A \(\mu\)-INVARIANT ONE HOMOLOGY 3-SPHERE THAT BOUNDS AN ORIENTABLE RATIONAL BALL

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In this note we show that the Brieskorn homology sphere \(\Sigma(2,3,7)\) bounds an orientable rational ball \(Q\). It is known that the \(\mu\)-invariant of \(\Sigma(2,3,7)\) is one as it bounds the plumbed 4-manifold \(W^4\)

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2 \\
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\]

Note that \(W^4\) has an even intersection form with signature \(\sigma(W^4) = 8\) and rank 10. Thus \(M^4 = Q \cup \Sigma_2 W^4\) is a closed orientable 4-manifold with even intersection form of signature 8 and rank 10. (Note that \(M^4\) cannot be a spin 4-manifold.) As a corollary we have the following recent theorem of N. Habegger [1]:

**Corollary.** Every even unimodular symmetric bilinear form \(F\) with

\[
|\frac{\text{rank}(F)}{\sigma(F)}| > \frac{5}{4}
\]

can be realized as the intersection form of a closed orientable 4-manifold.

**Theorem.** \(\Sigma(2,3,7)\) bounds an orientable rational ball \(Q^4\).

**Proof.** First we attach a 1-handle and a 2-handle to \(\Sigma(2,3,7) \times I\) to obtain a rational homology cobordism \(W_1\) between \(\Sigma(2,3,7)\) and a 3-manifold \(K^3\) which has the integral homology of \(L(4,-1)\). Then we describe an integral homology cobordism \(W_2\) between \(K^3\) and \(L(4,-1)\). Since \(L(4,-1)\) bounds a rational ball \(W_3\), we let \(Q = W_1 \cup W_2 \cup W_3\). This is done as follows.

It is well known that \(\Sigma(2,3,7)\) is obtained by \(+1\) surgery on the figure eight knot. Attach a 1-handle to \(\Sigma(2,3,7) \times I\) to obtain a cobordism from \(\Sigma(2,3,7)\) to \(\Sigma(2,3,7) \# S^2 \times S^1\):

\(^1\)Supported in part by NSF grant MCS 7900244A01.

\(^2\)Supported in part by NSF grant MCS 8002843A01.
Now attach a 2-handle
This describes the cobordism $W_1$. To see that it is a rational homology cobordism note that the attached 2-handle kills 4 times the generator of $H_1$ which was introduced by the 1-handle.

Now the link

is ribbon concordant to the link

by means of the ribbon

Thus $K^3$ is integral homology cobordant to

i.e. to $L(4,-1)$. Hence we have $W_2$.

Finally $L(4,-1)$ bounds a rational ball $W_3$. To see this attach the following 2-handle to $L(4,-1)$ to obtain $S^2 \times S^1$:

blow down $-1$
QUESTION. Does there exist a closed orientable 4-manifold with definite even intersection pairing and signature 8?

BIBLIOGRAPHY


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