat{K})$ with $H_*(S^2 \times S^1)$ using the homology cobordism $Z$.) Then by obstruction theory $f \cup \tilde{f}$ extends to $f: Z \times S^2 \times S^1 \times I$. Since $\tilde{f}$ extends over $B^3 \times S^1 + B^3 \times S^1$ we obtain a homology equivalence

$$f: M = X \cup Y \cup Z \cup B^3 \times S^1 \cup S^2 \times MB \cup S^2 \times S^1 \times I \# S^2 \times S^2 \cup S^2 \times S^1 \times I \cup B^3 \times S^1 = S^1 \times S^3 \# S^2 \times S^2.$$ 

Using Van Kampen's theorem one checks that $\pi_1(M^4) = \mathbb{Z}$ and hence $f$ induces an isomorphism on fundamental groups. Let $\tilde{f}: \tilde{M} \to S^1 \times S^3 \# S^2 \times S^2 \# S^2 \times S^2$ be the induced map on oriented double covers. As $f$ is degree one, the induced homomorphisms on homology with $\mathbb{Z}[\mathbb{Z}]$ coefficients split [W; Lemma 2.2]. However, all homology groups are free and in any dimension are the same rank, so $\tilde{f}$, hence $f$, induces an isomorphism on homology with local coefficients. So $f$ is a homotopy equivalence. It is easy to compute that $\rho(M) \equiv 8(\text{mod } 16)$ (see [FS$_2$; proof of Prop. 5.5]); hence $M$ is not s-cobordant to $S^1 \times S^3 \# S^2 \times S^2$.

We now show that the double cover $\tilde{M}$ is standard. Note that $\tilde{M}$ is obtained by gluing together two copies of $Y \cup Z \cup B^3 \times S^1$ by the involution $t: K \times K$. Since $t$ is contained in an $S^1$ action, $t$ is isotopic to the identity. Hence $\tilde{M}$ is the double of $Y \cup Z \cup B^3 \times S^1$. A handle decomposition for $Y \cup Z \cup B^3 \times S^1$ consists of a 0-handle, two 1-handles, and three 2-handles. (The cobordism $Z$ is constructed by attaching algebraically cancelling 2 and 3-handles to $\hat{K} \times I$.) So the framed link picture for $\tilde{M}$ is obtained by adding a meridional circle labelled "0" to each circle representing a 2-handle.

Using these it is easy to slide 2-handles to obtain

![Diagram](image)
i.e. $\mathcal{M} \cong S^3 \times S^1 \sharp S^2 \times S^2 \# S^1 \times S^2$.

BIBLIOGRAPHY

[A] S. Akbulut, A fake 4-manifold, these proceedings.


DEPARTMENT OF MATHEMATICS
TULANE UNIVERSITY
NEW ORLEANS, LA 70118

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
SALT LAKE CITY, UT 84112