Euler, Polyhedron, and Smooth 4-Manifolds

Ronald J. Stern
University of California, Irvine
June 1, 2007
Euler and the Beginnings of Combinatorial Topology

<table>
<thead>
<tr>
<th>Name</th>
<th>Image</th>
<th>( V ) vertices</th>
<th>( E ) edges</th>
<th>( F ) faces</th>
<th>( V - E + F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td><img src="image1" alt="Tetrahedron Image" /></td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Hexahedron or cube</td>
<td><img src="image2" alt="Hexahedron Image" /></td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Octahedron</td>
<td><img src="image3" alt="Octahedron Image" /></td>
<td>6</td>
<td>12</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>Dodecahedron</td>
<td><img src="image4" alt="Dodecahedron Image" /></td>
<td>20</td>
<td>30</td>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td>Icosahedron</td>
<td><img src="image5" alt="Icosahedron Image" /></td>
<td>12</td>
<td>30</td>
<td>20</td>
<td>2</td>
</tr>
</tbody>
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\( (V - E + F) \) for a convex polyhedron equals 2.

First mentioned in a letter to Goldbach dated 14 November 1750.
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<tr>
<td>Hexahedron or cube</td>
<td><img src="cube.png" alt="Image" /></td>
<td>8</td>
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<td>6</td>
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\((V[ertices] - E[dges] + F[aces])(convex polyhedron) = 2\)

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An Early Topological Invariant

Euler characteristic $= \chi = V[ertices] - E[edges] + F[aces]$
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<tr>
<td>Sphere</td>
<td><img src="image1.png" alt="Sphere Image" /></td>
<td>2</td>
</tr>
<tr>
<td>Torus</td>
<td><img src="image2.png" alt="Torus Image" /></td>
<td>0</td>
</tr>
<tr>
<td>Two-Holed Torus</td>
<td><img src="image3.png" alt="Two-Holed Torus Image" /></td>
<td>-2</td>
</tr>
<tr>
<td>Klein bottle</td>
<td><img src="image4.png" alt="Klein Bottle Image" /></td>
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Euler characteristic = \( \chi = V[ertices] - E[dges] + F[aces] \)

- Classifies orientable surfaces: \( \chi = 2 - 2g \)
- Classifies non-orientable surfaces: \( \chi = 1 - g \)
  - Möbius (1870) was the first to attempt the classification of surfaces: proved for orientable surfaces smoothly imbedded in \( R^3 \).
  - Classification for non-orientable surfaces first announced by W. von Dyck (1888): proof incomplete. Among other problems, at that time there was no satisfactory concept of an abstract surface, not imbedded in Euclidean space.
  - The first rigorous proof was given by M. Dehn and P. Heegard in 1907, assuming surfaces are polyhedra
  - Surfaces are polyhedra first proved by T. Rado (1925) thus completing the proof of the classification theorem.
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What is a polyhedron?
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Grünebaum (1994)- Are Your Polyhedra the Same as My Polyhedra?

The Original Sin in the theory of polyhedra goes back to Euclid, and through Kepler, Poinsot, Cauchy and many others ... [in that] at each stage ... the writers failed to define what are the 'polyhedra' .... There is very little in common between the meaning of the word in topology and geometry.
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- For us - a *polyhedron* $P$ is topological space given along with a specific decomposition into shapes that are topologically equivalent to simplices (convex hull of the coordinate unit vectors in $R^{n+1}$, for some $n$) and that are attached to each other in a regular way.

- More precisely - a *polyhedron* $P$ is a set of simplices $\kappa$ that satisfies the following conditions:
  
  Any face of a simplex from $\kappa$ is also in $\kappa$.
  
  The intersection of any two simplices $\sigma_1$ and $\sigma_2$ is a single face of both $\sigma_1$ and $\sigma_2$.

- $P$ is triangulated as a *simplicial complex*.

- The Euler characteristic of polyhedron $P$ is

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\chi(P) = \sum_{i=0}^{\infty} (-1)^i \alpha_i
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where $\alpha_i = \text{ the number of simplices of dimension } i$.
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Is every compact topological $n$-manifold a polyhedron?

- **Yes** for $n \leq 3$ ( $n = 2$: Rado, 1925; $n = 3$: Moise, 1952)
- **No** for $n = 4$ (Freedman, 1980)
- **Unknown** for $n > 4$

**Open Conjecture**
Every topological $n$-manifold, $n > 4$, is a polyhedron, i.e. can be triangulated as a simplicial complex.
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- If a topological $n$-manifold $X$ is a simplicial complex, then the link $\Sigma$ of every vertex is a homotopy $n - 1$-sphere.
- The Poincaré conjecture implies that if $n = 4$, then $\Sigma$ is the 3-sphere; hence $X$ is a PL manifold; hence a smooth manifold [Cairns, Whitehead, Hirsch, Milnor, Munkres, Lashof, Mazur, . . . , 1940 -1968]
- A 4-manifold $X$ is a polyhedron iff $X$ smoothable.
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- Rochlin invariant map $\mu$ fits into short exact sequence

$$0 \rightarrow \ker(\mu) \rightarrow \Theta^H_3 \xrightarrow{\mu} \mathbb{Z}_2 \rightarrow 0$$

$\Theta^H_3$ the homology cobordism group of oriented smooth homology 3–spheres and $\mu(\Sigma) = \text{signature}(W^4)/8 \mod 2$, $W^4$ spin with $\partial W^4 = \Sigma$.

- (Galewski-Stern, 1976) The triangulation obstruction of a compact TOP $n$–manifold $M$ is

$$\delta_M \in H^5(M; \ker(\mu))$$

For $n \geq 5$ $M$ can be triangulated iff $\delta_M(M) = 0$.

- $|H^4(M; \ker(\mu))|$ such triangulations up to concordance.

- Still unknown if there is an $M$ with $\delta_M \neq 0$. 
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$n > 4$: The triangulation obstruction

- For $n \geq 5$: $M^n$ can be triangulated iff $\delta_M \in H^5(M; \ker(\mu))$ vanishes.
- $|H^4(M; \ker(\mu))|$ such triangulations up to concordance.
- Still unknown if there is an $M$ with $\delta_M \neq 0$.
- (Galewski-Stern, Matumoto) $\delta_M = 0$ iff there exits an order two element $\Sigma \in \Theta_3^H$ with $\mu(\Sigma) = 1$; i.e $\Sigma \# \Sigma$ bounds an acyclic 4-manifold.
- Do not know any example of $\Sigma \in \Theta_3^H$ with non-zero finite order.
- $\ker(\mu)$ is infinitely generated (Furuta, Fintushel-Stern 1990 using Donaldson, 1982). Each generator has infinite order.
- $\delta_M$ is the image of the Kirby-Siebenmann PL-triangulation obstruction $\kappa(M) \in H^4(M; \mathbb{Z}_2)$ under the Bockstein

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Is every topological $n$-manifold a polyhedron?

- For $n \leq 3$: Yes since all are smooth.
- For $n = 4$: iff smooth
- For $n \geq 5$: iff there exists a homology 3-sphere $\Sigma$ with $\mu(\Sigma) = 1$ and $\Sigma \# \Sigma$ bounds an acyclic 4-manifold.
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What do we know about 4-manifolds?

Invariants

- Euler characteristic: $e(M) = \sum_{i=0}^{4} (-1)^i H^i(M; \mathbb{Z})$
- $Q_M : H^2(M; \mathbb{Z}) \times H^2(M; \mathbb{Z}) \to \mathbb{Z}; \ Q_M(\alpha, \beta) = (\alpha \cup \beta)[M]$
  is an integral, symmetric, unimodular, bilinear form.

Signature of $M = \sigma(M) = \text{Signature of } Q_M = b_+ - b_-$

Type: Even if $Q_M(\alpha, \alpha)$ even for all $\alpha$; otherwise Odd

- (Freedman, 1980) $Q_M$ classifies simply-connected topological 4-manifolds: There is one homeomorphism type if $Q_M$ even; there are two if $Q_M$ odd - exactly one of which has $M \times S^1$ smoothable.

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Wild and Open Conjecture

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every $n$-manifold has only finitely many distinct smooth $n$-manifolds which are homeomorphic to it.

- Need new invariants: Donaldson, Seiberg-Witten Invariants
  
  $SW : H^2(M; \mathbb{Z}) \rightarrow \mathbb{Z}$

- $SW(\beta) \neq 0$ for only finitely many $\beta$: called basic classes

- $\alpha \in H^2(M; \mathbb{Z})$, then

  \[2g(\Sigma_\alpha) - 2 \geq Q_M(\alpha, \alpha) + |Q_M(\alpha, \beta)|\]

  for every basic class $\beta$. (adjunction inequality[Kronheimer-Mrowka])

Basic classes are the smooth analogue of the canonical class of a complex surface.
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What we know about smooth 4-manifolds with $\mathcal{SW} \neq 0$

joint with Ron Fintushel
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All manifolds minimal
$c = 3\sigma + 2e$
$\chi_h = \frac{\sigma + e}{4}$
What we know about smooth 4-manifolds with $\mathcal{SW} \neq 0$
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surfaces of general type

$2\chi_h - 6 \leq c \leq 9\chi_h$

$c = 9\chi_h$

$c = 2\chi_h - 6$

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All lattice points have $\infty$ smooth structures except possibly near $c = 9\chi_h$ and on $\chi_h = 1$

For $n > 4$ TOP n-manifolds have finitely many smooth structures

$\mathbb{CP}^2 \bigoplus \mathbb{CP}^2 \bigoplus \mathbb{CP}^2$
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Elliptic Surfaces
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$\sigma = 3\sigma + 2e$
$\chi_h = \frac{\sigma + e}{4}$

$\chi_h \leq c \leq 9\chi_h$ surfaces of general type

$c = 2\chi_h - 6$ symplectic with one SW basic class

$c = \chi_h - 3$ symplectic with $(\chi_h - c - 2) SW$ basic classes

$0 \leq c \leq (\chi_h - 3)$

Elliptic Surfaces $E(n)$

$\sigma > 0$ $\sigma < 0$

$c = 9\chi_h$
$c = 8\chi_h$
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CP$^2$, $S^2 \times S^2$
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- $c = 8\chi_h$ for $\sigma = 0$
- $c > 9\chi_h$ ?? for $\sigma > 0$
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Happy Birthday Leonhard!