Surgery on nullhomologous tori and simply connected 4-manifolds with $b^+ = 1$

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Abstract

For $5 \leq k \leq 8$, we show that an infinite family of exotic smooth structures on $\mathbb{CP}^2 \# k \mathbb{CP}^2$ can be obtained by $1/n$-surgeries on a single embedded nullhomologous torus in a manifold $R_k$ which is homeomorphic to $\mathbb{CP}^2 \# k \mathbb{CP}^2$.

1. Introduction

In the past few years, there has been significant progress on the problem of finding exotic smooth structures on the manifolds $P_m = \mathbb{CP}^2 \# m \mathbb{CP}^2$. The initial step was taken by Park [16], who found the first exotic smooth structure on $P_7$, and whose ideas renewed the interest in this subject. Őzsváth and Szabó proved that Park’s manifold is minimal [15], and Stipsicz and Szabó used a technique similar to that of Park’s to construct an exotic structure on $P_6$ [19]. Shortly thereafter, the authors of this paper developed a technique for producing infinite families of smooth structures on $P_m$, $6 \leq m \leq 8$ (see [8]), and Park, Stipsicz, and Szabó showed that this can be applied to the case $m = 5$ (see [8, 18]).

It is the goal of this paper to better understand the underlying mechanism which produces infinitely many distinct smooth structures on $P_m$, $5 \leq m \leq 8$. As we explain below, all these constructions start with the elliptic surface $E(1) = P_6$: perform a knot surgery using a family of twist knots indexed by an integer $n$ [7], then blow the result up several times in order to find a suitable configuration of spheres that can be rationally blown down [5] to obtain a smooth structure on $P_m$ that is distinguished by the integer $n$. We shall explain how this can be accomplished by surgery on nullhomologous tori in a manifold $R_m$ homeomorphic to $P_m$, $5 \leq m \leq 8$. In other words, we shall find a nullhomologous torus $\Lambda_m$ in $R_m$ so that $1/n$-surgery on $\Lambda_m$ preserves the homeomorphism type of $R_m$, but changes the smooth structure of $R_m$ in a way that depends on $n$. Presumably, $R_m$ is diffeomorphic to $P_m$, but we have not yet been able to show this in general. Our hope is that by better understanding $\Lambda_m$ and its properties, one will be able to find similar nullhomologous tori in $P_m$, for $m < 5$.

2. A short history of simply connected 4-manifolds with $b^+ = 1$

It is a basic problem of 4-manifold topology to understand the smooth structures on the complex projective plane $\mathbb{CP}^2$. Thus, one is interested in knowing the smallest $m$ for which $P_m = \mathbb{CP}^2 \# m \mathbb{CP}^2$ admits an exotic smooth structure. The first such example was produced by Donaldson in the historic paper [2], where it was shown that the Dolgachev surface $E(1)_{2,3}$, the result of performing log transforms of orders 2 and 3 on the rational elliptic surface $E(1) = P_6$ [3], is homeomorphic but not diffeomorphic to $P_6$. This breakthrough example provided the first known instance of an exotic smooth structure on a simply connected 4-manifold.

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Later, work of Friedman and Morgan [9] showed that the integers \( p \) and \( q \), for \( p, q > 1 \), are smooth invariants. The fact that \( E(1)_{p,q} \) is not diffeomorphic to \( E(1)_{p',q'} \) for \( \{ p, q \} \neq \{ p', q' \} \), \( p, q, p', q' > 1 \) persists even after an arbitrary number of blowups [4]; however, no minimal exotic smooth structures are currently known for \( P_m \) where \( m \geq 10 \).

In the late 1980s, Kotschick [11] proved that the Barlow surface, known to be homeomorphic to \( P_8 \), is not diffeomorphic to it. However, in following years, the subject of simply connected smooth 4-manifolds with \( b^+ = 1 \) languished because of lack of suitable examples. As we mentioned above, largely due to the example of Park [16] of an exotic smooth structure on \( P_2 \), this topic has again become active.

Here is an outline of a version of Park’s example: consider \( E(1) \) with an elliptic fibration whose singular fibers are four nodal fibers and an \( I_8 \)-fiber. (An \( I_n \)-fiber is comprised of a circular plumbing of \( n \) 2-spheres of self-intersection \( -2 \); see [1].) This elliptic fibration has a section which is an exceptional curve \( E \) (of self-intersection \( -1 \)). Blow up \( E(1) \) four times, at the double points of the four nodal fibers. Then in \( E(1) \# 4\mathbb{CP}^2 \cong P_{13} \), we find a configuration of 2-spheres consisting of \( E \), four disjoint spheres of self-intersection \(-4 \), each intersecting \( E \) once, and the \( I_8 \)-fiber, which intersects \( E \) in exactly one 2-sphere, and the \( I_8 \)-fiber is disjoint from the 4-spheres of self-intersection \(-4 \).

The transverse intersections of \( E \) with the 4-spheres of self-intersection \(-4 \) can be smoothed to obtain a 2-sphere of self-intersection \(-9 \). Together with spheres from the \( I_8 \)-fiber, we obtain a linear configuration of 2-spheres:

\[
-9 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2
\]

A regular neighborhood of this configuration has as its boundary the lens space \( L(49, -6) \), and as we explain below, this lens space bounds a rational homology ball. This means that this configuration can be rationally blown down [5], reducing \( b^+ \) by 6. One obtains a manifold \( P \) with \( b^+ = 1 \) and \( b^- = 7 \). It is not difficult to show that \( P \) is simply connected, and so it is homeomorphic to \( P_7 \). It follows from Seiberg–Witten theory that \( P \) is not diffeomorphic to \( P_7 \). This will be explained below. While this is not precisely the description of the manifold that was given in [16] (and it is not even clear that \( P \) is diffeomorphic to the example of [16]), the construction given here is similar to that of Park.

Stipsicz and Szabó improved on Park’s example by finding a more complicated configuration in a larger blowup of \( E(1) \), yet one which could be rationally blown down to get an even smaller manifold, homeomorphic but not diffeomorphic to \( P_6 \).

For some time, even after these examples, it was suspected that \( P_m \), for \( m \leq 9 \), would support only finitely many distinct smooth structures. This was due to the fact that until [8] the only technique available for producing infinitely many distinct smooth structures on a given smooth 4-manifold \( X \) was to require that \( X \) contain a minimal genus torus with trivial normal bundle and representing a nontrivial homology class. It is known that \( P_m \), for \( m \leq 9 \), contain no such tori. Thus, it is the goal of this paper to better understand the techniques for producing infinitely many distinct smooth structures and to better understand the examples of [8].

3. Seiberg–Witten invariants, rational blowdowns, and knot surgery

3.1. Seiberg–Witten invariants

Let \( X \) be a simply connected oriented 4-manifold with \( b_2^+ = 1 \), a given orientation of \( H^2_+(X; \mathbb{R}) \), and a given metric \( g \). The Seiberg–Witten invariant depends on the metric \( g \) and a self-dual 2-form as follows. There is a unique \( g \)-self-dual harmonic 2-form \( \omega_g \in H^2_+(X; \mathbb{R}) \) with \( \omega_g^2 = 1 \) and corresponding to the positive orientation. (Often \( \omega_g \) is called a period point for the
metric $g$.) Fix a characteristic homology class $k \in H_2(X; \mathbb{Z})$. Given a pair $(A, \psi)$, where $A$ is a connection in the complex line bundle whose first Chern class is the Poincaré dual $\hat{k} = \frac{1}{2\pi}[F_A]$ of $k$ and $\psi$ is a section of the bundle $W^+$ of self-dual spinors for the associated $spin^c$ structure, the perturbed Seiberg–Witten equations are:

$$D_A \psi = 0$$
$$F_A^+ = q(\psi) + i\eta$$

where $F_A^+$ is the self-dual part of the curvature $F_A$, $D_A$ is the twisted Dirac operator, $\eta$ is a self-dual 2-form on $X$, and $q$ is a quadratic function. Write $SW_{X,g,\eta}(k)$ for the corresponding invariant. As the pair $(g, \eta)$ varies, $SW_{X,g,\eta}(k)$ can change only at those pairs $(g, \eta)$ for which there are solutions with $\psi = 0$. These solutions occur for pairs $(g, \eta)$ satisfying $(2\pi \hat{k} + \eta) \cdot \omega_g = 0$. This last equation defines a wall in $H^2(X; \mathbb{R})$.

The point $\omega_g$ determines a component of the double cone consisting of elements of $H^2(X; \mathbb{R})$ of positive square. We prefer to work with $H_2(X; \mathbb{R})$. The dual component is determined by the Poincaré dual $H$ of $\omega_g$. An element $H' \in H_2(X; \mathbb{R})$ of positive square lies in the same component as $H$ if $H' \cdot H > 0$. If $(2\pi \hat{k} + \eta) \cdot \omega_g \neq 0$ for a generic $\eta$, $SW_{X,g,\eta}(k)$ is well defined, and its value depends only on the sign of $(2\pi \hat{k} + \eta) \cdot \omega_g$. Write $SW_{X,H}^+(k)$ for $SW_{X,g,\eta}(k)$ if $(2\pi \hat{k} + \eta) \cdot \omega_g > 0$, and $SW_{X,H}^-(k)$ in the other case.

The invariant $SW_{X,H}(k)$ is defined by $SW_{X,H}(k) = SW_{X,H}^+(k)$ if $(2\pi \hat{k} + \eta) \cdot \omega_g > 0$, or dually, if $k \cdot H > 0$, and $SW_{X,H}(k) = SW_{X,H}^-(k)$ if $k \cdot H < 0$. The wall-crossing formula $[12, 13]$ states that if $H', H''$ are elements of positive square in $H_2(X; \mathbb{R})$ with $H' \cdot H > 0$ and $H'' \cdot H > 0$, then if $k \cdot H' < 0$ and $k \cdot H'' > 0$,

$$SW_{X,H'}(k) - SW_{X,H''}(k) = (-1)^{1 + \frac{d(k)}{2}}$$

where $d(k) = \frac{1}{4}(k^2 - (3 \text{ sign } + 2 \text{ e})(X))$ is the formal dimension of the Seiberg–Witten moduli spaces.

Furthermore, in case $b^- \leq 9$, the wall-crossing formula, together with the fact that $SW_{X,H}(k) = 0$ if $d(k) < 0$, implies that $SW_{X,H}(k) = SW_{X,H'}(k)$ for any $H'$ of positive square in $H_2(X; \mathbb{R})$ with $H' \cdot H > 0$. So in case $b^+ = 1$ and $b^- \leq 9$, there is a well-defined Seiberg–Witten invariant, $SW_X(k)$. If $SW_X(k) \neq 0$, $k$ is called a basic class of $X$.

It is convenient to view the Seiberg–Witten invariant as an element of the integral group ring $\mathbb{Z}H_2(X)$. For $k \in H_2(X)$, we let $t_k$ denote the corresponding element in $\mathbb{Z}H_2(X)$. Then, the Seiberg–Witten invariant of $X$ is

$$SW_{X,H} = \sum SW_{X,H}(k) \cdot t_k.$$  

An important property of the Seiberg–Witten invariant is that if $X$ admits a metric $g$ of positive scalar curvature, then for the Poincaré dual $H$ of $\omega_g$, we have $SW_{X,H} = 0$. In particular, for $m \leq 9$, $SW_{P_m} = 0$.

### 3.2. Rational blowdowns

Let $C_p$ be the smooth 4-manifold obtained by plumbing $(p-1)$ disk bundles over the 2-sphere according to the diagram:

$$
\begin{array}{cccc}
& -(p+2) & -2 & \\
\downarrow & u_0 & u_1 & \cdots & \cdots & \cdots & \cdots & \cdots & -2 & u_{p-2} \\
& -2 & 0 & \\
\end{array}
$$

Then, the classes of the 0-sections have self-intersections $u_0^2 = -(p+2)$ and $u_i^2 = -2$, $i = 1, \ldots, p-2$. The boundary of $C_p$ is the lens space $L(p^2, 1-p)$ which bounds a rational ball.
By ‘pseudosection’ we mean that the intersection number $SK_XK$ where $mT$ be a tubular neighborhood of $Bp$.

The two pieces are glued together in such a way that the homology class $\{ptT\}$ it intersects the fiber $XK$ where $XK$ is a simply connected $4$-manifold containing a homologically essential torus $T$ of self-intersection $0$, and let $K$ be a knot in $S^3$. Let $N(K)$ be a tubular neighborhood of $K$ in $S^3$, and let $T \times D^2$ be a tubular neighborhood of $T$ in $X$. Then the knot surgery manifold $X_K$ is defined by:

$$X_K = (X \setminus (T \times D^2)) \cup (S^1 \times (S^3 \setminus N(K))).$$

The two pieces are glued together in such a way that the homology class $\{pt \times \lambda\}$, where $\lambda$ is the class of a longitude of $K$. For example, if $X$ is a simply connected elliptic surface with a (spherical) section $S$ of self-intersection $n$ and one performs knot surgery on the fiber $T$ of this fibration, then the gluing condition implies that in $X_K$ there is a pseudosection $S_K$ of genus equal to the genus of the knot $K$ and with self-intersection $n$. By ‘pseudosection’ we mean that the intersection number $S_K \cdot F = 1$. However, $X_K$ need no longer be an elliptic surface. This surface $S_K$ is constructed by removing a disk from $S$ where it intersects the fiber $T$ and replacing this disk by a Seifert surface for the knot $K$.

One can also interpret $X_K$ as a fiber sum. Let $M_K$ denote the $3$-manifold obtained from $0$-framed surgery on $K$ in $S^3$. Then

$$X_K = X \#_{T=S^1 \times m} S^1 \times M_K$$

where $m$ is a meridian of $K$.

The gluing condition does not, in general, completely determine the diffeomorphism type of $X_K$; however, if we take $X_K$ to be any manifold constructed in this fashion and if, for example, $T$ has a cusp neighborhood, then the Seiberg–Witten invariant of $X_K$ is completely determined by the Seiberg–Witten invariant of $X$ and the symmetrized Alexander polynomial $\Delta^s$ of $K$.

**Knot Surgery Theorem** [7]. Let $X$ be a $4$-manifold which contains a homologically essential torus $T$ of self-intersection $0$ whose $H_1$ is generated by vanishing cycles, and let $K$ be a knot in $S^3$. The Seiberg–Witten invariant of the knot surgery manifold $X_K$ is given by

$$SW_{X_K} = SW_X \cdot \Delta(t^2)$$

where $t$ represents the homology class of the torus $T$. Furthermore, if $X$ and $X \setminus T$ are simply connected, then so is $X_K$.

4. **Double node neighborhoods and knot surgery**

A simply connected elliptic surface is fibered over $S^2$ with a smooth fiber torus and with singular fibers. The most generic type of singular fiber is a nodal fiber (an immersed $2$-sphere with one transverse positive double point). The monodromy of a nodal fiber is $D_a$, a Dehn twist.
around the ‘vanishing cycle’ \( a \in H_1(F; \mathbb{Z}) \), where \( F \) is a smooth fiber of the elliptic fibration. The vanishing cycle \( a \) is represented by a nonseparating loop on the smooth fiber and the nodal fiber is obtained by collapsing this vanishing cycle to a point to create a transverse self-intersection. The vanishing cycle bounds a ‘vanishing disk’, a disk of relative self-intersection \(-1\) with respect to the framing of its boundary given by pushing the loop off itself on the smooth fiber.

An \( I_2 \)-fiber consists of a pair of 2-spheres of self-intersection \(-2\), which are plumbed at two points. The monodromy of an \( I_2 \)-fiber is \( D^2 \), which is also the monodromy of a pair of nodal fibers with the same vanishing cycle. This means that an elliptic fibration, which contains an \( I_2 \)-fiber, can be perturbed to contain two nodal fibers with the same vanishing cycle.

A double node neighborhood \( D \) is a fibered neighborhood of an elliptic fibration, which contains exactly two nodal fibers with the same vanishing cycle. If \( F \) is a smooth fiber of \( D \), there is a vanishing class \( a \) that bounds vanishing disks in the two different nodal fibers, and these give rise to a sphere \( V \) of self-intersection \(-1\) in \( D \).

In [8] we showed how performing knot surgery in a double node neighborhood \( D \) in \( E(1) \) can give rise to an immersed pseudosection of self-intersection \(-1\) in \( E(1)_K \). Let us review a version of this construction. Consider the knot \( K \) of Figure 1. Of course, this is just the unknot, and we see a Seifert surface \( \Sigma \) of genus one. Let \( \Gamma \) be the loop which runs through both half-twists in the clasp. Then \( \Gamma \) satisfies the two key conditions of [8]:

(i) \( \Gamma \) bounds a disk in \( S^3 \) which intersects \( K \) at exactly two points.
(ii) The linking number in \( S^3 \) of \( \Gamma \) with its pushoff on \( \Sigma \) is \(+1\).

It follows from these properties that \( \Gamma \) bounds a punctured torus in \( S^3 \setminus K \).

It is known that \( E(1) \) admits an elliptic fibration with two nodal fibers: an \( I_2 \)-fiber and an \( I_8 \)-fiber [17]. As above, this fibration can be perturbed so that the \( I_2 \)-fiber gives us a double node neighborhood \( D \) with vanishing cycle \( a \). Consider the result of knot surgery in \( D \) using the knot \( K \) and the fiber \( F \) of \( E(1) \). In the knot surgery construction, one is free to make any choice of gluing as long as a longitude of \( K \) is sent to the boundary circle of a normal disk to \( F \). We choose the gluing so that the class of a meridian \( m \) of \( K \) is sent to the class of \( a \times \{ pt \} \) in \( H_1(\partial(D \setminus N(F)); \mathbb{Z}) = H_1(F \times \partial D^2; \mathbb{Z}) \). Note that the result of knot surgery

\[
E(1)_K = E(1) \setminus N(F) \cup S^1 \times (S^3 \setminus N(K)) = E(1) \setminus N(F) \cup T^2 \times D^2
\]

because \( K \) is the unknot. Since any diffeomorphism of \( \partial(E(1) \setminus N(F)) \) extends over all of \( E(1) \setminus N(F) \), we see that \( E(1)_K \) is diffeomorphic to \( E(1) \).

There is a genus one pseudosection \( S_K \) in \( E(1)_K \) which is formed using the genus one Seifert surface \( \Sigma \). The self-intersection of \( S_K \) is \(-1\). The loop \( \Gamma \) sits on \( S_K \), and by condition (i) it bounds a twice-punctured disk \( \Delta \) in \( \{ pt \} \times \partial(S^3 \setminus N(K)) \) where \( \partial \Delta = \Gamma \cup m_1 \cup m_2 \) where the \( m_i \) are the meridians of \( K \). The meridians \( m_i \) bound disjoint vanishing disks \( \Delta_i \) in \( D \setminus N(F) \).
since they are identified with disjoint loops each of which represents the class of $a \times \{pt\}$ in $H_1(\partial(D \setminus N(F)); \mathbb{Z})$. (Our use of the terminology ‘vanishing disk’ is not entirely standard. Sometimes these disks are referred to as ‘Lefschetz thimbles’.) Hence in $D_K$, the result of knot surgery on $D$, the loop $\Gamma \subset S_K$ bounds a disk $U = \Delta \cup \Delta_1 \cup \Delta_2$. By construction, the relative self-intersection of $U$ relative to the framing given by the pushoff of $\Gamma$ in $S_K$ is $+1 - 1 - 1 = -1$. (This uses condition (ii).) Furthermore, $U \cap S_K = \Gamma$. This means that the relative normal bundle of $U$ has Euler number $-1$; hence, it has a section which intersects $U$ in a single point.

Since $\Gamma$ is nonseparating in $S_K$, surgery on it kills $\pi_1(S_K)$. Ambient surgery may be performed in $D_K$ by removing an annular neighborhood of $\Gamma$ and replacing it with a pair of disks $U', U''$ as obtained above. These disks intersect in a single point, and this is precisely the complex-algebraic model of a nodal intersection. This means that we can represent the homology class of the pseudosection $[S_K]$ in $H_2(E(1)_K; \mathbb{Z})$ by an immersed sphere $S'$ with one positive double point.

With these as preliminaries, our goal for the remainder of this paper is to construct for every $5 \leq m \leq 8$, a manifold $R_m$ that is homeomorphic to $P_m$, has vanishing Seiberg–Witten invariants, and contains an embedded nullhomologous torus $\Lambda_m$ such that $1/n$-surgery on $\Lambda_m$ (with respect to the nullhomologous framing) yields a smooth structure on $P_m$ distinguished by the integer $n$. We conjecture that $R_m$ is diffeomorphic to $P_m$, but we are unable to show this at this time. We start with the $m = 8$ case in the next section.

5. Infinite families homeomorphic to $\mathbb{CP}^2 \# 8 \overline{\mathbb{CP}^2}$

Our goal is to construct a manifold $R_8$, homeomorphic to $P_8$, with trivial Seiberg–Witten invariants and containing an embedded nullhomologous torus $T$ such that $1/n$-surgery on $T$ yields a smooth structure on $P_8$ which is distinguished by the integer $n$.

The construction of the previous section shows that when $K$ is the unknot, then in $E(1) \cong E(1)_K$ one has the configuration consisting of the immersed 2-sphere $S'$ with a pair of disjoint nodal fibers, each intersecting $S'$ once transversely. Also, $S'$ intersects the $I_8$-fiber transversely at one point. This is illustrated in Figure 2.

At this stage there are three possibilities:

(1) Blow up the double point of $S'$. Then, in $P_{10}$ we obtain a configuration consisting of the total transform $S''$ of $S'$, which is a sphere of self-intersection $-5$, and the sphere of self-intersection $-2$ at which $S'$ intersects $I_8$ (see Figure 3). This is the configuration $C_3$ which can be rationally blown down to obtain a manifold $R_8$ with $b^+ = 1$ and $b^- = 8$. It is easy to see that $R_8$ is simply connected; so $R_8$ is homeomorphic to $P_8$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Figure 2.}
\end{figure}
(2) Blow up at the double point of $S'$ as well as at the double point of one of the nodal fibers. Then, in $P_{11}$ we get a configuration of 2-spheres consisting of $S''$, a transverse sphere $F'$ of self-intersection $-4$, and three spheres from the $I_8$-fiber (see Figure 4). Smoothing the intersection of $S''$ and $F'$ gives a sphere of self-intersection $-7$ and we obtain the configuration $C_5$ in $P_{11}$. Rationally blowing down $C_5$ gives a manifold $R_7$ homeomorphic to $P_7$.

(3) Blow up at the double point of $S'$ as well as at the double points of both nodal fibers. Then, in $P_{12}$ we get a configuration of 2-spheres consisting of $S''$, two disjoint transverse spheres $F'$, $F''$ of self-intersection $-4$, and five spheres from the $I_8$-fiber (see Figure 5). Smoothing the intersections of $S''$, $F'$ and $F''$ gives a sphere of self-intersection $-9$ and we obtain the configuration $C_7$ in $P_{12}$. Rationally blowing down $C_7$ gives a manifold $R_6$ homeomorphic to $P_6$.

We shall work with the the first case, and then indicate what needs to be done to take care of the other cases. In Case 1, we obtain a manifold $R = R_8$ which is homeomorphic to $P_8$. We conjecture that $R$ is actually diffeomorphic to $P_8$, but for now it will suffice to see that it shares with $P_8$ the property that its Seiberg–Witten invariant vanishes.

**Proposition 5.1.** \( \mathcal{SW}_R = 0 \).

**Proof.** We will show that $R$ contains an embedded torus of self-intersection $+1$. Then the adjunction inequality will imply that $\mathcal{SW}_R = 0$. The upshot of using the unknot as $K$ is that the loop $v$ of Figure 6 bounds a disk of relative framing 0 with respect to the shaded genus 1 Seifert surface.
It follows that a local picture of a neighborhood of the pseudosection $S_K$ is given in Figure 7. The component labelled ‘0’ is $v$, and the component on top labelled ‘$-1$’ is $\Gamma$ of Figure 1 arising from the double node construction.
Next, blow up to see $S'' = S - 2E$ (where $E$ is the exceptional class) in Figure 8(a). In Figure 8(b), we cancel the 1-handle and the 2-handle labelled $'−1'$. The new 2-handle in this figure, labelled $'−2'$, comes from the $I_8$-fiber as in Figure 3.

Rationally blow down the configuration $\{-5, -2\}$ to obtain Figure 9(a). The new loop labelled $'F'$ in Figure 9(b) is the intersection of a fiber of $E(1) = E(1)_K$ with the original section $S$ and, therefore, with $S''$. This fiber can be taken to be far away from the region where the double node construction was performed.

Cancel the 1-handle and the 2-handle labelled $'0'$ to obtain Figure 10(a), and then slide the handle labelled $'+2'$ over the $'−1'$ to obtain Figure 10(b).
Note that $F$ bounds a punctured torus of square 0 outside of the neighborhood in Figure 10. Together with the shaded annulus in the figure and the +1-disk that the other boundary component of the annulus bounds, we get a torus of self-intersection +1 in $\mathbb{R}^3$. As we have pointed out above, this implies that $SW_R = 0$.

Back in $E(1) = E(1)_K = E(1)\# F = S^1 \times S^1 \times M_K$ there is a nullhomologous torus $\Lambda = S^1 \times \lambda$ where $\lambda$ is the loop shown in Figure 11, which is a Kirby calculus depiction of $M_K = S^1 \times S^2$, since $K$ is the unknot. Since $\Lambda$ as well as the 3-manifold that it bounds ($S^1 \times$ punctured torus) are disjoint from the regions where our constructions were made, $\Lambda$ descends to a nullhomologous torus (which we still call $\Lambda$) in $\mathbb{R}^3$. Let $Q$ be the result of 0-surgery on $\Lambda \subset \mathbb{R}^3$, where the '0-framing' is taken from the 0-framing on $\lambda$ in Figure 11. After this surgery, the loop $\mu$ which bounds a normal disk to $\Lambda$, does not bound in $Q$. In fact, $H_2(Q)$ is the direct sum of $H_2(R)$ with a hyperbolic pair generated by $\Lambda_0$, the torus in $Q$ corresponding to $\Lambda$, and a dual class represented by a torus built from the punctured torus that the longitude to $\lambda$ (the surgery curve) bounds and the disk that the surgery curve bounds in $Q$. Thus $b^+(Q) = 2$.

**Theorem 5.2.** The Seiberg–Witten invariant of $Q$ is $SW_Q = t^{-1} - t$, where $t = t_{S^1 \times S^1} \in \mathbb{Z}H_2(Q)$.

**Proof.** The manifold $Q$ is obtained by:

1. double node surgery with $K =$ the unknot, blowing up, and then rationally blowing down,
2. 0-surgery on $\Lambda$. 

![Figure 10](image10.png)

![Figure 11](image11.png)
Since $\Lambda$ is disjoint from all the constructions in (1), the order in which (1) and (2) are performed is irrelevant. (Note that if we could exactly ‘see’ $\Lambda$ embedded in $P_8$, step (1) would be unnecessary, and we could then use $P_8$ rather than $R_8$.)

In $E(1)_K \cong E(1)$, surgery is firstly done on $\Lambda$. Recall that $E(1)_K$ is a fiber sum $E(1) = E(1)\#_F S^1 \times m S^1 \times MK$ and $MK$ is the manifold given in Figure 11. The result of 0-surgery on $\Lambda$ in $E(1)_K$ is the fiber sum $E(1)_K,0 = E(1)\#_F S^1 \times m S^1 \times Y$ where $Y$ is the 3-manifold obtained from 0-surgery on $\lambda$ in Figure 11.

We shall now need the sewn-up link exterior construction of Brakes and Hoste. We recall what this means. Let $L$ be a link in $S^3$ with two oriented components $L_1$ and $L_2$. Fix tubular neighborhoods $N_1 \cong S^1 \times D^2$ of $L_1$ with $S^1 \times (\text{pt on } \partial D^2)$ a longitude of $L_1$, that is, nullhomologous in $S^3 \setminus L_1$. For any $A \in GL(2; \mathbb{Z})$ with $\det A = -1$, we get a 3-manifold $s(L; A) = (S^3 \setminus \text{int}(N_1 \cup N_2))/A$ called a sewn-up link exterior by identifying $\partial N_1$ with $\partial N_2$ via a diffeomorphism inducing $A$ in homology. For $n \in \mathbb{Z}$, let $A_n = (\begin{smallmatrix} -1 & 0 \\ n & 1 \end{smallmatrix})$. A simple calculation shows that $H_1(s(L; A_n); \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_{n-2\ell}$ where $\ell$ is the linking number in $S^3$ of the two components $L_1, L_2$, of $L$. The second summand is generated by the meridian to either component.

We now refer to the proof of the Knot Surgery Theorem given in [7]. A key step in the proof (see [7, Figure 6]), which uses the work of Hoste [10], shows that $E(1)_{K,0}$ is diffeomorphic to $E(1)_L = E(1)\#_F S^1 \times m S^1 \times s(L, A_{-2})$ where $L = L_1 \cup L_2$ is the link of Figure 12, that is, $L$ is the Hopf link. The orientations on $L_1$ and $L_2$ are inherited from fixing an orientation on the knot $K$, for example, in Figure 11. Note that since the linking number of $L_1$ and $L_2$ is $-1$, we have $H_1(s(L; A_{-2}); \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.

In $s(L, A_{-2})$ we see an embedded torus transverse to the meridian $m$ obtained by sewing up a Seifert surface for $L$, and we also see a loop $\Gamma$ which satisfies conditions (i) and (ii) for double node surgery.

The proof of the Knot Surgery Theorem shows that the Seiberg–Witten invariant of $E(1)_L$ can be calculated via skein moves (macarena). Note that we are now calculating the Seiberg–Witten invariant of a manifold with $b^+ = 2$. This calculation is shown in Figure 13. This figure depicts the fact that $SW_{E(1)_L} = SW_{E(1)_{K,0}} - (t - t^{-1})^2 SW_{E(1)_U}$ where $U$ is the unlink and $K_0$ is the unknot (see [7, Section 3, equation (3)]).
A 2-sphere that separates the two components of the unlink gives rise to an essential 2-sphere in $E^{(1)}_{U}$ of self-intersection 0. Thus $SW_{E^{(1)}_{U}} = 0$. Since $E^{(1)}_{K_0} = E^{(1)}$, we have $SW_{E^{(1)}_{K_0}} = (t - t^{-1})^{-1}$. Thus, $SW_{E^{(1)}_{L}} = t^{-1} - t$. This completes our discussion of step (2).

Next, we carry out the constructions of step (1). In $E^{(1)}_{L}$ there is a genus one pseudosection to which we can apply the double node construction. This pseudosection is the connected sum of a section in $E^{(1)}$ with the torus in $\{\text{point}\} \times s(L, A_{-2})$ obtained by sewing up the shaded region in Figure 14. The necessary loop $\Gamma$ is shown in Figure 12.

The result of the double node construction is an immersed genus 0 pseudosection with one positive double point. Blow up at this double point to get an embedded 2-sphere $C$ of self-intersection $-5$ in $E^{(1)}_{L} \# \overline{\text{CP}}^2$. The blowup formula [6] implies that $SW_{E^{(1)}_{L} \# \overline{\text{CP}}^2} = (t^{-1} - t)(e + e^{-1})$ where $e$ is the class in the group ring corresponding to the new exceptional curve. Hence, the basic classes of $E^{(1)}_{L} \# \overline{\text{CP}}^2$ are $\pm F \pm E$. Now $\pm(F + E) \cdot C = \pm 3$ whereas $\pm(F - E) \cdot C = \mp 1$. It follows from the Rational Blowdown Theorem that only the basic classes $\pm(F + E)$ descend to the rational blowdown $Q$. Thus, $Q$ has two basic classes whose Seiberg–Witten invariants are those of $\pm(F + E)$ in $E^{(1)}_{L} \# \overline{\text{CP}}^2$, namely, $\mp 1$.

**Theorem 5.3.** There are infinitely many pairwise nondiffeomorphic 4-manifolds homeomorphic to $P_8 = \text{CP}^2 \# 8\overline{\text{CP}}^2$ obtained from $1/n$-surgery on the nullhomologous torus $\Lambda$ in $R$. 

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**Figure 13.**

**Figure 14.**
Proof. For \( n \geq 2 \), let \( X_n \) be the 4-manifold obtained from \( 1/n \)-surgery on \( \Lambda \) in \( R \). By this we mean \( S^1 \) times \( 1/n \)-surgery on \( \lambda \) in Figure 11. It is easy to see that \( X_n \) is simply connected and that there is an isomorphism \( \varphi : H_2(X_n; \mathbb{Z}) \to H_2(R; \mathbb{Z}) \), which is realized outside of a neighborhood of the surgery by the identity map. Thus, \( X_n \) is homeomorphic to \( P_8 \).

Morgan, Mrowka, and Szabó have calculated the effect of such a surgery on Seiberg–Witten invariants [14]. Given a class \( k \in H_2(X_n) \),

\[
SW_{X_n}(k) = SW_{R}(\varphi(k')) + n \sum_i SW_Q(k'' + i[\Lambda_0]).
\]

(Recall that the torus \( \Lambda \) is nullhomologous in \( R \) and the corresponding torus \( \Lambda_n \), the core of the surgery, is nullhomologous in \( X_n \).) Further, \( k'' \in H_2(Q) \) is any class that agrees with the restriction of \( k \) in \( H_2(R \setminus \Lambda \times D^2, \partial) \) in the following diagram.

\[
\begin{array}{ccc}
H_2(Q) & \longrightarrow & H_2(Q, \Lambda_0 \times D^2) \\
\downarrow \cong & & \downarrow \cong \\
H_2(R \setminus \Lambda \times D^2, \partial) & \xrightarrow{\cong} & H_2(X_n, \Lambda_n \times D^2)
\end{array}
\]

The Seiberg–Witten invariants of the two \( b^+ = 1 \) manifolds \( X_n \) and \( R \) are calculated in corresponding chambers.

Given \( k \in H_2(X_n) \) and \( H \) an element of positive self-intersection in \( H_2(X_n) \), the small perturbation chamber, that is, the sign \( \pm \) such that \( SW_{X_n}(k) = SW^H_{X_n}(k) \) is determined homologically. This means that the small perturbation chambers for \( k \) in \( X_n \) and for \( \varphi(k) \) in \( R \) correspond under \( \varphi \). According to the previous theorem, there are only two classes: \( \pm T \), \( T = [S^1 \times m] \) in \( Q \) with nontrivial Seiberg–Witten invariants, and \( SW_Q(\pm T) = \mp 1 \).

Thus, we have

\[
SW_{X_n} = SW_{R} + SW_{Q} = 0 + n(t^{-1} - t).
\]

This shows that the manifolds \( X_n \) are pairwise nondiffeomorphic. That they are all minimal follows from the blowup formula. \( \square \)

6. Infinite families with \( b^- = 5, 6, 7 \)

For \( b^- = 6, 7 \), the constructions and calculations to show that there is a nullhomologous torus \( \Lambda \) in \( R = R_6 \) or \( R_7 \) such that \( 1/n \)-surgery on \( \Lambda \) in \( R \) produces infinitely many distinct smooth structures on \( P_6 \) or \( P_7 \) are completely analogous to those in the last section. What differs is the proof that the Seiberg–Witten invariants of \( R = R_6 \) or \( R_7 \) vanish. We shall accomplish this by using an argument adapted from [15]. The important point in the argument is that we are starting our construction with \( E(1)_K = E(1) \); so all the exceptional curves are represented by spheres, etc. First consider the \( b^- = 7 \) case. The classes

\[
\begin{align*}
V_1 &= H - E_1 - E_2 - E_3, & V_2 &= H - E_2 - E_3 - E_4 \\
V_3 &= H - E_3 - E_4 - E_5, & V_4 &= H - E_6 - E_7 - E_8 - E_9 \\
V_5 &= F - E_5, & V_6 &= E_{11} - E_1 - E_2 \\
V_7 &= E_{10} - E_1 - E_2, & V_8 &= 2H - 3E_{11}
\end{align*}
\]

are all orthogonal to the configuration \( C_5 \) and generate \( H_2(P_{11} \setminus C_5; \mathbb{Z}) = H_2(R_7 \setminus B_5; \mathbb{Z}) \). In \( P_{11} \), the classes \( V_1, V_2, V_3 \) are represented by embedded spheres of self-intersection \(-2\), \( V_4, V_6, V_7 \) are represented by embedded spheres of self-intersection \(-3\), \( V_5 \) is represented by an embedded torus of self-intersection \(-3\), and \( V_8 \) is represented by an embedded torus with square \(-5\). According to the argument of [15], any basic class \( k \) of \( R_7 \) must satisfy the adjunction
inequality:
\[ V_i^2 + |k \cdot V_i| \leq 0, \quad i = 1, \ldots, 8. \]
Furthermore, \( k \) must satisfy
\[ k^2 \geq 2, \quad k^2 \equiv 2 \pmod{8}. \]
These follow from the fact that for any basic class, its corresponding moduli space must have nonnegative even dimension.

Since \( H^2(R_7; \mathbb{Z}) \) injects into \( H^2(R_7 \setminus B_5; \mathbb{Z}) \), any Seiberg–Witten basic class of \( R_7 \) is uniquely determined by its intersection numbers with \( V_1, \ldots, V_8 \). There is now a finite check for possible basic classes \( k \) of \( R_7 \) which must satisfy these three previous conditions. This check turns up 40 classes in \( H^2(R_7 \setminus B_5; \mathbb{Z}) \). Another class in \( H_2(P_{11}; \mathbb{Z}) \) which is orthogonal to \( C_5 \) is \( V_9 = 2H - 3E_{10} \). It is represented by an embedded torus of self-intersection \(-5\). Any basic class of \( R_7 \) must also satisfy the adjunction inequality with respect to \( V_9 \). This condition reduces the number of possible classes to 14. According to the Rational Blowdown Theorem, the Seiberg–Witten invariant of any such class is determined by the Seiberg–Witten invariant of an appropriate lift to \( P_{11} \). Such a lift determines a Seiberg–Witten moduli space for \( P_{11} \) whose formal dimension is the same as that of the moduli space for \( R_7 \) corresponding to the class being lifted. This is accomplished via an extension across \( C_5 \) for each of the 14 possibilities.

The class \( \hat{H} = V_1 + V_2 + V_3 + V_8 \) is orthogonal to \( C_5 \) and \( H^2 = 4 > 0 \), \( \hat{H} \cdot H = 6 > 0 \). Hence, \( \hat{H} \) serves as a period point for \( R_{11} \). Since \( SW_{P_{11}, \hat{H}} = 0 \), for any characteristic cohomology class \( k \) of \( P_{11} \):
\[
SW_{P_{11}, \hat{H}}(k) = \begin{cases} 
0, & \text{if the signs of } k \cdot \hat{H} \text{ and } k \cdot H \text{ agree} \\
\pm 1, & \text{if the signs of } k \cdot \hat{H} \text{ and } k \cdot H \text{ do not agree}.
\end{cases}
\]
Using this criterion on each of the 14 possibilities mentioned above shows that each has Seiberg–Witten invariant equal to 0. Thus, we have \( SW_{R_{11}} = 0 \).

The \( b^- = 6 \) case follows similarly, but the calculation turns out to be easier. The classes
\[
V_1 = E_{12} - E_1, \quad V_2 = E_{11} - E_1, \quad V_3 = E_{10} - E_1, \quad V_4 = E_3 - E_1, \\
V_5 = E_2 - E_1, \quad V_6 = H - 3E_1, \quad V_7 = 2F + H - E_3 - E_{10} - E_{11} - E_{12}
\]
generate \( H_2(P_{12} \setminus C_7; \mathbb{Z}) = H_2(R_6 \setminus B_7; \mathbb{Z}) \). In \( P_{12} \), the classes \( V_1, \ldots, V_5 \) are all represented by embedded spheres of self-intersection \(-2\), \( V_6 \) is represented by an embedded torus of self-intersection \(-8\), and \( V_7 \) is represented by an embedded surface of genus 4 with square 5. If \( k \) is a basic class of \( X \), then it must satisfy the adjunction inequality with respect to the classes \( V_1, \ldots, V_7 \); that is
\[
V_i^2 + |k \cdot V_i| \leq 0, \quad i = 1, \ldots, 6; \quad V_7^2 + |k \cdot V_7| \leq 2g - 2 = 6
\]
and as above, \( k \) must also satisfy
\[ k^2 \geq 3, \quad k^2 \equiv 3 \pmod{8}. \]
This time a check turns up no classes in \( H^2(R_6 \setminus B_7; \mathbb{Z}) \) which satisfy these conditions. Hence, \( SW_{R_6} = 0 \).

To obtain families of manifolds homeomorphic to \( P_3 \), we start with an elliptic fibration on \( E(1) \) with two nodal fibers, two \( I_2 \)-fibers and an \( I_6 \)-fiber (gain, see [17]). This time we have two double node neighborhoods. One goes through the same construction in each of these to obtain an immersed pseudosection with two double points. Blowing up each, we obtain a sphere of square \(-9\) in \( E(1) \# 2\mathbb{CP}^1 \), and adding on five of the spheres of the \( I_6 \)-singularity, gives a copy of the configuration \( C_7 \), which can be rationally blown down to obtain a manifold \( R_5 \) homeomorphic to \( P_3 \). This process can be carried out so the the \( +1 \)-torus of the proof of Proposition 5.1 descends to \( R_5 \); so we see that \( SW_{R_5} = 0 \). Furthermore, we get
the nullhomologous tori \( \Lambda_1, \Lambda_2 \), as before; one in each neighborhood. Performing 0-surgery on each gives a manifold \( Q \) with \( b^+ = 3 \) and \( SW = (t_1^{-1} - t_1)(t_2^{-1} - t_2) \). Perform +1-surgery on \( \Lambda_2 \) and \( 1/n \)-surgery on \( \Lambda_1 \) to obtain \( Y_n \) which is homeomorphic to \( P_3 \), but which has \( SW_{Y_n} = n(t_1^{-1} - t_1)(t_2^{-1} - t_2) \).

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