RESOLUTIONS OF HOMOLOGY MANIFOLDS:
A CLASSIFICATION THEOREM

ALLAN L. EDMONDS† AND RONALD J. STERN†

1. Introduction

Let \( BPL \) and \( BH \) denote the classifying spaces for stable PL block bundles \([9]\) and stable homology cobordism bundles \(([4], [5])\). There is a natural map \( j : BPL \to BH \) with homotopy fibre denoted by \( H/PL \). If \( M \) is a closed (integral) homology manifold, a resolution of \( M \) is a pair \((P, f)\), where \( P \) is a piecewise linear (PL) manifold and \( f: P \to M \) is a surjective PL map which is acyclic, i.e. \( \tilde{f}_*(f^{-1}(x)) = 0 \) for all \( x \in M \). Let \( \tau : M \to BH \) classify the homology tangent bundle of \( M \). We prove the following theorems.

**Existence Theorem.** There is a resolution of \( M \) if and only if \( \tau \) lifts through \( j \) to \( BPL \).

**Classification Theorem.** The set of concordance classes (naturally defined) of resolutions of \( M \) is in one-to-one correspondence with the set of vertical homotopy classes of lifts of \( \tau \) through \( j \) to \( BPL \).

The Existence and Classification Theorems are deduced using the

**Product Structure Theorem.** \( M \times [-1, 1]^k \) is resolvable if and only if \( M \) is resolvable.

Let \( \theta_3^H \) denote the abelian group obtained from the set of oriented 3-dimensional PL homology spheres using the operation of connected sum, modulo those which bound acyclic PL 4-manifolds. Then according to N. Martin \([3]\) (also see \([7]\)),

\[
\pi_i(H/PL) = \begin{cases} 
\theta_3^H & \text{if } i = 3 \\
0 & \text{otherwise.}
\end{cases}
\]

Thus, using standard obstruction theory, the Existence Theorem can be restated in the form of the usual resolution theorem due to Sullivan \([12]\) and Cohen \([2]\) (also see \([3]\) and \([10]\)): \( M \) is resolvable if and only if a class in \( H^4(M; \theta_3^H) \) is zero. Similarly, if \( M \) is resolvable, then there is a one-to-one correspondence between the set of concordance classes of resolutions of \( M \) and \( H^3(M; \theta_3^H) \). This version of the Classification Theorem has also been obtained by N. Martin.

In §2 we summarize some basic facts about homology manifolds and their resolutions; in §3 we prove the Product Structure Theorem; in §4 we investigate the tangential properties of resolutions; and, finally, in §5 we complete the proofs of the main theorems.

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2. Background

A compact polyhedron $M$ is called a homology $n$-manifold if there is a triangulation $K$ of $M$ such that for any $x \in M$ and subdivision $K_1$ of $K$ such that $x$ is a vertex, $H_\ast(\text{Link}(x, K_1))$ is isomorphic either to $H_\ast(S^{n-1})$ or to $H_\ast(\text{point})$. The boundary of $M$, $\partial M$, is the set of points $x$ such that $H_\ast(\text{Link}(x)) = H_\ast(\text{point})$ and is a closed (compact without boundary) homology $(n-1)$-manifold. We refer the reader to [6] for the basic properties of homology manifolds.

Two closed homology $n$-manifolds $M$ and $N$ are said to be $H$-cobordant if there is a homology $(n+1)$-manifold $W$ such that $\partial W$ is the disjoint union of $M$ and $N$ and $H_\ast(W, M) = H_\ast(W, N) = 0$. If $M$ and $N$ are homology $n$-manifolds with boundary, they are said to be $H$-cobordant if there is a homology $(n+1)$-manifold $W$ such that $\partial W = M \cup W_0 \cup N$ where $W_0$ is an $H$-cobordism from $\partial M$ to $\partial N$ and $H_\ast(W, M) = H_\ast(W, N) = 0$.

Let $M$ be a homology manifold and $N$ be a codimension zero PL submanifold of $\partial M$. A resolution of $M$ rel $N$ is a pair $(P, f)$, where $P$ is a PL manifold and $f: (P, \partial P) \to (M, \partial M)$ is a surjective PL map of pairs such that (i) $f^{-1}(\partial M) = \partial P$, (ii) $f|f^{-1}(N)$ is a PL homeomorphism, and (iii) $f$ is acyclic, i.e. $H_\ast(f^{-1}(x)) = 0$ for all $x \in M$. If $(P, f)$ is a resolution of $M$ and $X$ is a subcomplex of $M$, then $f_\ast f^{-1}(X): f^{-1}(X) \to X$ is an acyclic map and so, by the Vietoris–Begle Mapping Theorem [11; p. 344], induces an isomorphism $H_\ast(f^{-1}(X)) \to H_\ast(X)$.

Two such resolutions $(P_i, f_i), i = 0, 1$, of $M$ rel $N$ are concordant rel $N$ if there is a resolution $(Q, F)$ of $M \times I$ rel $N \times I$ such that $(F^{-1}(M \times i), F|F^{-1}(M \times i))$ is $(P_i, f_i)$ for $i = 0, 1$. If $\partial M = N = \emptyset$, we denote the set of concordance classes of resolutions of $M$ by $\text{Res}(M)$.

Sullivan [12] and Cohen [2] (see also Martin [3] and Sato [10]) have constructed an elegant obstruction theory for resolving a homology manifold. The theory shows that if $N$ is a codimension zero PL submanifold of the boundary of a homology manifold $M$, then there is an element $\sigma_M$ in $H^4(M, N; \mathbb{Z})$ as defined in §1) such that $\sigma_M = 0$ if and only if $M$ can be resolved rel $N$. See especially Martin [3] for this relative formulation.

We shall also use the elementary fact that the mapping cylinder of a surjective PL acyclic map between two homology manifolds is an $H$-cobordism and in particular a homology manifold.

For the basic properties of homology cobordism (disk) bundles over homology manifolds, we refer the reader to [4]. Recall from [4] that if $\xi$ is a homology cobordism bundle over a homology manifold $M$, then the total space $E(\xi)$ is also a homology manifold. Also if $\xi$ and $\zeta$ are equivalent homology cobordism bundles then $E(\xi)$ and $E(\zeta)$ are $H$-cobordant as homology manifolds.

Martin and Maunder in [4] and [5] have shown that there is a space $BH$ which classifies stable equivalence classes of homology cobordism bundles. There is a natural map $j: BPL \to BH$, where $BPL$ denotes the classifying space for stable PL block bundles [9]. We make $j$ into a fibration and call its fibre $H/PL$. It is not hard to see that two homology cobordism bundles $\xi$ and $\zeta$ are stably equivalent if and only if $\xi \times I^n$ is isomorphic to $\zeta \times I^n$, for some $m$ and $n$, in the sense of [4].

If $M$ is a homology manifold we denote its homology tangent bundle by $T(M)$. If $M$ is a PL manifold, we denote its PL tangent block bundle by $T(M)$. In either case, the tangent bundle is given by a regular neighbourhood of the diagonal in $M \times M$. 

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If $\xi$ is a homology cobordism bundle (resp. PL block bundle) over a complex $X$, we will also often denote by $\xi : X \to BH$ (resp., $\xi : X \to BPL$) the stable classifying map of the bundle.

Finally we recall the construction of the pullback of a homology cobordism bundle [5; p. 112]. Let $f : M \to N$ be a simplicial map of homology manifolds and let $\xi$ be a homology cobordism bundle over $N$. Let $C_f$ denote the simplicial mapping cylinder of $f$. Then, as in [4; 3.5], $\xi$ can be extended to a homology cobordism bundle $\xi$ over all of $C_f$ and $\xi$ is unique up to isomorphism. Then we define $f^*\xi$ to be $\xi | M$.

3. The Product Structure Theorem.

Let $M$ be a homology manifold such that $\partial M$ is a PL manifold. Our goal in this section is to relate the resolutions of $M \times B^k (B^k = [-1, 1]^k)$ with those of $M$.

**Theorem 3.1** (Product Structure Theorem). Let $(Q, g)$ be a resolution of $M \times B^k$ rel $\partial M \times B^k$. Then there is a resolution $(P, f)$ of $M$, where $P$ is a PL submanifold of $Q$ with trivial normal block bundle, and a commutative diagram

\[
\begin{array}{ccc}
Q & \longrightarrow & M \times B^k \\
\uparrow g & & \uparrow h \\
P & \longrightarrow & M
\end{array}
\]

where $h$ is a proper PL embedding isotopic to the standard embedding $M \times 0$.

**Proof.** A finite induction shows that it suffices to find a resolution $(P, f)$ of $M \times B^{k-1}$ rel $\partial M \times B^{k-1}$, where $P$ is a properly embedded PL submanifold of $Q$ with trivial normal bundle, and a commutative diagram

\[
\begin{array}{ccc}
Q & \longrightarrow & M \times B^k \\
\uparrow g & & \uparrow h \\
P & \longrightarrow & M \times B^{k-1}
\end{array}
\]

in which $h$ is a proper embedding properly isotopic to the standard inclusion $M \times B^{k-1} \subset M \times B^k$.

To this end we may assume that $Q$ and $M \times B^k$ are triangulated so that $g$ is simplicial and so that the standard $M \times B^{k-1}$ is a full subcomplex of $M \times B^k$. Let $N$ be the simplicial neighbourhood of $M \times B^{k-1}$ in the first derived subdivision of $M \times B^k$. Since inverse images of dual cells by simplicial maps are manifolds [1; 5.6], we see that $\mathcal{Q}_0 = g^{-1}(N)$ is a codimension zero submanifold of $Q$ and that $g | \mathcal{Q}_0 : \mathcal{Q}_0 \to N$ is a resolution rel $N \cap (\partial M \times B^k)$. Now the frontier of $N$ breaks into two pieces; let $N^+$ be one of them. Let $P = g^{-1}(N^+)$, which must be one of the two components of the frontier of $\mathcal{Q}_0$ in $Q$. Let $f = g | P$. Then the theory of derived neighbourhoods shows that there is a proper PL homeomorphism

\[(M \times B^{k-1}, \partial M \times B^{k-1}) \to (N^+, N^+ \cap \partial M \times B^k)\]
which is isotopic to the standard inclusion $M \times B^{k-1} \subset M \times B^k$. Finally, $P$ has trivial normal bundle in $Q$, being a boundary component of a codimension zero submanifold.

4. Tangential Properties of Resolutions

If $M$ is a homology manifold and $\xi$ is a homology cobordism bundle over $M$, then a (stable) reduction of $\xi$ to a PL block bundle is a homology cobordism bundle $\eta$ over $M \times I$ such that $\eta | M \times 0 = \xi$ stably and $\eta | M \times 1$ is a PL block bundle. Two reductions of $\xi$ are equivalent if there is a reduction of $\xi \times I$ over $(M \times I) \times I$ between them. Let $H/PL(\xi)$ denote the set of equivalence classes of stable reductions of $\xi$.

Also, let Lift$(\xi)$ denote the set of vertical homotopy classes of lifts to $BPL$ of $\xi : M \rightarrow BH$ through $j : BPL \rightarrow BH$. Then the following lemma is an exercise in the definitions, using the homotopy lifting property.

**Lemma 4.1.** There is a one-to-one correspondence between the elements of $H/PL(\xi)$ and those of Lift$(\xi)$.

**Lemma 4.2.** If $N$ and $M$ are homology manifolds and $f : N \rightarrow M$ induces an isomorphism on homology, then the natural map

$$f^* : H/PL(\tau(M)) \rightarrow H/PL(f^* \tau(M))$$

is a bijection.

**Proof** (of 4.2). If $\eta$ is a reduction of $\tau(M)$, $f^*[\eta]$ is given by $[(f \times 1)^*\eta]$. Obstruction theory shows that there are lifts of $\tau(M)$ to $BPL$ if and only if there are lifts of $f^*\tau(M)$ to $BPL$. So assume such lifts exist and choose one lift $\alpha$ of $\tau(M)$. Then comparison with $\alpha$ and with $f^*\alpha$ and application of the homotopy lifting property for fibrations shows that we have a commutative diagram

$$\begin{array}{ccc}
H/PL(\tau(M)) & \rightarrow & H/PL(f^* \tau(M)) \\
\downarrow & & \downarrow \\
\text{Lift}(\tau(M)) & \rightarrow & \text{Lift}(f^* \tau(M)) \\
\downarrow & & \downarrow \\
[M; H/PL] & \rightarrow & [N; H/PL]
\end{array},$$

where horizontal arrows are induced by $f$ and the vertical arrows are bijections. But the lower horizontal arrow is a bijection by obstruction theory (all coefficients are simple since $\pi_1(H/PL) = 0$), completing the proof.

**Theorem 4.3.** A resolution $(P, f)$ of a homology manifold $M$ determines a well defined element of $H/PL(\tau(M))$ which depends only on the class of $(P, f)$ in Res$(M)$.

**Proof.** The mapping cylinder $C_f$ of $f$ is a homology manifold (see §2). Let $\pi : P \times I \rightarrow C_f$ be the natural quotient map. Then $\pi^* (C_f)$ is a stable reduction of $f^* \tau(M)$ to the PL block bundle $T(P)$. Using (4.2), let $\alpha$ be the unique element of $H/PL(\tau(M))$ corresponding to $f^*\tau(M)$. The same construction applied to a concordance shows that the class of $\alpha$ in $H/PL(\tau(M))$ depends only on the concordance class of $(P, f)$ in Res$(M)$.

By (4.3) we obtain a well-defined function

$$\psi : \text{Res}(M) \rightarrow H/PL(\tau(M)).$$
5. Bijectivity of $\psi$.

We first show that the existence of a reduction of $\tau(M)$ to a PL block bundle implies that $M$ is resolvable. Carefully done this shows that $\psi$ is surjective. A suitable relative version of the above then shows that $\psi$ is injective.

If $(P,f)$ is a resolution of $M$ let $[P,f]$ denote its class in $\text{Res}(M)$; if $\eta$ is a stable reduction of $\tau(M)$ to a PL block bundle, let $[\eta]$ denote its class in $H/\text{PL}(\tau(M))$.

**Theorem 5.1 (Existence Theorem).** Let $M$ be a compact homology $n$-manifold with $\partial M = \emptyset$. If $\eta$ is a stable reduction of $\tau(M)$ to a PL block bundle, then there is a resolution $(P,f)$ of $M$ such that $\psi[P,f] = [\eta]$.

**Proof.** Let $M$ be embedded in some Euclidean space $\mathbb{R}^k$ and let $N$ be a closed regular neighbourhood of $M$ in $\mathbb{R}^k$. Then $N$ is a parallelizable PL $k$-manifold and there is a PL deformation retraction $r : N \to M$. The induced stable reduction $\xi = (r \times 1)^*\eta$ of $r^*\tau(M)$ to a PL block bundle is a homology cobordism bundle over $N \times I$ such that $\xi_0 = \xi | N \times 0$ is a PL block bundle and $\xi_1 = \xi | N \times 1$ is stably isomorphic to $r^*\tau(M)$. The total space $E(\xi)$ of $\xi$ is an $H$-cobordism between the PL manifold $E(\xi_0)$ and the homology manifold $E(\xi_1)$. Also $E(\xi_1)$ is PL homeomorphic to $M \times I^m$ for some $m$, since $E(\xi_1) = E(r^*\tau(M)) \times I^q$ for some $q$, and $E(r^*\tau(M))$ can be seen to be a regular neighbourhood of $M \times 0$ in $M \times \mathbb{R}^k$ (compare [8; 5.12]).

Since $H^*(E(\xi), E(\xi_0)) = 0$, the obstruction to resolving $E(\xi)$ rel $E(\xi_0)$ is zero, so let $(Q, g)$ be such a resolution. Pulling back a small regular neighbourhood of $E(\xi_1) \cong M \times I^m$ in $\partial E(\xi)$ we obtain by restriction, as in the proof of the Product Structure Theorem, a resolution $(R, h)$ of $M \times I^m$, and hence a resolution $(P, f)$ of $M$.

It remains to see that $\psi[P, f] = [\eta]$. For this we use the fact that we resolved the entire $H$-cobordism $E(\xi)$. Now by (4.3), $(Q, g)$ determines a reduction of $\tau(E(\xi))$ to a PL block bundle, hence a reduction of $\tau(E(\xi)) | N \times I$ to a PL block bundle. But $\tau(E(\xi)) | N \times I \cong \tau(N \times I^I) \oplus \xi$ which stably is just $\xi$ since $N$ is parallelizable. Thus $(Q, g)$ determines a stable reduction $\beta$ over $(N \times I) \times I$ with $\beta | N \times I \times 1$ a PL block bundle and $\beta | N \times I \times 0 = \xi$. Furthermore, $\beta | N \times 1 \times I$ is a PL block bundle, being obtained from the trivial reduction induced by the identity map and the isomorphism $\tau(E(\xi)) | N \times 1 = T(E(\xi)) | N \times 1$. Also, $\beta | N \times 0 \times I$ is stably $r^*\mu$ where $\mu$ is the reduction of $\tau(M)$ induced by $(P, f)$. This follows because, by the Product Structure Theorem, $P$ has a trivial normal PL block bundle in $Q$ and there is a commutative diagram

$$
\begin{array}{ccc}
Q & \xrightarrow{g} & M \times I^m \\
\uparrow & & \uparrow h \\
P & \xrightarrow{f} & M 
\end{array}
$$

where $h$ is a PL embedding which is PL isotopic to the standard embedding $M \times 0$.

Thus $\beta$ determines an equivalence between the stable reductions $\xi = r^*\eta$ and $r^*\mu$ and hence between $\eta$ and $\mu$ by (4.2) as desired.

In a similar fashion one proves the following relative version of (5.1).
**Theorem 5.2.** Let $M$ be a homology manifold with $\partial M$ a PL manifold. If $\eta$ is a stable reduction of $\tau(M)$ to a PL block bundle such that $\eta|\partial M \times I = T(\partial M) \times I$ as stable PL block bundles, then there is a resolution $(P, f)$ of $M$ rel $\partial M$.

**Corollary 5.3.** (Classification Theorem). If $M$ is a homology manifold with $\partial M = \emptyset$, then $\psi : \text{Res}(M) \to H/\text{PL}(\tau(M))$ is bijective. Hence $\text{Res}(M) \approx \text{Lift}(\tau(M))$.

**Proof.** Surjectivity follows from (5.1). To prove injectivity, let $(P_i, f_i)$, $i = 0, 1$, be two resolutions of $M$ such that the corresponding stable reductions $\eta_0$ and $\eta_1$ are equivalent. Then we obtain a homology cobordism bundle $\eta$ over $M \times I \times I$ such that $\eta| (M \times I \times i) = \eta_i$ stably, for $i = 0, 1$, $\eta| (M \times 1 \times I)$ is a PL block bundle, and $\eta| (M \times 0 \times I)$ is $\tau(M) \times I$ stably.

Let $C_i$ denote the mapping cylinder of $f_i$, $i = 0, 1$, and consider the homology manifold $N = C_0 \cup M \times I \cup C_1$, where we identify the zero ends of $C_0$ and $C_1$ with $M \times 0$ and $M \times 1$ respectively. Using the natural retraction of $N$ onto $M \times I$, we obtain from $\eta$ a reduction of $\tau(N)$ to a PL bundle rel $P_0 \cup P_1$. Thus we obtain by (5.2) a resolution $(Q, g)$ of $N$ rel $\partial N = P_0 \cup P_1$.

Define a map $r : N \to M \times I$ by

$$
[r[x, t]] = \begin{cases} 
(f_0(x), (1-t)/3) & \text{if } [x, t] \in C_0, \\
(f_1(x), (2+t)/3) & \text{if } [x, t] \in C_1, \\
(x, (1+t)/3) & \text{if } [x, t] \in M \times I.
\end{cases}
$$

Then $(Q, rg)$ is a resolution of $M \times I$ and a concordance between $(P_0, f_0)$ and $(P_1, f_1)$.

Using similar techniques one also proves the following more general relative classification theorem.

**Theorem 5.4.** Let $M$ be a homology manifold with $\partial M$ a PL manifold. Then there is a one-to-one correspondence between concordance classes of resolutions of $M$ rel $\partial M$ and vertical homotopy classes of lifts of $\tau(M) : M \to BH$ to $BPL$ rel a fixed lift $T(\partial M) : \partial M \to BPL$ of $\tau(\partial M) : \partial M \to BH$.

**References**


Cornell University,
Ithaca, New York 14853,
U.S.A.

University of Utah,
Salt Lake City, Utah 84112
U.S.A.