Pinwheels
Smooth structures
and
4-manifolds with Euler characteristic 3

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Joint work with Ron Fintushel

Progress from Earlier Topology Festival Talks

Q: (1976): Are all closed topological M^n, n >4, simplicial complexes?
Galewski-S, T. Matumoto: Yes iff there is a Kervire-Milnor-Rochlin invariant one homology 3-sphere S with S#S bounding an acyclic smooth 4-manifold.
(Also a classification up to natural equivalence)

No progress in last 36 years-but introduced to low-dimensional topology

Q: (1979): Does every smooth pseudo-free S^1-action on S^5 have at most 3 exceptional orbits.

Montgomery-Yang Problem: F-S gave non-linear examples with 3 such orbits

No progress in last 33 years

Rediscovered recently by Kollar and other algebraic geometers

Q: (2012): Does every smooth closed 4-manifold have infinitely many distinct smooth structures?

Do smooth 4-manifold have infinitely many distinct smooth structures?

n >4: Only finitely many

Over years F-S (focused on $\pi_1=0$) developed techniques to support YES.

Constructions

Invariants

Artful blend of both:
Constructing examples that change known invariants

Algebraic Topological: Intersection form
e, sign, type

Gauge theoretic: Donaldson-Seiberg-Witten
orient M: e + sign = 0 (4)
i.e. almost complex structure

Need to know don't change homeo type
Freedman-Donaldson: For simply-connected: e, sign, type determine homeo type

Part of the art taught us that the smaller the Euler characteristic, newer constructions evolve

Let's go for broke: smallest e with $b_1=0$; e=3, sign=1.

e.g. CP^2

Fake Projective Planes

Call any oriented manifold with $e=3$, $b_1=0$, sign=1 and not CP^2 a fake projective plane.

Surfaces of general type: Rich algebraic and arithmetic geometry history: Classified in 2010

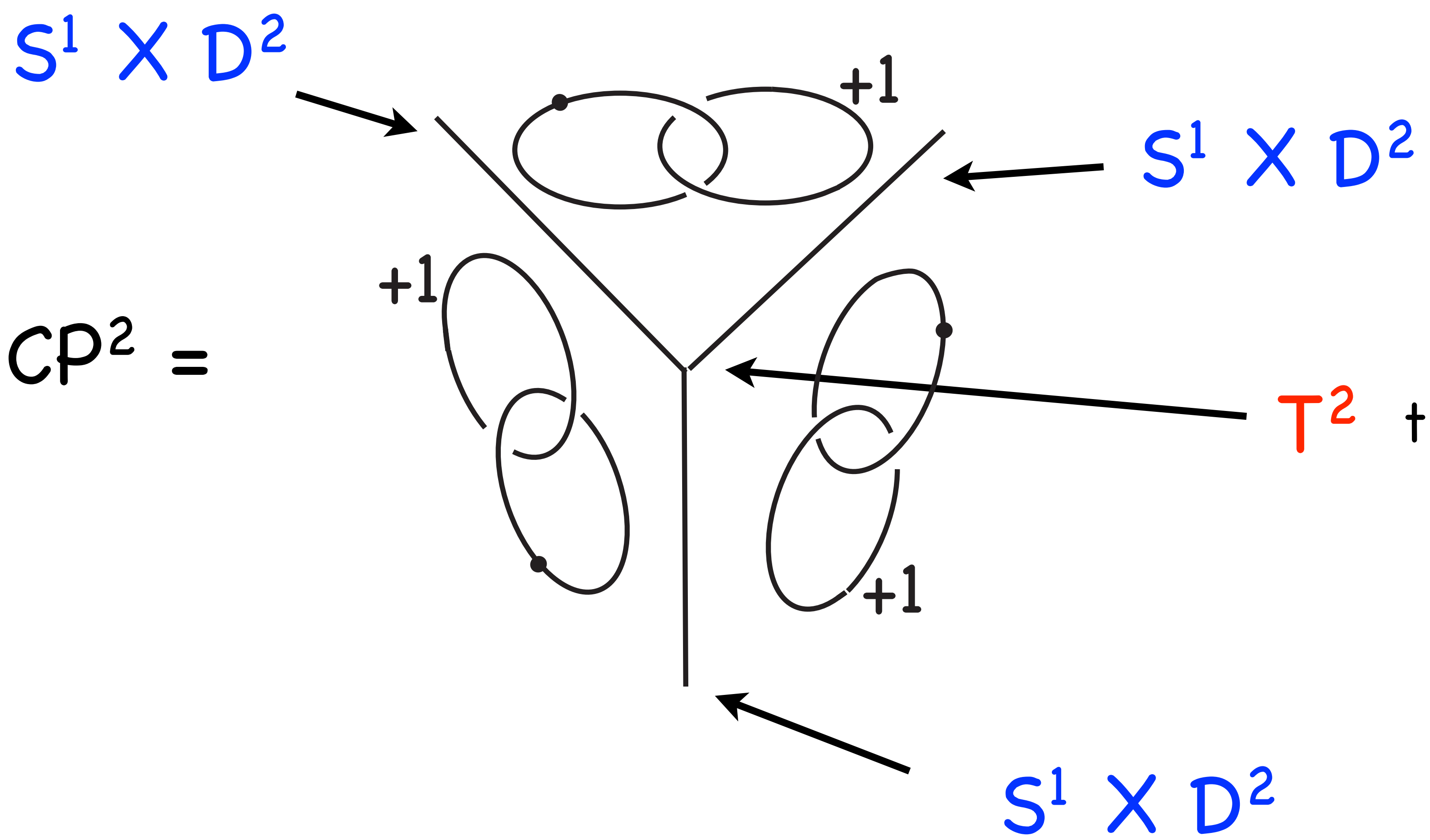
- Yau (1977): π_1 infinite discrete subgroup of PU(2,1) i.e. complex ball quotient
- Mumford (1979); Using p-adic uniformization techniques constructed one example: Mumford plane
- Ishida (1987): Quotient of an 8-fold cover of M is a Doglachev surface I
- Keum(2006): Start with I and found a different fake projective plane as a 7-fold cover.
- Prasad-Yeung (2007): Began classification of π_1 -one of 28 types
- Cartwright-Steger(2010): 50 such examples up to isometry (100 up to biholomorhpism)
- All have H_1 non-trivial finite

Issue: None are explicit constructions

Let's start from scratch!

How to build Fake Projective Planes

e=3, so cover with 3 balls



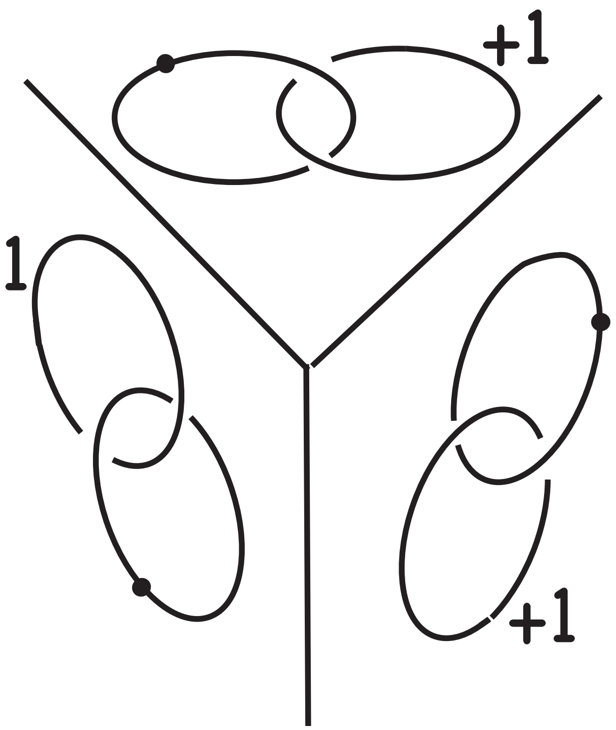
Actually $T^2 \times S^1$: this is a condition on the glueing; fill in with $T^2 \times D^2$

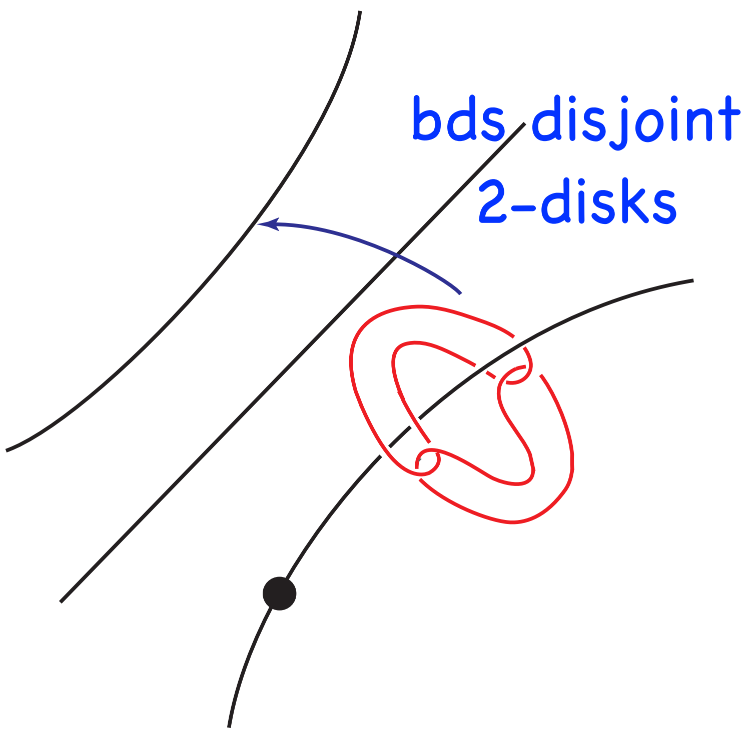
Note: Boundary is that of two intersection spheres: Condition: One self-intersection 0, other self intersection -1: Glue 0 to -1

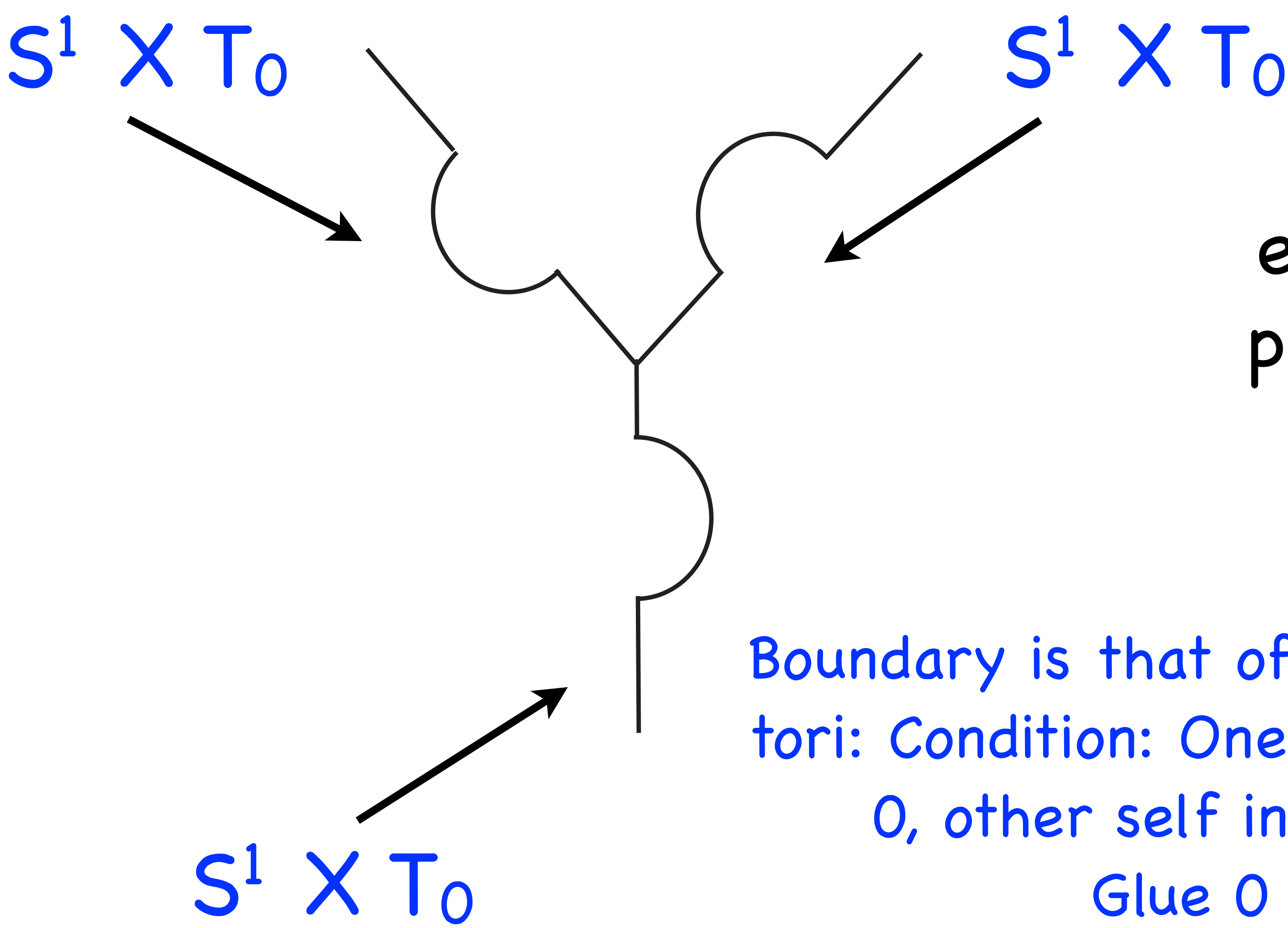
M. Symington: Can do symplectically (3-fold sum): In $CP^2 \# -CP^2$ consider the symplectic spheres: h-e, e . The complement of this configuration is the 4-ball. Can glue to obtain a symplectic CP^2.

How to build Fake Projective Planes

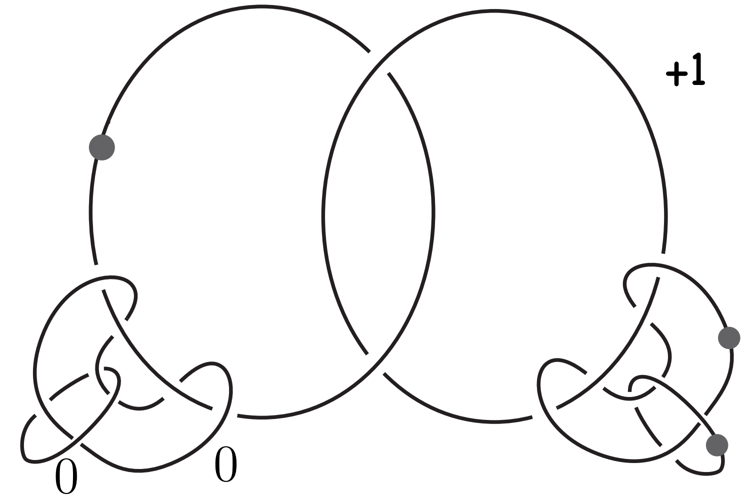
CP^2 =







each piece

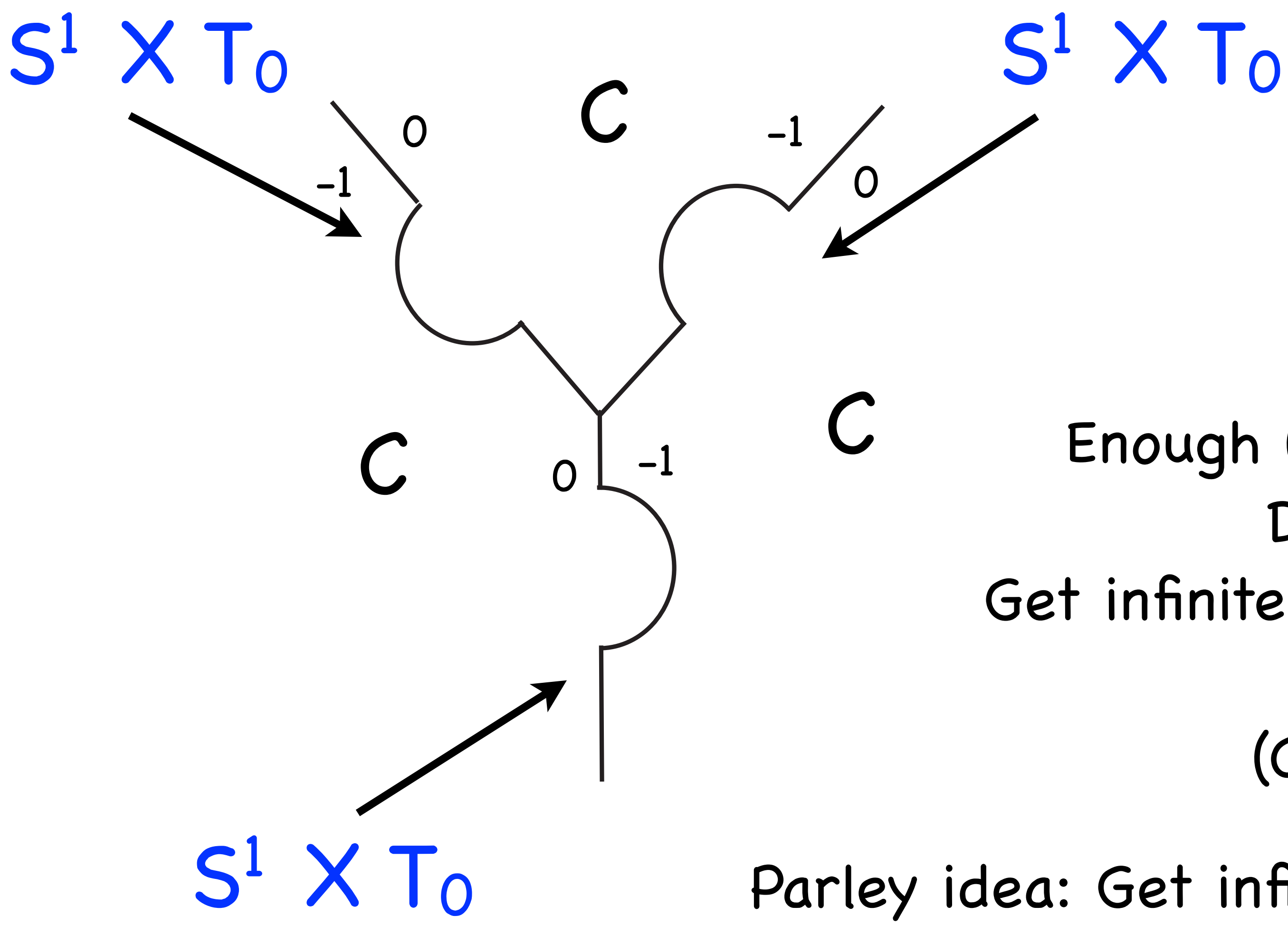


Boundary is that of two intersecting tori: Condition: One self-intersection 0, other self intersection -1
Glue 0 to -1

So find such tori in e=0 manifold: then glue

How to build fake $CP^2\#-3CP^2$

In $(T^2 \times T^2)\ \#\ -CP^2$ find tori T and S-e
Complement C (e=2)



Get e =6: "Model" for

$CP^2\#-3CP^2$

But $b_1=6$

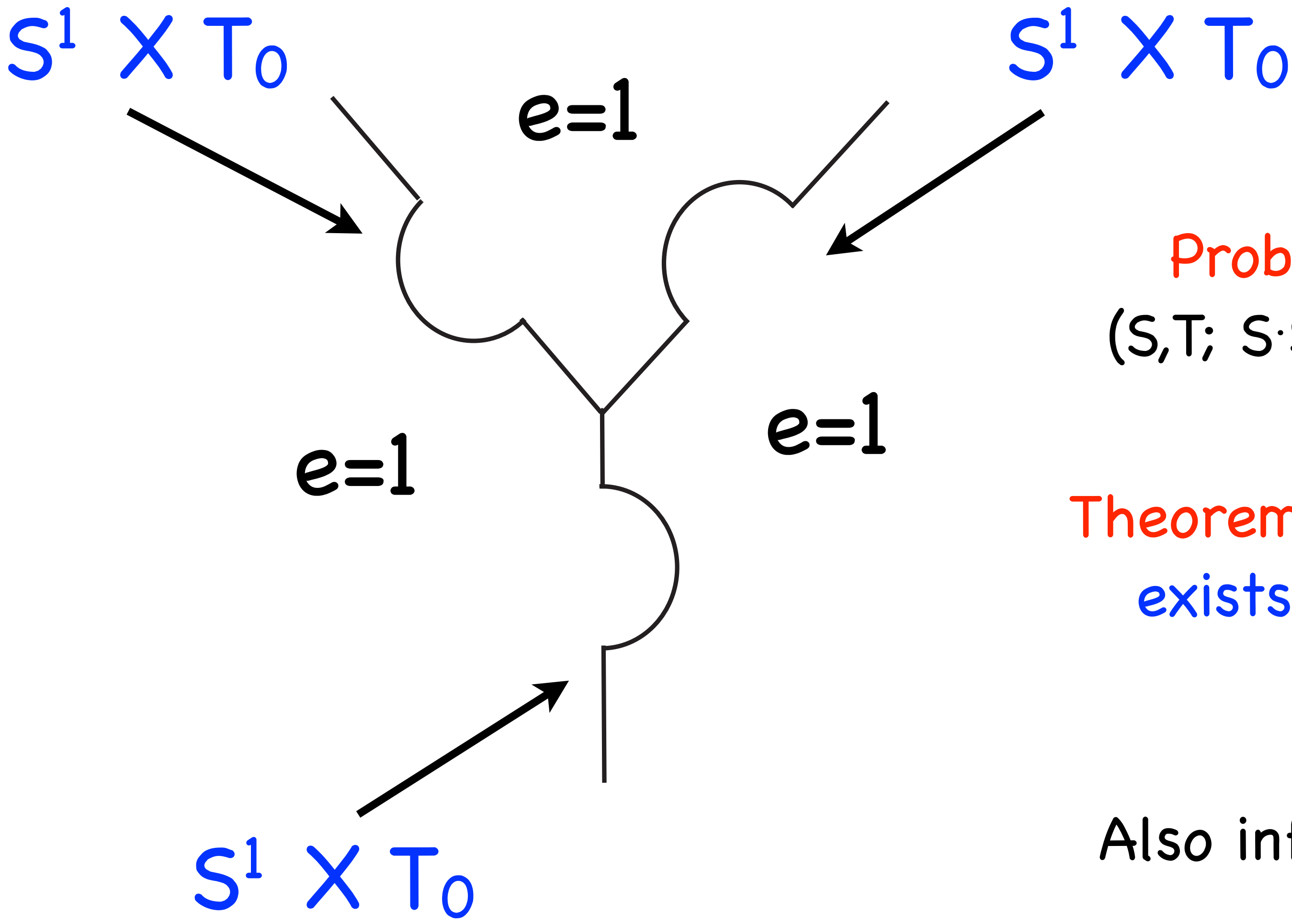
Enough (Lagrangian) tori to surger so $b_1=0$
Do carefully-surger so $\pi_1=0$
Get infinitely many distinct smooth structures on
$CP^2\#-3CP^2$
(Call this "pinwheel surgery")

Parley idea: Get infinitely many smooth structures on
$CP^2\#-nCP^2$ n=2,3,4,5,6,7,8,9

Furthermore: Find one (null-homologous) torus in $CP^2\#-nCP^2$:Surger it to get
infinitely many distinct smooth structures
What about n=0?

Problem: find transveral symplectic tori (S,T; S·S=-1, T·T=0) in e=0 symplectic manifold
not so easy.

How to build fake projective planes



Problem: find transversal symplectic tori (S,T; $S \cdot S=-1$, $T \cdot T=0$) in e=0 symplectic manifold

Theorem (F-S): Such a symplectic 4- manifold exists. Yields infinitely many distinct fake symplectic projective planes

Distinguished by H_1

Also inftly many smooth and not symplectic

Distinguished by SW

Also on each lattice point of BMY line

Most likely all a K(G,1)

Q: Does every smooth 4-manifold have infinitely many distinct smooth structures?

(For "Small" fundamental groups)

Q: If X is a smooth (symplectic) K(G,1) it has a unique smooth structure

Q: Symplectic Yau theorem: Any symplectic manifold on the BMY line is a K(G,1).

THE END!