Smooth 4-manifolds: BIG and small

Ronald J. Stern
University of California, Irvine
August 11, 2008

Joint work with Ron Fintushel
The Pre–4-manifold Kirby

Annulus Conjecture, Torus trick, Hauptvermutung using Engulfing, Surgery theory

- Phd 1965 – Advisor: Eldon Dyer, University of Chicago
- Assistant Professor UCLA: 1966-69; Full Professor UCLA: 1969-71; Full Professor UCB: 1971 —
- 1971 Fifth Veblen Prize: The annulus conjecture is true: a region in n-space bounded by two locally flat \((n - 1)\)-spheres is an annulus \((n > 5)\): *Stable Homeomorphisms and the Annulus Conjecture*, Annals of Math 89 (1969), 574–82.
- With Larry Siebenmann: The Hauptvermutung is false: PL structures (up to isotopy) on a PL manifold \(M\) correspond to elements of \(H^3(M; \mathbb{Z}_2)\) \((n > 4)\)
- The triangulation conjecture is false: a topological manifold has no PL structure when an obstruction in \(H^4(M; \mathbb{Z}_2)\) is non-zero \((n > 4)\)
- Simple homotopy type is a topological invariant \((n > 4)\)


- Key new idea: Torus Trick; works in dimension \(> 4\), so led to Rob’s 4-dimensional interests and his 4-manifold legacy —
- 50 PhD students; 82 grandchildren; 16 great grandchildren
The Pre– 4-manifold Kirby
Annulus Conjecture, Torus trick, Hauptvermutung using Engulfing, Surgery theory

- Phd 1965 – Advisor: Eldon Dyer, University of Chicago
- Assistant Professor UCLA: 1966-69; Full Professor UCLA: 1969-71; Full Professor UCB: 1971
- 1971 Fifth Veblen Prize: The annulus conjecture is true: a region in $n$-space bounded by two locally flat $(n-1)$-spheres is an annulus ($n > 5$): Stable Homeomorphisms and the Annulus Conjecture, Annals of Math 89 (1969), 574–82.
- With Larry Siebenmann: The Hauptvermutung is false: PL structures (up to isotopy) on a PL manifold $M$ correspond to elements of $H^3(M; \mathbb{Z}_2)$ ($n > 4$)
- The triangulation conjecture is false: a topological manifold has no PL structure when an obstruction in $H^4(M; \mathbb{Z}_2)$ is non-zero ($n > 4$)
- Simple homotopy type is a topological invariant ($n > 4$)


- Key new idea: Torus Trick; works in dimension $> 4$, so led to Rob’s 4-dimensional interests and his 4-manifold legacy —
- 50 PhD students; 82 grandchildren; 16 great grandchildren
The Pre–4-manifold Kirby

Annulus Conjecture, Torus trick, Hauptvermutung using Engulfing, Surgery theory

- PhD 1965 – Advisor: Eldon Dyer, University of Chicago
- Assistant Professor UCLA: 1966-69; Full Professor UCLA: 1969-71; Full Professor UCB: 1971 —
- 1971 Fifth Veblen Prize: The annulus conjecture is true: a region in n-space bounded by two locally flat \((n-1)\)–spheres is an annulus \((n > 5)\): *Stable Homeomorphisms and the Annulus Conjecture*, Annals of Math 89 (1969), 574–82.
- With Larry Siebenmann: The Hauptvermutung is false: PL structures (up to isotopy) on a PL manifold \(M\) correspond to elements of \(H^3(M; \mathbb{Z}_2)\) \((n > 4)\)
- The triangulation conjecture is false: a topological manifold has no PL structure when an obstruction in \(H^4(M; \mathbb{Z}_2)\) is non-zero \((n > 4)\)
- Simple homotopy type is a topological invariant \((n > 4)\)


- Key new idea: Torus Trick; works in dimension \(> 4\), so led to Rob’s 4-dimensional interests and his 4-manifold legacy —
- 50 PhD students; 82 grandchildren; 16 great grandchildren
The Pre– 4-manifold Kirby
Annulus Conjecture, Torus trick, Hauptvermutung using Engulfing, Surgery theory

- Phd 1965 – Advisor: Eldon Dyer, University of Chicago
- Assistant Professor UCLA: 1966-69; Full Professor UCLA: 1969-71; Full Professor UCB: 1971 —
- 1971 Fifth Veblen Prize: The annulus conjecture is true: a region in n-space bounded by two locally flat \((n - 1)\)-spheres is an annulus \((n > 5)\): *Stable Homeomorphisms and the Annulus Conjecture*, Annals of Math 89 (1969), 574–82.
- With Larry Siebenmann: The Hauptvermutung is false: PL structures (up to isotopy) on a PL manifold \(M\) correspond to elements of \(H^3(M; \mathbb{Z}_2)\) \((n > 4)\)
- The triangulation conjecture is false: a topological manifold has no PL structure when an obstruction in \(H^4(M; \mathbb{Z}_2)\) is non-zero \((n > 4)\)
- Simple homotopy type is a topological invariant \((n > 4)\)
- Key new idea: Torus Trick; works in dimension \(\geq 4\), so led to Rob’s 4-dimensional interests and his 4-manifold legacy —
- 50 PhD students; 82 grandchildren; 16 great grandchildren
Basic facts about 4-manifolds

Invariants

- Euler characteristic: \( e(X) = \sum_{i=0}^{4} (-1)^i \text{rk}(H^i(M; \mathbb{Z})) \)
- Intersection form: \( H^2(X; \mathbb{Z}) \otimes H^2(X; \mathbb{Z}) \to \mathbb{Z}; \)

\[ \alpha \cdot \beta = (\alpha \cup \beta)[X] \]

is an integral, symmetric, unimodular, bilinear form.

Signature of \( X = \text{sign}(X) = \) Signature of intersection form
\[ = b^+ - b^- \]

Type: Even if \( \alpha \cdot \alpha \) even for all \( \alpha \); otherwise Odd

(Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has \( X \times S^1 \) smoothable.

(Donaldson, 1982) Two simply connected smooth 4-manifolds are homeomorphic iff they have the same \( e \), \( \text{sign} \), and \( \text{type} \).
Basic facts about 4-manifolds

Invariants

▶ Euler characteristic: \( e(X) = \sum_{i=0}^{4} (-1)^i \text{rk}(H^i(M; \mathbb{Z})) \)

▶ Intersection form: \( H^2(X; \mathbb{Z}) \otimes H^2(X; \mathbb{Z}) \to \mathbb{Z}; \)

\[
\alpha \cdot \beta = (\alpha \cup \beta)[X]
\]

is an integral, symmetric, unimodular, bilinear form.

Signature of \( X = \text{sign}(X) = \text{Signature of intersection form} = b^+ - b^- \)

Type: Even if \( \alpha \cdot \alpha \) even for all \( \alpha \); otherwise Odd

▶ (Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has \( X \times S^1 \) smoothable.

▶ (Donaldson, 1982) Two simply connected smooth 4-manifolds are homeomorphic iff they have the same \( e, \text{sign}, \) and \( \text{type}. \)
Basic facts about 4-manifolds

Invariants

- Euler characteristic: \( e(X) = \sum_{i=0}^{4} (-1)^i \text{rk}(H^i(M; \mathbb{Z})) \)

- Intersection form: \( H^2(X; \mathbb{Z}) \otimes H^2(X; \mathbb{Z}) \rightarrow \mathbb{Z}; \)

\[
\alpha \cdot \beta = (\alpha \cup \beta)[X]
\]

is an integral, symmetric, unimodular, bilinear form.

Signature of \( X = \text{sign}(X) = \) Signature of intersection form
\( = b^+ - b^- \)

Type: Even if \( \alpha \cdot \alpha \) even for all \( \alpha \); otherwise Odd

(Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has \( X \times S^1 \) smoothable.

(Donaldson, 1982) Two simply connected smooth 4-manifolds are homeomorphic iff they have the same \( e, \) \( \text{sign}, \) and \( \text{type}. \)
Basic facts about 4-manifolds

Invariants

- Euler characteristic: \( e(X) = \sum_{i=0}^{4} (-1)^i r(k(H^i(M;\mathbb{Z}))) \)

- Intersection form: \( H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \to \mathbb{Z}; \)
  \[ \alpha \cdot \beta = (\alpha \cup \beta)[X] \]

is an integral, symmetric, unimodular, bilinear form.

Signature of \( X = \text{sign}(X) = \text{Signature of intersection form} \)
\[ = b^+ - b^- \]

Type: Even if \( \alpha \cdot \alpha \) even for all \( \alpha \); otherwise Odd

(Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has \( X \times S^1 \) smoothable.

(Donaldson, 1982) Two simply connected smooth 4-manifolds are homeomorphic iff they have the same \( e \), sign, and type.
Basic facts about 4-manifolds

Invariants

▶ Euler characteristic: \( e(X) = \sum_{i=0}^{4} (-1)^i \text{rk}(H^i(M; \mathbb{Z})) \)

▶ Intersection form: \( H^2(X; \mathbb{Z}) \otimes H^2(X; \mathbb{Z}) \to \mathbb{Z}; \alpha \cdot \beta = (\alpha \cup \beta)[X] \)

is an integral, symmetric, unimodular, bilinear form.

Signature of \( X = \text{sign}(X) = \text{Signature of intersection form} \)

\( = b^+ - b^- \)

Type: Even if \( \alpha \cdot \alpha \) even for all \( \alpha \); otherwise Odd

▶ (Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has \( X \times S^1 \) smoothable.

▶ (Donaldson, 1982) Two simply connected smooth 4-manifolds are homeomorphic iff they have the same \( e, \text{sign}, \) and \( \text{type} \).
Basic facts about 4-manifolds

Invariants

- Euler characteristic: \( e(X) = \sum_{i=0}^{4} (-1)^i \text{rk}(H^i(M; \mathbb{Z})) \)
- Intersection form: \( H^2(X; \mathbb{Z}) \otimes H^2(X; \mathbb{Z}) \to \mathbb{Z} \)

\[ \alpha \cdot \beta = (\alpha \cup \beta)[X] \]

is an integral, symmetric, unimodular, bilinear form.

Signature of \( X = \text{sign}(X) = \text{Signature of intersection form} = b^+ - b^- \)

**Type:** Even if \( \alpha \cdot \alpha \) even for all \( \alpha \); otherwise Odd

- (Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has \( X \times S^1 \) smoothable.
- (Donaldson, 1982) Two simply connected smooth 4-manifolds are homeomorphic iff they have the same \( e, \text{sign}, \) and type.
What do we know about smooth 4-manifolds?

Much—but so very little

Wild Conjecture

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for \( n > 4 \), every \( n \)-manifold has only finitely many distinct smooth \( n \)-manifolds which are homeomorphic to it.

Main goal: Discuss invariants and techniques developed to study this conjecture

► Need more invariants: Donaldson, Seiberg-Witten Invariants

\[ SW : \{ \text{characteristic elements of } H_2(X; \mathbb{Z}) \} \rightarrow \mathbb{Z} \]

► \( SW(\beta) \neq 0 \) for only finitely many \( \beta \): called basic classes.

► For each surface \( \Sigma \subset X \) with \( g(\Sigma) > 0 \) and \( \Sigma \cdot \Sigma \geq 0 \)

\[ 2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta| \]

for every basic class \( \beta \). (adjunction inequality[Kronheimer-Mrowka])

Basic classes = smooth analogue of the canonical class of a complex surface

► \( SW(\kappa) = \pm 1, \kappa \) the first Chern class of a symplectic manifold [Taubes].
What do we know about smooth 4-manifolds?

Much—but so very little

Wild Conjecture

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every $n$-manifold has only finitely many distinct smooth $n$-manifolds which are homeomorphic to it.

Main goal: Discuss invariants and techniques developed to study this conjecture

Need more invariants: Donaldson, Seiberg-Witten Invariants

$\text{SW} : \{\text{characteristic elements of} \ H_2(X; \mathbb{Z})\} \rightarrow \mathbb{Z}$

$\text{SW}(\beta) \neq 0$ for only finitely many $\beta$: called basic classes.

For each surface $\Sigma \subset X$ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma \geq 0$

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|$$

for every basic class $\beta$. (adjunction inequality[Kronheimer-Mrowka])

Basic classes = smooth analogue of the canonical class of a complex surface

$\text{SW}(\kappa) = \pm 1$, $\kappa$ the first Chern class of a symplectic manifold [Taubes].
What do we know about smooth 4-manifolds?
Much—but so very little

Wild Conjecture

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for \( n > 4 \), every \( n \)-manifold has only finitely many distinct smooth \( n \)-manifolds which are homeomorphic to it.

Main goal: Discuss invariants and techniques developed to study this conjecture

- Need more invariants: Donaldson, Seiberg-Witten Invariants
  \( SW: \{ \text{characteristic elements of } H_2(X; \mathbb{Z}) \} \to \mathbb{Z} \)
  - \( SW(\beta) \neq 0 \) for only finitely many \( \beta \): called basic classes.
  - For each surface \( \Sigma \subset X \) with \( g(\Sigma) > 0 \) and \( \Sigma \cdot \Sigma \geq 0 \)
    \[
    2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|
    \]
    for every basic class \( \beta \). (adjunction inequality[Kronheimer-Mrowka])
  - Basic classes = smooth analogue of the canonical class of a complex surface
  - \( SW(\kappa) = \pm 1 \), \( \kappa \) the first Chern class of a symplectic manifold [Taubes].
What do we know about smooth 4-manifolds?

Much—but so very little

Wild Conjecture

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for \( n > 4 \), every \( n \)-manifold has only finitely many distinct smooth \( n \)-manifolds which are homeomorphic to it.

Main goal: Discuss invariants and techniques developed to study this conjecture

- Need more invariants: Donaldson, Seiberg-Witten Invariants
  \[ SW : \{ \text{characteristic elements of } H_2(X; \mathbb{Z}) \} \to \mathbb{Z} \]

  - \( SW(\beta) \neq 0 \) for only finitely many \( \beta \): called basic classes.
  - For each surface \( \Sigma \subset X \) with \( g(\Sigma) > 0 \) and \( \Sigma \cdot \Sigma \geq 0 \)
    \[ 2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta| \]
    for every basic class \( \beta \). (adjunction inequality\([\text{Kronheimer-Mrowka}]\))

  Basic classes = smooth analogue of the canonical class of a complex surface

- \( SW(\kappa) = \pm 1 \), \( \kappa \) the first Chern class of a symplectic manifold \([\text{Taubes}]\).
What do we know about smooth 4-manifolds?

Much—but so very little

Wild Conjecture

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for \( n > 4 \), every \( n \)-manifold has only finitely many distinct smooth \( n \)-manifolds which are homeomorphic to it.

Main goal: Discuss invariants and techniques developed to study this conjecture

- Need more invariants: Donaldson, Seiberg-Witten Invariants
  \( SW : \{ \text{characteristic elements of} \; H_2(X; \mathbb{Z}) \} \rightarrow \mathbb{Z} \)

  - \( SW(\beta) \neq 0 \) for only finitely many \( \beta \): called basic classes.
  - For each surface \( \Sigma \subset X \) with \( g(\Sigma) > 0 \) and \( \Sigma \cdot \Sigma \geq 0 \)

    \[
    2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|
    \]

    for every basic class \( \beta \). (adjunction inequality [Kronheimer-Mrowka])

    Basic classes = smooth analogue of the canonical class of a complex surface

  - \( SW(\kappa) = \pm 1 \), \( \kappa \) the first Chern class of a symplectic manifold [Taubes].
What do we know about smooth 4-manifolds?

Much—but so very little

Wild Conjecture

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every $n$-manifold has only finitely many distinct smooth $n$-manifolds which are homeomorphic to it.

Main goal: Discuss invariants and techniques developed to study this conjecture

▶ Need more invariants: Donaldson, Seiberg-Witten Invariants

$SW : \{\text{characteristic elements of } H_2(X; \mathbb{Z})\} \rightarrow \mathbb{Z}$

▶ $SW(\beta) \neq 0$ for only finitely many $\beta$: called basic classes.

▶ For each surface $\Sigma \subset X$ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma \geq 0$

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|$$

for every basic class $\beta$. (adjunction inequality [Kronheimer-Mrowka])

Basic classes = smooth analogue of the canonical class of a complex surface

▶ $SW(\kappa) = \pm 1$, $\kappa$ the first Chern class of a symplectic manifold [Taubes].
What do we know about smooth 4-manifolds?

Much—but so very little

Wild Conjecture

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every $n$-manifold has only finitely many distinct smooth $n$-manifolds which are homeomorphic to it.

Main goal: Discuss invariants and techniques developed to study this conjecture

- Need more invariants: Donaldson, Seiberg-Witten Invariants

  $SW : \{\text{characteristic elements of } H_2(X; \mathbb{Z})\} \rightarrow \mathbb{Z}$

- $SW(\beta) \neq 0$ for only finitely many $\beta$: called basic classes.
- For each surface $\Sigma \subset X$ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma \geq 0$

  $$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|$$

  for every basic class $\beta$. (adjunction inequality[Kronheimer-Mrowka])

  Basic classes = smooth analogue of the canonical class of a complex surface

- $SW(\kappa) = \pm 1$, $\kappa$ the first Chern class of a symplectic manifold [Taubes].
What do we know about smooth 4-manifolds?

Much—but so very little

Wild Conjecture

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for \( n > 4 \), every \( n \)-manifold has only finitely many distinct smooth \( n \)-manifolds which are homeomorphic to it.

Main goal: Discuss invariants and techniques developed to study this conjecture

▶ Need more invariants: Donaldson, Seiberg-Witten Invariants

\( SW : \{ \text{characteristic elements of} \ H_2(X; \mathbb{Z}) \} \to \mathbb{Z} \)

▶ \( SW(\beta) \neq 0 \) for only finitely many \( \beta \): called basic classes.

▶ For each surface \( \Sigma \subset X \) with \( g(\Sigma) > 0 \) and \( \Sigma \cdot \Sigma \geq 0 \)

\[
2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|
\]

for every basic class \( \beta \). (adjunction inequality[Kronheimer-Mrowka])

Basic classes = smooth analogue of the canonical class of a complex surface

▶ \( SW(\kappa) = \pm 1 \), \( \kappa \) the first Chern class of a symplectic manifold [Taubes].
Oriented minimal \((\pi_1 = 0)\) 4-manifolds with \(SW \neq 0\)

Geography
Oriented minimal ($\pi_1 = 0$) 4-manifolds with $c^2 \neq 0$

**Geography**

\[ c = 3\text{sign} + 2e \]

\[ \chi_h = \frac{\text{sign} + e}{4} \]
Oriented minimal ($\pi_1 = 0$) 4-manifolds with $SW \neq 0$ Geography

\[ c = 3\text{sign} + 2e \]
\[ \chi_h = \frac{\text{sign} + e}{4} \]
Oriented minimal ($\pi_1 = 0$) 4-manifolds with $SW \neq 0$

Geography

$c = 3\text{sign} + 2e$

$\chi_h = \frac{\text{sign} + e}{4}$

$surfaces\ of\ general\ type$

$2\chi_h - 6 \leq c \leq 9\chi_h$

$c = 2\chi_h - 6$

$c = 9\chi_h$
Oriented minimal \((\pi_1 = 0)\) 4-manifolds with \(SW \neq 0\)

**Geography**

\[ c = 3\text{sign} + 2e \]
\[ X_h = \frac{\text{sign} + e}{4} \]

\[ c = 9X_h \]

surfaces of general type

\[ 2X_h - 6 \leq c \leq 9X_h \]

\[ c = 2X_h - 6 \]

\[ CP^2 \]

Elliptic Surfaces \(E(n)\)

\[ CP^2 \#_k CP^2 \]

\[ 1 \leq k \leq 9 \]

\[ S^2 \times S^2 \]

\[ S^4 \]
Oriented minimal ($\pi_1 = 0$) 4-manifolds with $\mathcal{SW} \neq 0$

Geography

$c = 3\text{sign} + 2e$

$\chi_h = \frac{\text{sign} + e}{4}$

Elliptic Surfaces $E(n)$

$C = 9\chi_h$

$C = 8\chi_h$

$\text{sign} = 0$

$\text{sign} > 0$

$\text{sign} < 0$

surfaces of general type

$2\chi_h - 6 \leq c \leq 9\chi_h$

$C = 2\chi_h - 6$

$\mathbb{CP}^2$

$S^2 \times S^2$

All lattice points have $\infty$ smooth structures except possibly near $c = 9\chi_h$ and on $\chi_h = 1$.

For $n > 4$ TOP $n$-manifolds have finitely many smooth structures.

$\mathbb{CP}^2 \# k \mathbb{CP}^2$

$1 \leq k \leq 9$

$\mathbb{CP}^2 \# S^2 \times S^2$

$S^2 \times S^2$
Oriented minimal ($\pi_1 = 0$) 4-manifolds with $SW \neq 0$

**Geography**

\[ c = 3\text{sign} + 2e \]

\[ \chi_h = \frac{\text{sign} + e}{4} \]

\[ 2\chi_h - 6 \leq c \leq 9\chi_h \]

Elliptic Surfaces $E(n)$
Oriented minimal ($\pi_1 = 0$) 4-manifolds with $SW \neq 0$

**Geography**

\[ c = 3\text{sign} + 2e \]
\[ \chi_h = \frac{\text{sign} + e}{4} \]

- $c = 9\chi_h$
- $c = 8\chi_h$, sign $= 0$
- $c = 2\chi_h - 6$
- $c = \chi_h - 3$

Surfaces of general type:
\[ 2\chi_h - 6 \leq c \leq 9\chi_h \]

Symplectic with one $SW$ basic class:
\[ \chi_h - 3 \leq c \leq 2\chi_h - 6 \]

Symplectic with $(\chi_h - c - 2)$ $SW$ basic classes:
\[ 0 \leq c \leq (\chi_h - 3) \]

Elliptic Surfaces $E(n)$
Oriented minimal ($\pi_1 = 0$) 4-manifolds with $SW \neq 0$

**Geography**

\[ c = 3\text{sign} + 2e \]
\[ \chi_h = \frac{\text{sign} + e}{4} \]

\[ \sum \chi_h = 3 \text{sign} + 2 \]

\[ \chi_h = \frac{\text{sign} + e}{4} \]

- **Surfaces of general type**
  \[ 2\chi_h - 6 \leq c \leq 9\chi_h \]
- **Symplectic with one $SW$ basic class**
  \[ \chi_h - 3 \leq c \leq 2\chi_h - 6 \]
- **Symplectic with \((\chi_h - c - 2)\) $SW$ basic classes**
  \[ 0 \leq c \leq (\chi_h - 3) \]

- **Elliptic Surfaces $E(n)$**
  \[ c < 0 \]

All lattice points have $\infty$ smooth structures except possibly near $c = 9\chi_h$ and on $\chi_h = 1$. For $n > 4$ TOP $n$-manifolds have finitely many smooth structures.

\[ CP^2 \]
\[ S^2 \times S^2 \]
\[ CP^2 \# k \overline{CP^2} \]
\[ 1 \leq k \leq 9 \]
Oriented minimal ($\pi_1 = 0$) 4-manifolds with $SW \neq 0$

Geography

- $c = 3\text{sign} + 2e$
- $\chi_h = \frac{\text{sign} + e}{4}$

All lattice points have $\infty$ smooth structures except possibly near $c = 9\chi_h$ and on $\chi_h = 1$.

$c > 9\chi_h$ ??

$\chi_h = 3\text{sign} + 2e$

$2\chi_h - 6 \leq c \leq 9\chi_h$

Elliptic Surfaces $E(n)$

$1 \leq k \leq 9$

$c = \chi_h - 3$

symplectic with one $SW$ basic class

$\chi_h - 3 \leq c \leq 2\chi_h - 6$

$c = 2\chi_h - 6$

symplectic with $(\chi_h - c - 2)$ $SW$ basic classes

$0 \leq c \leq (\chi_h - 3)$
Oriented minimal \((\pi_1 = 0)\) 4-manifolds with \(\mathcal{SW} \neq 0\)

**Geography**

\[
\begin{align*}
  c &= 3\text{sign} + 2e \\
  \chi_h &= \frac{\text{sign} + e}{4}
\end{align*}
\]

All lattice points have \(\infty\) smooth structures except possibly near \(c = 9\chi_h\) and on \(\chi_h = 1\)

For \(n > 4\) TOP \(n\)-manifolds have finitely many smooth structures

\[
\begin{align*}
  c &= 9\chi_h \\
  c &= 8\chi_h \quad \text{sign} = 0 \\
  c &= 2\chi_h - 6 \\
  c &= \chi_h - 3 \\
  2\chi_h - 6 &\leq c \leq 9\chi_h \\
  \chi_h - 3 &\leq c \leq 2\chi_h - 6 \\
  0 &\leq c \leq (\chi_h - 3)
\end{align*}
\]
Oriented minimal \((\pi_1 = 0)\) 4-manifolds with \(SW \neq 0\)

**Geography**

\[
c = 3\text{sign} + 2e
\]

\[
\chi_h = \frac{\text{sign} + e}{4}
\]

All lattice points have \(\infty\) smooth structures except possibly near \(c = 9\chi_h\) and on \(\chi_h = 1\)

For \(n > 4\) TOP \(n\)-manifolds have finitely many smooth structures

**Surfaces of general type**

\[
2\chi_h - 6 \leq c \leq 9\chi_h
\]

**Symplectic with one SW basic class**

\[
\chi_h - 3 \leq c \leq 2\chi_h - 6
\]

**Symplectic with \((\chi_h - c - 2)\) SW basic classes**

\[
0 \leq c \leq (\chi_h - 3)
\]
Every 4-manifold has zero or infinitely many distinct smooth structures

- One way to try to prove this conjecture is to find a “dial” to change the smooth structure at will.
- Since Rob’s last party — An effective dial: Surgery on null-homologous tori

\( T \): any self-intersection 0 torus \( \subset X \), Tubular nbd \( N_T \cong T^2 \times D^2 \).

**Surgery on** \( T \): \( X \setminus N_T \cup \varphi T^2 \times D^2 \), \( \varphi : \partial(T^2 \times D^2) \rightarrow \partial(X \setminus N_T) \)

\( \varphi(pt \times \partial D^2) = \) surgery curve

Result determined by \( \varphi_{\ast}[pt \times \partial D^2] \in H_1(\partial(X \setminus N_T)) = \mathbb{Z}^3 \)

Choose basis \( \{ \alpha, \beta, [\partial D^2] \} \) for \( H_1(\partial N_T) \) where \( \{ \alpha, \beta \} \) are pushoffs of a basis for \( H_1(T) \).

\[ \varphi_{\ast}[pt \times \partial D^2] = p\alpha + q\beta + r[\partial D^2] \]

Write \( X \setminus N_T \cup \varphi T^2 \times D^2 = X_T(p, q, r) \)

This operation does not change \( e(X) \) or \( \sigma(X) \)

**Note:** \( X_T(0, 0, 1) = X \)

Need formula for the Seiberg-Witten invariant of \( X_T(p, q, r) \) to determine when the smooth structure changes:

Due to Morgan, Mrowka, and Szabó (1996).
Every 4-manifold has zero or infinitely many distinct smooth structures

▶ One way to try to prove this conjecture is to find a “dial” to change the smooth structure at will.

▶ Since Rob’s last party — An effective dial: Surgery on null-homologous tori

\[ T: \text{any self-intersection 0 torus } \subset X, \text{ Tubular nbd } N_T \cong T^2 \times D^2. \]

**Surgery on** \( T: \ X \setminus N_T \cup_\varphi T^2 \times D^2, \ \varphi : \partial(T^2 \times D^2) \to \partial(X \setminus N_T) \)

\[ \varphi(pt \times \partial D^2) = \text{ surgery curve} \]

Result determined by \( \varphi_*[pt \times \partial D^2] \in H_1(\partial(X \setminus N_T)) = \mathbb{Z}^3 \)

Choose basis \( \{\alpha, \beta, [\partial D^2]\} \) for \( H_1(\partial N_T) \) where \( \{\alpha, \beta\} \) are pushoffs of a basis for \( H_1(T) \).

\[ \varphi_*[pt \times \partial D^2] = p\alpha + q\beta + r[\partial D^2] \]

Write \( X \setminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r) \)

This operation does not change \( e(X) \) or \( \sigma(X) \)

Note: \( X_T(0, 0, 1) = X \)

Need formula for the Seiberg-Witten invariant of \( X_T(p, q, r) \) to determine when the smooth structure changes:

Due to Morgan, Mrowka, and Szabó (1996).
Every 4-manifold has zero or infinitely many distinct smooth structures

- One way to try to prove this conjecture is to find a “dial” to change the smooth structure at will.
- Since Rob’s last party — An effective dial: Surgery on null-homologous tori

\( T \): any self-intersection 0 torus \( \subset X \), Tubular nbd \( N_T \cong T^2 \times D^2 \).

Surgery on \( T \):
\[
X \setminus N_T \cup \varphi T^2 \times D^2, \quad \varphi : \partial (T^2 \times D^2) \to \partial (X \setminus N_T)
\]
\[
\varphi(pt \times \partial D^2) = \text{surgery curve}
\]
Result determined by \( \varphi_*[pt \times \partial D^2] \in H_1(\partial (X \setminus N_T)) = \mathbb{Z}^3 \)

Choose basis \( \{\alpha, \beta, [\partial D^2]\} \) for \( H_1(\partial N_T) \) where \( \{\alpha, \beta\} \) are pushoffs of a basis for \( H_1(T) \).

\[
\varphi_*[pt \times \partial D^2] = p\alpha + q\beta + r[\partial D^2]
\]

Write \( X \setminus N_T \cup \varphi T^2 \times D^2 = X_T(p, q, r) \)

This operation does not change \( e(X) \) or \( \sigma(X) \)

Note: \( X_T(0, 0, 1) = X \)

Need formula for the Seiberg-Witten invariant of \( X_T(p, q, r) \) to determine when the smooth structure changes:

Due to Morgan, Mrowka, and Szabó (1996).
Every 4-manifold has zero or infinitely many distinct smooth structures

- One way to try to prove this conjecture is to find a “dial” to change the smooth structure at will.
- Since Rob’s last party — An effective dial: Surgery on null-homologous tori

**T:** any self-intersection 0 torus ⊂ X, Tubular nbd $N_T \cong T^2 \times D^2$.

**Surgery on T:** $X \setminus N_T \cup_\varphi T^2 \times D^2$, \( \varphi : \partial(T^2 \times D^2) \to \partial(X \setminus N_T) \)

\( \varphi(pt \times \partial D^2) = \text{ surgery curve} \)

Result determined by \( \varphi_*[pt \times \partial D^2] \in H_1(\partial(X \setminus N_T)) = \mathbb{Z}^3 \)

Choose basis \( \{\alpha, \beta, [\partial D^2]\} \) for \( H_1(\partial N_T) \) where \( \{\alpha, \beta\} \) are pushoffs of a basis for \( H_1(T) \).

\[ \varphi_*[pt \times \partial D^2] = p\alpha + q\beta + r[\partial D^2] \]

Write \( X \setminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r) \)

This operation does not change \( e(X) \) or \( \sigma(X) \)

Note: \( X_T(0, 0, 1) = X \)

Need formula for the Seiberg-Witten invariant of \( X_T(p, q, r) \) to determine when the smooth structure changes:

Due to Morgan, Mrowka, and Szabó (1996).
Every 4-manifold has zero or infinitely many distinct smooth structures

- One way to try to prove this conjecture is to find a “dial” to change the smooth structure at will.
- Since Rob’s last party — An effective dial: Surgery on null-homologous tori

**T**: any self-intersection 0 torus \( \subset X \), Tubular nbd \( N_T \cong T^2 \times D^2 \).

**Surgery on ** \( T \): \( X \setminus N_T \cup_\varphi T^2 \times D^2 \), \( \varphi : \partial(T^2 \times D^2) \to \partial(X \setminus N_T) \)

\( \varphi(pt \times \partial D^2) = \) surgery curve

Result determined by \( \varphi_*[pt \times \partial D^2] \in H_1(\partial(X \setminus N_T)) = \mathbb{Z}^3 \)

Choose basis \( \{\alpha, \beta, [\partial D^2]\} \) for \( H_1(\partial N_T) \) where \( \{\alpha, \beta\} \) are pushoffs of a basis for \( H_1(T) \).

\[ \varphi_*[pt \times \partial D^2] = p\alpha + q\beta + r[\partial D^2] \]

Write \( X \setminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r) \)

This operation does not change \( e(X) \) or \( \sigma(X) \)

Note: \( X_T(0,0,1) = X \)

Need formula for the Seiberg-Witten invariant of \( X_T(p, q, r) \) to determine when the smooth structure changes:

Due to Morgan, Mrowka, and Szabó (1996).
Every 4-manifold has zero or infinitely many distinct smooth structures

► One way to try to prove this conjecture is to find a “dial” to change the smooth structure at will.
► Since Rob’s last party — An effective dial: Surgery on null-homologous tori

\( T \): any self-intersection 0 torus \( \subset X \), Tubular nbd \( N_T \cong T^2 \times D^2 \).

**Surgery on** \( T \): \( X \setminus N_T \cup_\varphi T^2 \times D^2 \), \( \varphi : \partial (T^2 \times D^2) \to \partial (X \setminus N_T) \)

\( \varphi (pt \times \partial D^2) = \) surgery curve

Result determined by \( \varphi_* [pt \times \partial D^2] \in H_1 (\partial (X \setminus N_T)) = \mathbb{Z}^3 \)

Choose basis \{\( \alpha, \beta, [\partial D^2] \)\} for \( H_1 (\partial N_T) \) where \{\( \alpha, \beta \)\} are pushoffs of a basis for \( H_1 (T) \).

\( \varphi_* [pt \times \partial D^2] = p\alpha + q\beta + r[\partial D^2] \)

Write \( X \setminus N_T \cup_\varphi T^2 \times D^2 = X_T (p, q, r) \)

This operation does not change \( e(X) \) or \( \sigma (X) \)

**Note:** \( X_T (0, 0, 1) = X \)

Need formula for the Seiberg-Witten invariant of \( X_T (p, q, r) \) to determine when the smooth structure changes:

Due to Morgan, Mrowka, and Szabó (1996).
Every 4-manifold has zero or infinitely many distinct smooth structures

➤ One way to try to prove this conjecture is to find a “dial” to change the smooth structure at will.

➤ Since Rob’s last party —— An effective dial: Surgery on null-homologous tori

\( T \): any self-intersection 0 torus \( \subset X \), Tubular nbd \( N_T \cong T^2 \times D^2 \).

Surgery on \( T \): \( X \setminus N_T \cup_{\varphi} T^2 \times D^2 \), \( \varphi : \partial (T^2 \times D^2) \to \partial (X \setminus N_T) \)

\( \varphi (pt \times \partial D^2) \) = surgery curve

Result determined by \( \varphi_* [pt \times \partial D^2] \in H_1 (\partial (X \setminus N_T)) = \mathbb{Z}^3 \)

Choose basis \( \{ \alpha, \beta, [\partial D^2] \} \) for \( H_1 (\partial N_T) \) where \( \{ \alpha, \beta \} \) are pushoffs of a basis for \( H_1 (T) \).

\( \varphi_* [pt \times \partial D^2] = p\alpha + q\beta + r[\partial D^2] \)

Write \( X \setminus N_T \cup_{\varphi} T^2 \times D^2 = X_T (p, q, r) \)

This operation does not change \( e(X) \) or \( \sigma (X) \)

Note: \( X_T (0, 0, 1) = X \)

Need formula for the Seiberg-Witten invariant of \( X_T (p, q, r) \) to determine when the smooth structure changes:

Due to Morgan, Mrowka, and Szabó (1996).
The Morgan, Mrowka, Szabó Formula

\[
\sum_i \text{SW}_{X_T(p,q,r)}(k + 2i[T_{(p,q,r)}]) = p \sum_i \text{SW}_{X_T(1,0,0)}(k' + 2i[T_{(1,0,0)}])
\]

\[
+ q \sum_i \text{SW}_{X_T(0,1,0)}(k'' + 2i[T_{(0,1,0)}]) + r \sum_i \text{SW}_X(k''' + 2i[T])
\]

\(k\) characteristic element of \(H_2(X_T(p,q,r))\)

\[
H_2(X_T(p, q, r)) \rightarrow H_2(X_T(p, q, r), N_{T(p,q,r)})
\]

\[
\downarrow \cong
\]

\[
H_2(X \setminus N_T, \partial)
\]

\[
\uparrow \cong
\]

\[
H_2(X_T(1, 0, 0)) \rightarrow H_2(X_T(1, 0, 0), N_{T(1,0,0)})
\]

\[
k \rightarrow \bar{k}
\]

\[
\downarrow
\]

\[
\hat{k} = \hat{k}'
\]

\[
k' \rightarrow \bar{k}'
\]

- All basic classes of \(X_T(p,q,r)\) arise in this way.
- Useful to determine situations when sums collapse to single summand.
The Morgan, Mrowka, Szabó Formula

\[ \sum_i \text{SW}_{X_T(p,q,r)}(k + 2i[T_{(p,q,r)}]) = p \sum_i \text{SW}_{X_T(1,0,0)}(k' + 2i[T_{(1,0,0)}]) \]

\[ + q \sum_i \text{SW}_{X_T(0,1,0)}(k'' + 2i[T_{(0,1,0)}]) + r \sum_i \text{SW}_{X}(k''' + 2i[T]) \]

\[ k \text{ characteristic element of } H_2(X_T(p,q,r)) \]

\[ H_2(X_T(p, q, r)) \rightarrow H_2(X_T(p, q, r), N_{T_{(p,q,r)}}) \]

\[ k \rightarrow \bar{k} \]

\[ \Downarrow \cong \]

\[ H_2(X \setminus N_T, \partial) \]

\[ \hat{k} = \hat{k}' \]

\[ \Uparrow \cong \]

\[ H_2(X_T(1, 0, 0)) \rightarrow H_2(X_T(1, 0, 0), N_{T_{(1,0,0)}}) \]

\[ k' \rightarrow \bar{k}' \]

- All basic classes of \( X_T(p, q, r) \) arise in this way.
- Useful to determine situations when sums collapse to single summand.
The Morgan, Mrowka, Szabó Formula

\[
\sum_i \text{SW}_{X_T(p,q,r)}(k + 2i[T_{(p,q,r)}]) = p \sum_i \text{SW}_{X_T(1,0,0)}(k' + 2i[T_{(1,0,0)}]) + q \sum_i \text{SW}_{X_T(0,1,0)}(k'' + 2i[T_{(0,1,0)}]) + r \sum_i \text{SW}_{X}(k''' + 2i[T])
\]

\(k\) characteristic element of \(H_2(X_T(p,q,r))\)

\[
\begin{align*}
H_2(X_T(p, q, r)) & \to H_2(X_T(p, q, r), N_{T(p,q,r)}) & k & \to \bar{k} \\
\downarrow \cong & & \downarrow & & \hat{k} = \hat{k}' \\
 & & & \uparrow \cong & & \\
H_2(X \setminus N_T, \partial) & & & \uparrow & & \\
H_2(X_T(1, 0, 0)) & \to H_2(X_T(1, 0, 0), N_{T(1,0,0)}) & k' & \to \bar{k}'
\end{align*}
\]

- All basic classes of \(X_T(p, q, r)\) arise in this way.
- Useful to determine situations when sums collapse to single summand.
The Morgan, Mrowka, Szabó Formula

$$\sum_i \text{SW}_{X_T(p,q,r)}(k + 2i[T_{(p,q,r)}]) = p \sum_i \text{SW}_{X_T(1,0,0)}(k' + 2i[T_{(1,0,0)}])$$

$$+ q \sum_i \text{SW}_{X_T(0,1,0)}(k'' + 2i[T_{(0,1,0)}]) + r \sum_i \text{SW}_{X}(k''' + 2i[T])$$

$k$ characteristic element of $H_2(X_T(p,q,r))$

$$H_2(X_T(p, q, r)) \rightarrow H_2(X_T(p, q, r), N_{T_{(p,q,r)}}) \quad k \rightarrow \bar{k}$$

$$\downarrow \cong$$

$$H_2(X \setminus N_T, \partial)$$

$$\hat{k} = \hat{k}'$$

$$\uparrow \cong$$

$$H_2(X_T(1, 0, 0)) \rightarrow H_2(X_T(1, 0, 0), N_{T_{(1,0,0)}}) \quad k' \rightarrow \bar{k}'$$

- All basic classes of $X_T(p, q, r)$ arise in this way.
- Useful to determine situations when sums collapse to single summand.
Surgery on Tori

Reducing to one summand

\[ \text{SW}_X T(p,q,r) = p \text{SW}_X T(1,0,0) + q \text{SW}_X T(0,1,0) + r \text{SW}_X \]

- When torus \( T \) is nullhomologous, and
- when a core torus is essential, there is a torus that intersects it algebraically nontrivially.

Some observations about null-homologous tori:

- With null-homologous framing: \( H_1(X T(p,q,1)) = H_1(X) \),
  So for an effective dial want, say, \( \text{SW}_X T(1,0,0) \neq 0 \);
- \( b_1(X T(1,0,0)) = b_1(X T(0,1,0)) = b_1(X) + 1 \).

Dual situations for surgery on tori \( T \)

a. \( T \) primitive, \( \alpha \subset T \) essential in \( X \setminus T \).
   \[ \Rightarrow \quad T(p,q,r) \text{ nullhomologous in } X T(1,0,r) \].

b. \( T \) nullhomologous, \( \alpha \) bounds in \( X \setminus N_T \)
   \[ \Rightarrow \quad (1,0,0) \text{ surgery on } T \text{ gives (a)}. \]
Surgery on Tori

Reducing to one summand

\[ \text{SW}_{X_T(p,q,r)} = p\text{SW}_{X_T(1,0,0)} + q\text{SW}_{X_T(0,1,0)} + r\text{SW}_X \]

- When torus \( T \) is nullhomologous, and
- when a core torus is essential, there is a torus that intersects it algebraically nontrivially.

Some observations about null-homologous tori:

- With null-homologous framing: \( H_1(X_T(p,q,1)) = H_1(X) \),
  So for an effective dial want, say, \( \text{SW}_{X_T(1,0,0)} \neq 0 \);
- \( b_1(X_T(1,0,0)) = b_1(X_T(0,1,0)) = b_1(X) + 1 \).

Dual situations for surgery on tori \( T \)

a. \( T \) primitive, \( \alpha \subset T \) essential in \( X \setminus T \).
  \[ \Rightarrow \quad T_{(1,0,r)} \text{ nullhomologous in } X_T(1,0,r). \]

b. \( T \) nullhomologous, \( \alpha \) bounds in \( X \setminus N_T \)
  \[ \Rightarrow \quad (1,0,0) \text{ surgery on } T \text{ gives (a)}. \]
Surgery on Tori
Reducing to one summand

$$SW_{X_{T(p,q,r)}} = pSW_{X_{T(1,0,0)}} + qSW_{X_{T(0,1,0)}} + rSW_X$$

- When torus $T$ is nullhomologous, and
- when a core torus is essential, there is a torus that intersects it algebraically nontrivially.

Some observations about null-homologous tori:

- With null-homologous framing: $H_1(X_{T(p,q,1)}) = H_1(X)$,
  So for an effective dial want, say, $SW_{X_{T(1,0,0)}} \neq 0$;
- $b_1(X_{T(1,0,0)}) = b_1(X_{T(0,1,0)}) = b_1(X) + 1$.

Dual situations for surgery on tori $T$

a. $T$ primitive, $\alpha \subset T$ essential in $X \setminus T$.
   $\Rightarrow \quad T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.
b. $T$ nullhomologous, $\alpha$ bounds in $X \setminus N_T$
   $\Rightarrow \quad (1,0,0)$ surgery on $T$ gives (a).
Old Application: Knot Surgery

$K$: Knot in $S^3$, $T$: square 0 essential torus in $X$

$X_K = X \setminus N_T \cup S^1 \times (S^3 \setminus N_K)$

Note: $S^1 \times (S^3 \setminus N_K)$ has the homology of $T^2 \times D^2$.

Facts about knot surgery

- If $X$ and $X \setminus T$ both simply connected; so is $X_K$ (So $X_K$ homeo to $X$)
- If $K$ is fibered and $X$ and $T$ both symplectic; so is $X_K$.
- $SW_{X_K} = SW_X \cdot \Delta_K(t^2)$

Conclusions

- If $X$, $X \setminus T$, simply connected and $SW_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to $X$.
- If in addition $X$, $T$ symplectic, $K$ fibered, then there is an infinite family of distinct symplectic manifolds all homeomorphic to $X$.

e.g. $X = K3$, $SW_X = 1$, $SW_{X_K} = \Delta_K(t^2)$
Old Application: Knot Surgery

\( K: \) Knot in \( S^3 \), \( T: \) square 0 essential torus in \( X \)

\[
X_K = X \setminus N_T \cup S^1 \times (S^3 \setminus N_K)
\]

Note: \( S^1 \times (S^3 \setminus N_K) \) has the homology of \( T^2 \times D^2 \).

Facts about knot surgery

\[ \text{If } X \text{ and } X \setminus T \text{ both simply connected; so is } X_K \] 

(So \( X_K \) homeo to \( X \))

\[ \text{If } K \text{ is fibered and } X \text{ and } T \text{ both symplectic; so is } X_K. \]

\[ SW_{X_K} = SW_X \cdot \Delta_K(t^2) \]

Conclusions

\[ \text{If } X, \ X \setminus T, \text{ simply connected and } SW_X \neq 0, \text{ then there is an} \] 

infinite family of distinct manifolds all homeomorphic to \( X \).

\[ \text{If in addition } X, \ T \text{ symplectic, } K \text{ fibered, then there is an} \]

infinite family of distinct symplectic manifolds all homeomorphic to \( X \).

e.g. \( X = K3, \ SW_X = 1, \ SW_{X_K} = \Delta_K(t^2) \)
Old Application: Knot Surgery

$K$: Knot in $S^3$, $T$: square 0 essential torus in $X$

$X_K = X \setminus N_T \cup S^1 \times (S^3 \setminus N_K)$

Note: $S^1 \times (S^3 \setminus N_K)$ has the homology of $T^2 \times D^2$.

Facts about knot surgery

- If $X$ and $X \setminus T$ both simply connected; so is $X_K$.
  (So $X_K$ homeo to $X$)
- If $K$ is fibered and $X$ and $T$ both symplectic; so is $X_K$.
- $SW_{X_K} = SW_X \cdot \Delta_K(t^2)$

Conclusions

- If $X$, $X \setminus T$, simply connected and $SW_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to $X$.
- If in addition $X$, $T$ symplectic, $K$ fibered, then there is an infinite family of distinct symplectic manifolds all homeomorphic to $X$.

e.g. $X = K3$, $SW_X = 1$, $SW_{X_K} = \Delta_K(t^2)$
Old Application: Knot Surgery

\( K \): Knot in \( S^3 \), \( T \): square 0 essential torus in \( X \)

\[ X_K = X \setminus N_T \cup S^1 \times (S^3 \setminus N_K) \]

Note: \( S^1 \times (S^3 \setminus N_K) \) has the homology of \( T^2 \times D^2 \).

Facts about knot surgery

\( \blacktriangleright \) If \( X \) and \( X \setminus T \) both simply connected; so is \( X_K \)

(\( \text{So } X_K \text{ homeo to } X \))

\( \blacktriangleright \) If \( K \) is fibered and \( X \) and \( T \) both symplectic; so is \( X_K \).

\( \blacktriangleright \) \( SW_{X_K} = SW_X \cdot \Delta_K(t^2) \)

Conclusions

\( \blacktriangleright \) If \( X \), \( X \setminus T \), simply connected and \( SW_X \neq 0 \), then there is an infinite family of distinct manifolds all homeomorphic to \( X \).

\( \blacktriangleright \) If in addition \( X \), \( T \) symplectic, \( K \) fibered, then there is an infinite family of distinct symplectic manifolds all homeomorphic to \( X \).

e.g. \( X = K3, \quad SW_X = 1, \quad SW_{X_K} = \Delta_K(t^2) \)
Knot surgery and nullhomologous tori

Knot surgery on torus $T$ in 4-manifold $X$ with knot $K$:

$$X_K = X \#_{T = S^1 \times m} S^1 \times \lambda = \text{nullhomologous torus}$$

Used to change crossings:

Now apply Morgan-Mrowka-Szabó formula + tricks

- Weakness of construction: Requires $T$ to be homologically essential
  Open conjecture: If $\chi(X) > 1$, $SW_X \neq 0$, then $X$ contains a homologically essential torus $T$ with trivial normal bundle.

- If $X$ homeomorphic to $\mathbb{CP}^2$ blown up at 8 or fewer points, then $X$ contains no such torus - so what can we do for these small manifolds?
Knot surgery and nullhomologous tori

Knot surgery on torus $T$ in 4-manifold $X$ with knot $K$:

$$X_K = X \# T = S^1 \times_m S^1$$

$\Lambda = S^1 \times \lambda = \text{nullhomologous torus}$ — Used to change crossings:

Now apply Morgan-Mrowka-Szabó formula + tricks

- Weakness of construction: Requires $T$ to be homologically essential
  Open conjecture: If $\chi(X) > 1$, $SW_X \neq 0$, then $X$ contains a homologically essential torus $T$ with trivial normal bundle.

- If $X$ homeomorphic to $\mathbb{C}P^2$ blown up at 8 or fewer points, then $X$ contains no such torus - so what can we do for these small manifolds?
Knot surgery and nullhomologous tori

Knot surgery on torus $T$ in 4-manifold $X$ with knot $K$:

$$X_K = X \# T = S^1 \times m \quad S^1 x \quad m$$

$\Lambda = S^1 \times \lambda = \text{nullhomologous torus} \quad \text{— Used to change crossings:} \quad \text{Now apply Morgan-Mrowka-Szabó formula + tricks}$

- Weakness of construction: Requires $T$ to be homologically essential
  Open conjecture: If $\chi(X) > 1$, $\mathcal{SW}_X \neq 0$, then $X$ contains a homologically essential torus $T$ with trivial normal bundle.

- If $X$ homeomorphic to $\mathbb{C}P^2$ blown up at 8 or fewer points, then $X$ contains no such torus - so what can we do for these small manifolds?
Knot surgery and nullhomologous tori

Knot surgery on torus $T$ in 4-manifold $X$ with knot $K$:

$$X_K = X \#_{T=S^1 \times m} S^1 \times \lambda$$

$\Lambda = S^1 \times \lambda =$ nullhomologous torus — Used to change crossings:

Now apply Morgan-Mrowka-Szabó formula + tricks

- **Weakness of construction:** Requires $T$ to be homologically essential
  
  Open conjecture: If $\chi(X) > 1$, $SW_X \neq 0$, then $X$ contains a homologically essential torus $T$ with trivial normal bundle.

- If $X$ homeomorphic to $\mathbb{CP}^2$ blown up at 8 or fewer points, then $X$ contains no such torus - so what can we do for these small manifolds?
Knot surgery and nullhomologous tori

Knot surgery on torus $T$ in 4-manifold $X$ with knot $K$:

$$X_K = X \#_{T = S^1 \times m} \frac{S^1}{S^1 \times \lambda} = \text{nullhomologous torus}$$

- Used to change crossings:

- Now apply Morgan-Mrowka-Szabó formula + tricks

- **Weakness of construction:** Requires $T$ to be homologically essential
  
  Open conjecture: If $\chi(X) > 1$, $SW_X \neq 0$, then $X$ contains a homologically essential torus $T$ with trivial normal bundle.

- If $X$ homeomorphic to $\mathbb{CP}^2$ blown up at 8 or fewer points, then $X$ contains no such torus - so what can we do for these small manifolds?
Knot surgery and nullhomologous tori

Knot surgery on torus $T$ in 4-manifold $X$ with knot $K$:

$$X_K = X \#_{T=S^1 \times \lambda} S^1 \times m$$

$\Lambda = S^1 \times \lambda = \text{nullhomologous torus} — \text{Used to change crossings}$:

Now apply Morgan-Mrowka-Szabó formula + tricks

- **Weakness of construction:** Requires $T$ to be homologically essential
  Open conjecture: If $\chi(X) > 1$, $SW_X \neq 0$, then $X$ contains a homologically essential torus $T$ with trivial normal bundle.

- If $X$ homeomorphic to $\mathbb{CP}^2$ blown up at 8 or fewer points, then $X$ contains no such torus - so what can we do for these small manifolds?
Oriented minimal $\pi_1 = 0$ 4-manifolds with $SW \neq 0$.

**Geography**

$$c = 3\text{sign} + 2e$$

$$\chi_h = \frac{\text{sign} + e}{4}$$

All lattice points have $\infty$ smooth structures except possibly near $c = 9\chi_h$ and on $\chi_h = 1$.

For $n > 4$ TOP $n$-manifolds have finitely many smooth structures.

- $c = 2\chi_h - 6$
  - Symplectic with one $SW$ basic class
  - $\chi_h - 3 \leq c \leq 2\chi_h - 6$

- $c = \chi_h - 3$
  - Symplectic with $(\chi_h - c - 2)$ $SW$ basic classes
  - $0 \leq c \leq (\chi_h - 3)$

- $c = 9\chi_h$
  - Surfaces of general type
  - $2\chi_h - 6 \leq c \leq 9\chi_h$

- $c = 8\chi_h$
  - $\text{sign} = 0$

- $c > 9\chi_h$
  - $\text{sign} > 0$
  - $\text{sign} < 0$

- $\text{Elliptic Surfaces } E(n)$
  - $c < 0$
Oriented minimal $\pi_1 = 0$ 4-manifolds with $SW \neq 0$

Geography

$c = 3\text{sign} + 2e$

$\chi_h = \frac{\text{sign} + e}{4}$

All lattice points have $\infty$ smooth structures except possibly near $c = 9\chi_h$ and on $\chi_h = 1$

For $n > 4$ TOP $n$-manifolds have finitely many smooth structures

Elliptic Surfaces $E(n)$

$c < 0$ ??
Oriented minimal \( \pi_1 = 0 \) 4-manifolds with \( SW \neq 0 \)

**Geography**

\[
c = 3 \text{sign} + 2e
\]

\[
\chi_h = \frac{\text{sign} + e}{4}
\]

All lattice points have \( \infty \) smooth structures except possibly near \( c = 9\chi_h \) and on \( \chi_h = 1 \)

For \( n > 4 \) TOP n-manifolds have finitely many smooth structures

\[
2\chi_h - 6 \leq c \leq 9\chi_h
\]

surfaces of general type

\[
c = 8\chi_h
\]

\( \text{sign} = 0 \)

\[
c = 9\chi_h
\]

\( \text{sign} < 0 \)

\[
c > 9\chi_h
\]

\( \text{sign} > 0 \)

symplectic with one \( SW \) basic class

\[
\chi_h - 3 \leq c \leq 2\chi_h - 6
\]

symplectic with \( (\chi_h - c - 2) \) \( SW \) basic classes

\[
0 \leq c \leq (\chi_h - 3)
\]

\( S^4 \times S^2 \), \( \mathbb{CP}^2 \# k \mathbb{CP}^2 \)

\( 0 \leq k \leq 8 \)

Elliptic Surfaces \( E(n) \)

\[
c < 0?
\]
Reverse Engineering

- Difficult to find useful nullhomologous tori like $\Lambda$ used in knot surgery.

- Recall: $SW_{X_T(p,q,r)} = pSW_{X_T(1,0,0)} + qSW_{X_T(0,1,0)} + rSW_X$

- With null-homologous framing: $H_1(X_T(p,q,1)) = H_1(X)$. So want, say, $SW_{X_T(1,0,0)} \neq 0$;

- $b_1(X_T(1,0,0)) = b_1(X_T(0,1,0)) = b_1(X) + 1$.

- Recall: Dual situations for surgery on tori $T$
  
  a. $T$ primitive, $\alpha \subset T$ essential in $X \setminus T$.
  
     $\quad \Rightarrow \quad T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.

  b. $T$ nullhomologous, $\alpha$ bounds in $X \setminus N_T$

     $\quad \Rightarrow \quad (1,0,0)$ surgery on $T$ gives (a).

IDEA: First construct $X_T(1,0,0)$ so that $SW_{X_T(1,0,0)} \neq 0$ and then surger to reduce $b_1$. 
Reverse Engineering

- Difficult to find useful nullhomologous tori like $\Lambda$ used in knot surgery.

- Recall: $SW_{X_{T(p,q,r)}} = pSW_{X_{T(1,0,0)}} + qSW_{X_{T(0,1,0)}} + rSW_X$

- With null-homologous framing: $H_1(X_{T(p,q,1)}) = H_1(X)$. So want, say, $SW_{X_{T(1,0,0)}} \neq 0$;

- $b_1(X_{T(1,0,0)}) = b_1(X_{T(0,1,0)}) = b_1(X) + 1$.

- Recall: Dual situations for surgery on tori $T$
  a. $T$ primitive, $\alpha \subset T$ essential in $X \setminus T$.
     $\Rightarrow$ $T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.
  b. $T$ nullhomologous, $\alpha$ bounds in $X \setminus N_T$
     $\Rightarrow$ $(1,0,0)$ surgery on $T$ gives (a).

IDEA: First construct $X_{T(1,0,0)}$ so that $SW_{X_{T(1,0,0)}} \neq 0$ and then surger to reduce $b_1$. 
Reverse Engineering

- Difficult to find useful nullhomologous tori like $\Lambda$ used in knot surgery.

- Recall: $\text{SW}_{X_T(p, q, r)} = p\text{SW}_{X_T(1, 0, 0)} + q\text{SW}_{X_T(0, 1, 0)} + r\text{SW}_{X}$

- With null-homologous framing: $H_1(X_{T(p, q, 1)}) = H_1(X)$. So want, say, $\text{SW}_{X_T(1, 0, 0)} \neq 0$;

- $b_1(X_T(1, 0, 0)) = b_1(X_T(0, 1, 0)) = b_1(X) + 1$.

- Recall: Dual situations for surgery on tori $T$
  a. $T$ primitive, $\alpha \subset T$ essential in $X \setminus T$.
     $\Rightarrow \ T_{(1, 0, r)}$ nullhomologous in $X_T(1, 0, r)$.
  b. $T$ nullhomologous, $\alpha$ bounds in $X \setminus N_T$
     $\Rightarrow \ (1, 0, 0)$ surgery on $T$ gives (a).

IDEA: First construct $X_{T(1, 0, 0)}$ so that $\text{SW}_{X_T(1, 0, 0)} \neq 0$ and then surger to reduce $b_1$. 
Reverse Engineering

- Difficult to find useful nullhomologous tori like $\Lambda$ used in knot surgery.

- Recall: $SW_{X_{T(p,q,r)}} = pSW_{X_{T(1,0,0)}} + qSW_{X_{T(0,1,0)}} + rSW_X$

- With null-homologous framing: $H_1(X_{T(p,q,1)}) = H_1(X)$. So want, say, $SW_{X_{T(1,0,0)}} \neq 0$;

- $b_1(X_{T(1,0,0)}) = b_1(X_{T(0,1,0)}) = b_1(X) + 1$.

- Recall: Dual situations for surgery on tori $T$
  a. $T$ primitive, $\alpha \subset T$ essential in $X \setminus T$.
     \[ \Rightarrow \quad T_{(1,0,r)} \text{ nullhomologous in } X_T(1,0,r). \]
  b. $T$ nullhomologous, $\alpha$ bounds in $X \setminus N_T$
     \[ \Rightarrow \quad (1,0,0) \text{ surgery on } T \text{ gives (a).} \]

IDEA: First construct $X_{T(1,0,0)}$ so that $SW_{X_{T(1,0,0)}} \neq 0$ and then surger to reduce $b_1$. 
Reverse Engineering

- Difficult to find useful nullhomologous tori like Λ used in knot surgery.

- Recall: $SW_{X_{T(p,q,r)}} = pSW_{X_{T(1,0,0)}} + qSW_{X_{T(0,1,0)}} + rSW_X$

- With null-homologous framing: $H_1(X_{T(p,q,1)}) = H_1(X)$. So for effective dial want, say, $SW_{X_{T(1,0,0)}} \neq 0$;

- $b_1(X_{T(1,0,0)}) = b_1(X_{T(0,1,0)}) = b_1(X) + 1$.

  IDEA: First construct $X_{T(1,0,0)}$ so that $SW_{X_{T(1,0,0)}} \neq 0$ and then surger to reduce $b_1$.

- Procedure to insure the existence of effective null-homologous tori

  1. Find model manifold $M$ with same Euler number and signature as desired manifold, but with $b_1 \neq 0$ and with $SW \neq 0$.

  2. Find $b_1$ disjoint essential tori in $M$ containing generators of $H_1$. Surger to get manifold $X$ with $H_1 = 0$. Want result of each surgery to have $SW \neq 0$ (except perhaps the very last).

  3. $X$ will contain a “useful” nullhomologous torus.
Reverse Engineering

- Difficult to find useful nullhomologous tori like $\Lambda$ used in knot surgery.

- Recall: $SW_{X_T(p,q,r)} = pSW_{X_T(1,0,0)} + qSW_{X_T(0,1,0)} + rSW_X$

- With null-homologous framing: $H_1(X_T(p,q,1)) = H_1(X)$. So for effective dial want, say, $SW_{X_T(1,0,0)} \neq 0$;

- $b_1(X_T(1,0,0)) = b_1(X_T(0,1,0)) = b_1(X) + 1$.

  IDEA: First construct $X_T(1,0,0)$ so that $SW_{X_T(1,0,0)} \neq 0$ and then surger to reduce $b_1$.

- Procedure to insure the existence of effective null-homologous tori

  1. Find model manifold $M$ with same Euler number and signature as desired manifold, but with $b_1 \neq 0$ and with $SW \neq 0$.

  2. Find $b_1$ disjoint essential tori in $M$ containing generators of $H_1$. Surger to get manifold $X$ with $H_1 = 0$. Want result of each surgery to have $SW \neq 0$ (except perhaps the very last).

  3. $X$ will contain a “useful” nullhomologous torus.
Luttinger Surgery

- For model manifolds with $H_1 \neq 0$: nature hands you symplectic manifolds.
- We seek tori that will kill $b_1$. Nature hands you Lagrangian tori.

$X$: symplectic manifold  
$T$: Lagrangian torus in $X$

Preferred framing for $T$: Lagrangian framing  
w.r.t. which all pushoffs of $T$ remain Lagrangian

$(1/n)$-surgeries w.r.t. this framing are again symplectic  
(Luttinger; Auroux, Donaldson, Katzarkov)

If $S_\beta^1 = \text{Lagrangian pushoff}, X_T(0, \pm 1, 0): \text{symplectic mfd}$

$\implies$ if $b^+ > 1$, $X_T(0, \pm 1, 0)$ has $SW \neq 0$

Then have infinitely many $H_1 = 0$ manifolds - keep fingers crossed $\pi_1 = 0$. 
Luttinger Surgery

- For model manifolds with $H_1 \neq 0$: nature hands you symplectic manifolds.
- We seek tori that will kill $b_1$. Nature hands you Lagrangian tori.

$X$: symplectic manifold  $T$: Lagrangian torus in $X$

Preferred framing for $T$: Lagrangian framing w.r.t. which all pushoffs of $T$ remain Lagrangian

$(1/n)$-surgeries w.r.t. this framing are again symplectic (Luttinger; Auroux, Donaldson, Katzarkov)

If $S^1_\beta$ = Lagrangian pushoff, $X_T(0, \pm 1, 0)$: symplectic mfd

$\implies$ if $b^+ > 1$, $X_T(0, \pm 1, 0)$ has $SW \neq 0$

Then have infinitely many $H_1 = 0$ manifolds - keep fingers crossed $\pi_1 = 0$. 
Luttinger Surgery

- For model manifolds with $H_1 \neq 0$: nature hands you symplectic manifolds.
- We seek tori that will kill $b_1$. Nature hands you Lagrangian tori.

$X$: symplectic manifold  
$T$: Lagrangian torus in $X$

Preferred framing for $T$: Lagrangian framing
w.r.t. which all pushoffs of $T$ remain Lagrangian

$(1/n)$-surgeries w.r.t. this framing are again symplectic
(Luttinger; Auroux, Donaldson, Katzarkov)

If $S^1_\beta = $ Lagrangian pushoff, $X_T(0, \pm 1, 0)$: symplectic mfd

$\implies$ if $b^+ > 1$, $X_T(0, \pm 1, 0)$ has $SW \neq 0$

Then have infinitely many $H_1 = 0$ manifolds - keep fingers crossed $\pi_1 = 0$. 
Luttinger Surgery

- For model manifolds with $H_1 \neq 0$: nature hands you symplectic manifolds.
- We seek tori that will kill $b_1$. Nature hands you Lagrangian tori.

$X$: symplectic manifold  $T$: Lagrangian torus in $X$

Preferred framing for $T$: Lagrangian framing
w.r.t. which all pushoffs of $T$ remain Lagrangian

$(1/n)$-surgeries w.r.t. this framing are again symplectic
(Luttinger; Auroux, Donaldson, Katzarkov)

If $S^1_\beta = \text{Lagrangian pushoff}, X_T(0, \pm 1, 0): \text{symplectic mfd}$

$\implies$ if $b^+ > 1, X_T(0, \pm 1, 0)$ has $SW \neq 0$

Then have infinitely many $H_1 = 0$ manifolds - keep fingers crossed $\pi_1 = 0$. 
Luttinger Surgery

- For model manifolds with $H_1 \neq 0$: nature hands you symplectic manifolds.
- We seek tori that will kill $b_1$. Nature hands you Lagrangian tori.

**X**: symplectic manifold  **T**: Lagrangian torus in $X$

Preferred framing for $T$: Lagrangian framing w.r.t. which all pushoffs of $T$ remain Lagrangian

$(1/n)$-surgeries w.r.t. this framing are again symplectic (Luttinger; Auroux, Donaldson, Katzarkov)

If $S^1_\beta = $ Lagrangian pushoff, $X_T(0, \pm 1, 0)$: symplectic mfd

\[ \implies \text{if } b^+ > 1, \text{ } X_T(0, \pm 1, 0) \text{ has } SW \neq 0 \]

Then have infinitely many $H_1 = 0$ manifolds - keep fingers crossed $\pi_1 = 0$. 
Luttinger Surgery

- For model manifolds with $H_1 \neq 0$: nature hands you symplectic manifolds.
- We seek tori that will kill $b_1$. Nature hands you Lagrangian tori.

$X$: symplectic manifold  \quad T$: Lagrangian torus in $X$

Preferred framing for $T$: Lagrangian framing
w.r.t. which all pushoffs of $T$ remain Lagrangian

$(1/n)$-surgeries w.r.t. this framing are again symplectic
(Luttinger; Auroux, Donaldson, Katzarkov)

If $S^1_{\beta} = \text{Lagrangian pushoff}, X_T(0, \pm 1, 0): \text{symplectic mfd}$

$\implies$ if $b^+ > 1$, $X_T(0, \pm 1, 0)$ has $SW \neq 0$

Then have infinitely many $H_1 = 0$ manifolds - keep fingers crossed $\pi_1 = 0$. 
Luttinger Surgery

- For model manifolds with $H_1 \neq 0$: nature hands you symplectic manifolds.
- We seek tori that will kill $b_1$. Nature hands you Lagrangian tori.

$X$: symplectic manifold  \hspace{1em} T: \text{Lagrangian torus in } X$

Preferred framing for $T$: Lagrangian framing
w.r.t. which all pushoffs of $T$ remain Lagrangian

$(1/n)$-surgeries w.r.t. this framing are again symplectic
(Luttinger; Auroux, Donaldson, Katzarkov)

If $S^1_\beta = \text{Lagrangian pushoff}, \ X_T(0, \pm 1, 0): \text{symplectic mfd}$

$\implies$ if $b^+ > 1, \ X_T(0, \pm 1, 0) \text{ has SW} \neq 0$

Then have infinitely many $H_1 = 0$ manifolds - keep fingers crossed $\pi_1 = 0.$
Reverse Engineering in Action
Infinite families of fake $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$

Need Model Manifolds for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$
i.e. symplectic manifolds $X_k$ with same $e$ and sign as $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, and $b_1 \geq 1$ disjoint lagrangian tori carrying basis for $H_1$.

- Surger lagrangian tori to decrease $b_1$.
- Resulting manifold has $H_1 = 0$ - but with a dial.
- Get infinite family of distinct manifolds all homology equivalent to $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$
- Keep fingers crossed that result has $\pi_1 = 0$, so all homeomorphic to $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$
Reverse Engineering in Action

Infinite families of fake $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$

Need Model Manifolds for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$

i.e. symplectic manifolds $X_k$ with same $e$ and sign as $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, and $b_1 \geq 1$ disjoint lagrangian tori carrying basis for $H_1$.

- Surger lagragian tori to decrease $b_1$.
- Resulting manifold has $H_1 = 0$ - but with a dial.
- Get infinite family of distinct manifolds all homology equivalent to $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$
- Keep fingers crossed that result has $\pi_1 = 0$, so all homeomorphic to $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$
Reverse Engineering in Action
Infinite families of fake $\mathbb{CP}^2 \# k \bar{\mathbb{CP}}^2$

Need Model Manifolds for $\mathbb{CP}^2 \# k \bar{\mathbb{CP}}^2$

i.e. symplectic manifolds $X_k$ with same $e$ and sign as $\mathbb{CP}^2 \# k \bar{\mathbb{CP}}^2$, and $b_1 \geq 1$ disjoint lagrangian tori carrying basis for $H_1$.

- Surger lagragian tori to decrease $b_1$.
- Resulting manifold has $H_1 = 0$ - but with a dial.
- Get infinite family of distinct manifolds all homology equivalent to $\mathbb{CP}^2 \# k \bar{\mathbb{CP}}^2$
- Keep fingers crossed that result has $\pi_1 = 0$, so all homeomorphic to $\mathbb{CP}^2 \# k \bar{\mathbb{CP}}^2$
Need Model Manifolds for $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$

i.e. symplectic manifolds $X_k$ with same $e$ and sign as $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$, and $b_1 \geq 1$ disjoint lagrangian tori carrying basis for $H_1$.

- Surger lagrangian tori to decrease $b_1$.
- Resulting manifold has $H_1 = 0$ - but with a dial.

- Get infinite family of distinct manifolds all homology equivalent to $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$
- Keep fingers crossed that result has $\pi_1 = 0$, so all homeomorphic to $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$
Reverse Engineering in Action

Infinite families of fake $\mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2$

Need Model Manifolds for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2$

i.e. symplectic manifolds $X_k$ with same $e$ and sign as $\mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2$, and $b_1 \geq 1$ disjoint lagrangian tori carrying basis for $H_1$.

- Surger lagragian tori to decrease $b_1$.
- Resulting manifold has $H_1 = 0$ - but with a dial.
- Get infinite family of distinct manifolds all homology equivalent to $\mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2$
- Keep fingers crossed that result has $\pi_1 = 0$, so all homeomorphic to $\mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2$
Need Model Manifolds for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$

i.e. symplectic manifolds $X_k$ with same $e$ and sign as $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, and $b_1 \geq 1$ disjoint lagrangian tori carrying basis for $H_1$.

- Surger lagrangian tori to decrease $b_1$.
- Resulting manifold has $H_1 = 0$ - but with a dial.

- Get infinite family of distinct manifolds all homology equivalent to $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$
- Keep fingers crossed that result has $\pi_1 = 0$, so all homeomorphic to $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$
Model Manifolds for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$

Basic Pieces: $X_0, X_1, X_2, X_3, X_4$

$X_r \# \Sigma_2 X_s$ is a model for $\mathbb{CP}^2 \# (r + s + 1) \overline{\mathbb{CP}^2}$

$X_0$: $\Sigma_2 \subset T^2 \times \Sigma_2$ representing $(0, 1)$
$X_1$: $\Sigma_2 \subset T^2 \times T^2 \# \overline{\mathbb{CP}^2}$ representing $(2, 1) - 2e$
$X_2$: $\Sigma_2 \subset T^2 \times T^2 \# 2 \overline{\mathbb{CP}^2}$ representing $(1, 1) - e_1 - e_2$
$X_3$: $\Sigma_2 \subset S^2 \times T^2 \# 3 \overline{\mathbb{CP}^2}$ representing $(1, 3) - 2e_1 - e_2 - e_3$
$X_4$: $\Sigma_2 \subset S^2 \times T^2 \# 4 \overline{\mathbb{CP}^2}$ representing $(1, 2) - e_1 - e_2 - e_3 - e_4$

Exception: $X_0 \# \Sigma_2 X_0 = \Sigma_2 \times \Sigma_2$ is a model for $S^2 \times S^2$

Enough lagrangian tori to kill $H_1$; The art is to find tori and show result has $\pi_1 = 0$

- First successful implementation of this strategy for $\mathbb{CP}^2 \# 3 \overline{\mathbb{CP}^2}$ (i.e. find tori, show surgery on model manifold results in $\pi_1 = 0$) obtained by Baldridge-Kirk; Akhmedov-Park
- Full implementation (i.e. infinite families) for $\mathbb{CP}^2 \# 3 \overline{\mathbb{CP}^2}$: Fintushel-Park-Stern using the 2-fold symmetric product $\text{Sym}^2(\Sigma_3)$ as model.
- Full implementation (i.e. infinite families) for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 4$ by Baldridge-Kirk, Akhmedov-Park, Fintushel-Park-Stern, Akhmedov-Baykur-Baldridge-Kirk-Park, Akhmedov-Baykur-Park.
Model Manifolds for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$

Basic Pieces: $X_0, X_1, X_2, X_3, X_4$

$X_r \# \Sigma_2 X_s$ is a model for $\mathbb{CP}^2 \# (r + s + 1) \overline{\mathbb{CP}^2}$

$X_0$: $\Sigma_2 \subset T^2 \times \Sigma_2$ representing $(0, 1)$

$X_1$: $\Sigma_2 \subset T^2 \times T^2 \# \overline{\mathbb{CP}^2}$ representing $(2, 1) - 2e$

$X_2$: $\Sigma_2 \subset T^2 \times T^2 \# 2\overline{\mathbb{CP}^2}$ representing $(1, 1) - e_1 - e_2$

$X_3$: $\Sigma_2 \subset S^2 \times T^2 \# 3\overline{\mathbb{CP}^2}$ representing $(1, 3) - 2e_1 - e_2 - e_3$

$X_4$: $\Sigma_2 \subset S^2 \times T^2 \# 4\overline{\mathbb{CP}^2}$ representing $(1, 2) - e_1 - e_2 - e_3 - e_4$

Exception: $X_0 \# \Sigma_2 X_0 = \Sigma_2 \times \Sigma_2$ is a model for $S^2 \times S^2$

Enough lagrangian tori to kill $H_1$; The art is to find tori and show result has $\pi_1 = 0$

- First successful implementation of this strategy for $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$ (i.e. find tori, show surgery on model manifold results in $\pi_1 = 0$) obtained by Baldridge-Kirk; Akhmedov-Park
- Full implementation (i.e. infinite families) for $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$: Fintushel-Park-Stern using the 2-fold symmetric product $\text{Sym}^2(\Sigma_3)$ as model.
- Full implementation (i.e. infinite families) for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 4$ by Baldridge-Kirk, Akhmedov-Park, Fintushel-Park-Stern, Akhmedov-Baykur-Baldridge-Kirk-Park, Akhmedov-Baykur-Park.
Model Manifolds for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$

Basic Pieces: $X_0, X_1, X_2, X_3, X_4$

$X_r \# \Sigma_2 X_s$ is a model for $\mathbb{CP}^2 \# (r + s + 1) \overline{\mathbb{CP}^2}$

$X_0$: $\Sigma_2 \subset T^2 \times \Sigma_2$ representing $(0, 1)$

$X_1$: $\Sigma_2 \subset T^2 \times T^2 \# \overline{\mathbb{CP}^2}$ representing $(2, 1) - 2e$

$X_2$: $\Sigma_2 \subset T^2 \times T^2 \# 2\overline{\mathbb{CP}^2}$ representing $(1, 1) - e_1 - e_2$

$X_3$: $\Sigma_2 \subset S^2 \times T^2 \# 3\overline{\mathbb{CP}^2}$ representing $(1, 3) - 2e_1 - e_2 - e_3$

$X_4$: $\Sigma_2 \subset S^2 \times T^2 \# 4\overline{\mathbb{CP}^2}$ representing $(1, 2) - e_1 - e_2 - e_3 - e_4$

Exception: $X_0 \# \Sigma_2 X_0 = \Sigma_2 \times \Sigma_2$ is a model for $S^2 \times S^2$

Enough lagrangian tori to kill $H_1$; The art is to find tori and show result has $\pi_1 = 0$

- First successful implementation of this strategy for $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$ (i.e. find tori, show surgery on model manifold results in $\pi_1 = 0$) obtained by Baldridge-Kirk; Akhmedov-Park
- Full implementation (i.e. infinite families) for $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$: Fintushel-Park-Stern using the 2-fold symmetric product $\text{Sym}^2(\Sigma_3)$ as model.
- Full implementation (i.e. infinite families) for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 4$ by Baldridge-Kirk, Akhmedov-Park, Fintushel-Park-Stern, Akhmedov-Baykur-Baldridge-Kirk-Park, Akhmedov-Baykur-Park.
Model Manifolds for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$

Basic Pieces: $X_0, X_1, X_2, X_3, X_4$

$X_r \# \Sigma_2 X_s$ is a model for $\mathbb{CP}^2 \# (r + s + 1) \overline{\mathbb{CP}^2}$

$X_0: \Sigma_2 \subset T^2 \times \Sigma_2$ representing $(0, 1)$

$X_1: \Sigma_2 \subset T^2 \times T^2 \# \overline{\mathbb{CP}^2}$ representing $(2, 1) - 2e$

$X_2: \Sigma_2 \subset T^2 \times T^2 \# 2\overline{\mathbb{CP}^2}$ representing $(1, 1) - e_1 - e_2$

$X_3: \Sigma_2 \subset S^2 \times T^2 \# 3\overline{\mathbb{CP}^2}$ representing $(1, 3) - 2e_1 - e_2 - e_3$

$X_4: \Sigma_2 \subset S^2 \times T^2 \# 4\overline{\mathbb{CP}^2}$ representing $(1, 2) - e_1 - e_2 - e_3 - e_4$

Exception: $X_0 \# \Sigma_2 X_0 = \Sigma_2 \times \Sigma_2$ is a model for $S^2 \times S^2$

Enough lagrangian tori to kill $H_1$; The art is to find tori and show result has $\pi_1 = 0$

- First successful implementation of this strategy for $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$ (i.e. find tori, show surgery on model manifold results in $\pi_1 = 0$) obtained by Baldridge-Kirk; Akhmedov-Park
- Full implementation (i.e. infinite families) for $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$: Fintushel-Park-Stern using the 2-fold symmetric product $\text{Sym}^2(\Sigma_3)$ as model.
- Full implementation (i.e. infinite families) for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 4$ by Baldridge-Kirk, Akhmedov-Park, Fintushel-Park-Stern, Akhmedov-Baykur-Baldridge-Kirk-Park, Akhmedov-Baykur-Park.
Model Manifolds for $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$

Basic Pieces: $X_0, X_1, X_2, X_3, X_4$

$X_r \# \Sigma_s X_s$ is a model for $\mathbb{C}P^2 \# (r + s + 1) \overline{\mathbb{C}P^2}$

$X_0$: $\Sigma_2 \subset T^2 \times \Sigma_2$ representing $(0, 1)$

$X_1$: $\Sigma_2 \subset T^2 \times T^2 \# \overline{\mathbb{C}P^2}$ representing $(2, 1) - 2e$

$X_2$: $\Sigma_2 \subset T^2 \times T^2 \# 2\overline{\mathbb{C}P^2}$ representing $(1, 1) - e_1 - e_2$

$X_3$: $\Sigma_2 \subset S^2 \times T^2 \# 3\overline{\mathbb{C}P^2}$ representing $(1, 3) - 2e_1 - e_2 - e_3$

$X_4$: $\Sigma_2 \subset S^2 \times T^2 \# 4\overline{\mathbb{C}P^2}$ representing $(1, 2) - e_1 - e_2 - e_3 - e_4$

Exception: $X_0 \# \Sigma_2 X_0 = \Sigma_2 \times \Sigma_2$ is a model for $S^2 \times S^2$

Enough lagrangian tori to kill $H_1$; The art is to find tori and show result has $\pi_1 = 0$

- First successful implementation of this strategy for $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ (i.e. find tori, show surgery on model manifold results in $\pi_1 = 0$) obtained by Baldridge-Kirk; Akhmedov-Park

- Full implementation (i.e. infinite families) for $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$: Fintushel-Park-Stern using the 2-fold symmetric product $\text{Sym}^2(\Sigma_3)$ as model.

- Full implementation (i.e. infinite families) for $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$, $k \geq 4$ by Baldridge-Kirk, Akhmedov-Park, Fintushel-Park-Stern, Akhmedov-Baykur-Baldridge-Kirk-Park, Akhmedov-Baykur-Park.
Alternate approaches successful for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}, \; k \geq 5$

**Rational Blowdown**

1989: First fake $\mathbb{CP}^2 \# 8 \overline{\mathbb{CP}^2}$; D. Kotschick showed Barlow surface exotic.

2004: First fake $\mathbb{CP}^2 \# 7 \overline{\mathbb{CP}^2}$; Jongil Park used rational blowdown (gave others courage to pursue small 4-manifolds).

2004: Related ideas used by Stipsicz-Szabó to produce fake $\mathbb{CP}^2 \# 6 \overline{\mathbb{CP}^2}$

2005: Fintushel-Stern introduced *double-node surgery* to produce infinitely many fake such structures.

2005: Two hours later Park-Stipsicz-Szabó used double-node surgery to produce infinitely many exotic $\mathbb{CP}^2 \# 5 \overline{\mathbb{CP}^2}$.

Jongil Park led effort to use rational blowdown techniques to find exotic complex structures on $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}, \; k \geq 5$

2006: Reverse Engineering introduced

2007: $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}, \; k \geq 3$, Baldridge-Kirk, Ahkmedov-Park, Fintushel-Park-Stern, Ahkmedov-Bakyur-Baldridge-Kirk-Park, Ahkmedov-Bakyur-Park.
Alternate approaches successful for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 5$

Rational Blowdown

1989: First fake $\mathbb{CP}^2 \# 8 \overline{\mathbb{CP}^2}$; D. Kotschick showed Barlow surface exotic.

2004: First fake $\mathbb{CP}^2 \# 7 \overline{\mathbb{CP}^2}$; Jongil Park used rational blowdown (gave others courage to pursue small 4-manifolds).

2004: Related ideas used by Stipsicz-Szabó to produce fake $\mathbb{CP}^2 \# 6 \overline{\mathbb{CP}^2}$

2005: Fintushel-Stern introduced *double-node surgery* to produce infinitely many fake such structures.

2005: Two hours later Park-Stipsicz-Szabó used double-node surgery to produce infinitely many exotic $\mathbb{CP}^2 \# 5 \overline{\mathbb{CP}^2}$.

Jongil Park led effort to use rational blowdown techniques to find exotic complex structures on $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 5$

2006: Reverse Engineering introduced

2007: $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 3$, Baldridge-Kirk, Ahkmedov-Park, Fintushel-Park-Stern, Ahkmedov-Bakyur-Baldridge-Kirk-Park, Ahkmedov-Bakyur-Park.
Alternate approaches successful for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 5$

**Rational Blowdown**

1989: First fake $\mathbb{CP}^2 \# 8 \overline{\mathbb{CP}^2}$; D. Kotschick showed Barlow surface exotic.

2004: First fake $\mathbb{CP}^2 \# 7 \overline{\mathbb{CP}^2}$; Jongil Park used rational blowdown (gave others courage to pursue small 4-manifolds).

2004: Related ideas used by Stipsicz-Szabó to produce fake $\mathbb{CP}^2 \# 6 \overline{\mathbb{CP}^2}$

2005: Fintushel-Stern introduced *double-node surgery* to produce infinitely many fake such structures

2005: Two hours later Park-Stipsicz-Szabó used double-node surgery to produce infinitely many exotic $\mathbb{CP}^2 \# 5 \overline{\mathbb{CP}^2}$.

Jongil Park led effort to use rational blowdown techniques to find exotic **complex structures** on $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 5$

2006: Reverse Engineering introduced

2007: $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 3$, Baldridge-Kirk, Ahkmedov-Park, Fintushel-Park-Stern, Ahkmedov-Bakyur-Baldridge-Kirk-Park, Ahkmedov-Bakyur-Park.
Alternate approaches successful for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2, \ k \geq 5$

**Rational Blowdown**

1989: First fake $\mathbb{CP}^2 \# 8 \overline{\mathbb{CP}}^2$; D. Kotschick showed Barlow surface exotic.

2004: First fake $\mathbb{CP}^2 \# 7 \overline{\mathbb{CP}}^2$; Jongil Park used rational blowdown (gave others courage to pursue small 4-manifolds).

2004: Related ideas used by Stipsicz-Szabó to produce fake $\mathbb{CP}^2 \# 6 \overline{\mathbb{CP}}^2$.

2005: Fintushel-Stern introduced *double-node surgery* to produce infinitely many fake such structures.

2005: Two hours later Park-Stipsicz-Szabó used double-node surgery to produce infinitely many exotic $\mathbb{CP}^2 \# 5 \overline{\mathbb{CP}}^2$.

Jongil Park led effort to use rational blowdown techniques to find exotic *complex structures* on $\mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2, \ k \geq 5$.

2006: Reverse Engineering introduced

2007: $\mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2, \ k \geq 3$, Baldridge-Kirk, Ahkmedov-Park, Fintushel-Park-Stern, Ahkmedov-Bakyur-Baldridge-Kirk-Park, Ahkmedov-Bakyur-Park.
Alternate approaches successful for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 5$

Rational Blowdown

1989: First fake $\mathbb{CP}^2 \# 8 \overline{\mathbb{CP}^2}$; D. Kotschick showed Barlow surface exotic.

2004: First fake $\mathbb{CP}^2 \# 7 \overline{\mathbb{CP}^2}$; Jongil Park used rational blowdown (gave others courage to pursue small 4-manifolds).

2004: Related ideas used by Stipsicz-Szabó to produce fake $\mathbb{CP}^2 \# 6 \overline{\mathbb{CP}^2}$

2005: Fintushel-Stern introduced *double-node surgery* to produce infinitely many fake such structures

2005: Two hours later Park-Stipsicz-Szabó used double-node surgery to produce infinitely many exotic $\mathbb{CP}^2 \# 5 \overline{\mathbb{CP}^2}$.

Jongil Park led effort to use rational blowdown techniques to find exotic complex structures on $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 5$

2006: Reverse Engineering introduced

2007: $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 3$, Baldridge-Kirk, Ahkmedov-Park, Fintushel-Park-Stern, Ahkmedov-Bakyur-Baldridge-Kirk-Park, Ahkmedov-Bakyur-Park.
Alternate approaches successful for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 5$

**Rational Blowdown**

1989: First fake $\mathbb{CP}^2 \# 8 \overline{\mathbb{CP}^2}$; D. Kotschick showed Barlow surface exotic.

2004: First fake $\mathbb{CP}^2 \# 7 \overline{\mathbb{CP}^2}$; Jongil Park used rational blowdown (gave others courage to pursue small 4-manifolds).

2004: Related ideas used by Stipsicz-Szabó to produce fake $\mathbb{CP}^2 \# 6 \overline{\mathbb{CP}^2}$

2005: Fintushel-Stern introduced *double-node surgery* to produce infinitely many fake such structures

2005: Two hours later Park-Stipsicz-Szabó used double-node surgery to produce infinitely many exotic $\mathbb{CP}^2 \# 5 \overline{\mathbb{CP}^2}$.

Jongil Park led effort to use rational blowdown techniques to find exotic complex structures on $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 5$

2006: Reverse Engineering introduced

2007: $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 3$, Baldridge-Kirk, Ahkmedov-Park, Fintushel-Park-Stern, Ahkmedov-Bakyur-Baldridge-Kirk-Park, Ahkmedov-Bakyur-Park.
Alternate approaches successful for $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$, $k \geq 5$

Rational Blowdown

1989: First fake $\mathbb{C}P^2 \# 8 \overline{\mathbb{C}P^2}$; D. Kotschick showed Barlow surface exotic.

2004: First fake $\mathbb{C}P^2 \# 7 \overline{\mathbb{C}P^2}$; Jongil Park used rational blowdown (gave others courage to pursue small 4-manifolds).

2004: Related ideas used by Stipsicz-Szabó to produce fake $\mathbb{C}P^2 \# 6 \overline{\mathbb{C}P^2}$

2005: Fintushel-Stern introduced \textit{double-node surgery} to produce infinitely many fake such structures.

2005: Two hours later Park-Stipsicz-Szabó used double-node surgery to produce infinitely many exotic $\mathbb{C}P^2 \# 5 \overline{\mathbb{C}P^2}$.

Jongil Park led effort to use rational blowdown techniques to find exotic \textbf{complex structures} on $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$, $k \geq 5$

2006: Reverse Engineering introduced

2007: $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$, $k \geq 3$, Baldridge-Kirk, Ahkmedov-Park, Fintushel-Park-Stern, Ahkmedov-Bakyur-Baldridge-Kirk-Park, Ahkmedov-Bakyur-Park.
Alternate approaches successful for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 5$

Rational Blowdown

1989: First fake $\mathbb{CP}^2 \# 8 \overline{\mathbb{CP}^2}$; D. Kotschick showed Barlow surface exotic.

2004: First fake $\mathbb{CP}^2 \# 7 \overline{\mathbb{CP}^2}$; Jongil Park used rational blowdown (gave others courage to pursue small 4-manifolds).

2004: Related ideas used by Stipsicz-Szabó to produce fake $\mathbb{CP}^2 \# 6 \overline{\mathbb{CP}^2}$

2005: Fintushel-Stern introduced double-node surgery to produce infinitely many fake such structures

2005: Two hours later Park-Stipsicz-Szabó used double-node surgery to produce infinitely many exotic $\mathbb{CP}^2 \# 5 \overline{\mathbb{CP}^2}$.

Jongil Park led effort to use rational blowdown techniques to find exotic complex structures on $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 5$

2006: Reverse Engineering introduced

2007: $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 3$, Baldridge-Kirk, Ahkmedov-Park, Fintushel-Park-Stern, Ahkmedov-Bakyur-Baldridge-Kirk-Park, Ahkmedov-Bakyur-Park.
Oriented minimal $\pi_1 = 0$ 4-manifolds with $SW \neq 0$

Geography

\[
c = 3\text{sign} + 2e
\]

\[
\chi_h = \frac{\text{sign} + e}{4}
\]

All lattice points have $\infty$ smooth structures except possibly near $c = 9\chi$ and on $\chi_h = 1$

For $n > 4$ TOP $n$-manifolds have finitely many smooth structures

$\chi = \chi + 2$ and $c = \chi + 3$

$\chi = 3\text{sign}$ and $c = 8\chi$

$\chi = \text{sign} + 2\epsilon$

$s = \text{sign} + 2\epsilon$

$s = 3\text{sign} + 2\epsilon$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$

$s = 9\chi$

$s = 8\chi$
Next Challenges

• Model for $\mathbb{CP}^2$; topological construction of the Mumford plane.

• What about $S^2 \times S^2; \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2; \mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$? ($\pi_1$ issues)

• Are the fake $\mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2$ obtained by surgery on null-homologous torus in the standard $\mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2$?

  (see Fintushel-Stern: *Surgery on nullhomologous tori and simply connected 4-manifolds with $b^+ = 1$, Journal of Topology 1 (2008), 1-15, for first attempts)

• More generally are all 4- manifolds obtained from either $\ell \mathbb{CP}^2 \# k \overline{\mathbb{CP}}^2$ or $nE(2)\#m(S^2 \times S^2)$ via a sequence of surgeries on null-homologous tori?

  Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on (null-homologous) tori.
Next Challenges

- Model for $\mathbb{CP}^2$; topological construction of the Mumford plane.
- What about $S^2 \times S^2$; $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$; $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$? ($\pi_1$ issues)
- Are the fake $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$ obtained by surgery on null-homologous torus in the standard $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$? (see Fintushel-Stern: *Surgery on nullhomologous tori and simply connected 4-manifolds with $b^+ = 1$*, Journal of Topology 1 (2008), 1-15, for first attempts)
- More generally are all 4- manifolds obtained from either $\ell \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$ or $nE(2)\# m(S^2 \times S^2)$ via a sequence of surgeries on null-homologous tori?

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on (null-homologous) tori.
Next Challenges

- Model for \(\mathbb{CP}^2\); topological construction of the Mumford plane.
- What about \(S^2 \times S^2; \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}; \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}\)? (\(\pi_1\) issues)
- Are the fake \(\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}\) obtained by surgery on null-homologous torus in the standard \(\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}\)? (see Fintushel-Stern: *Surgery on nullhomologous tori and simply connected 4-manifolds with \(b^+ = 1\), Journal of Topology 1 (2008), 1-15, for first attempts)
- More generally are all 4- manifolds obtained from either \(\ell \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}\) or \(nE(2)\#m(S^2 \times S^2)\) via a sequence of surgeries on null-homologous tori?

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on (null-homologous) tori.
Next Challenges

- Model for $\mathbb{C}P^2$; topological construction of the Mumford plane.
- What about $S^2 \times S^2$; $\mathbb{C}P^2 \# \bar{\mathbb{C}P}^2$; $\mathbb{C}P^2 \# 2\bar{\mathbb{C}P}^2$? ($\pi_1$ issues)
- Are the fake $\mathbb{C}P^2 \# k \bar{\mathbb{C}P}^2$ obtained by surgery on null-homologous torus in the standard $\mathbb{C}P^2 \# k \bar{\mathbb{C}P}^2$? (see Fintushel-Stern: *Surgery on nullhomologous tori and simply connected 4-manifolds with $b^+ = 1$*, Journal of Topology 1 (2008), 1-15, for first attempts)
- More generally are all 4-manifolds obtained from either $\ell \mathbb{C}P^2 \# k \bar{\mathbb{C}P}^2$ or $nE(2)\#m(S^2 \times S^2)$ via a sequence of surgeries on null-homologous tori?

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on (null-homologous) tori.
Next Challenges

- Model for $\mathbb{CP}^2$; topological construction of the Mumford plane.
- What about $S^2 \times S^2$; $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$; $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$? ($\pi_1$ issues)
- Are the fake $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$ obtained by surgery on null-homologous torus in the standard $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$?
  (see Fintushel-Stern: *Surgery on nullhomologous tori and simply connected 4-manifolds with $b^+ = 1$, Journal of Topology 1 (2008), 1-15, for first attempts*)
- More generally are all 4- manifolds obtained from either $\ell \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$ or $nE(2) \# m (S^2 \times S^2)$ via a sequence of surgeries on null-homologous tori?

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on (null-homologous) tori.
Next Challenges

- Model for $\mathbb{CP}^2$; topological construction of the Mumford plane.
- What about $S^2 \times S^2$; $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$; $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$? ($\pi_1$ issues)
- Are the fake $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$ obtained by surgery on null-homologous torus in the standard $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$? (see Fintushel-Stern: *Surgery on nullhomologous tori and simply connected 4-manifolds with $b^+ = 1$, Journal of Topology* 1 (2008), 1-15, for first attempts)
- More generally are all 4- manifolds obtained from either $\ell \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$ or $nE(2)\# m(S^2 \times S^2)$ via a sequence of surgeries on null-homologous tori?

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on (null-homologous) tori.
Focus Problem

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on (null-homologous) tori.

Aside: When is log transform on an essential torus a sequence of log transforms on null-homologous tori?

- Then, euler characteristic, signature, and type will classify smooth 4-manifolds up to surgery on (null-homologous) tori.
- In other words, algebraic topology will classify smooth 4-manifolds up to
Focus Problem

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on (null-homologous) tori.

Aside: When is log transform on an essential torus a sequence of log transforms on null-homologous tori?

- Then, euler characteristic, signature, and type will classify smooth 4-manifolds up to surgery on (null-homologous) tori.

- In other words, algebraic topology will classify smooth 4-manifolds up to
Focus Problem

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on (null-homologous) tori.

Aside: When is log transform on an essential torus a sequence of log transforms on null-homologous tori?

- Then, euler characteristic, signature, and type will classify smooth 4-manifolds up to surgery on (null-homologous) tori.
- In other words, algebraic topology will classify smooth 4-manifolds up to
Focus Problem

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on (null-homologous) tori.

Aside: When is log transform on an essential torus a sequence of log transforms on null-homologous tori?

- Then, euler characteristic, signature, and type will classify smooth 4-manifolds up to surgery on (null-homologous) tori.
- In other words, algebraic topology will classify smooth 4-manifolds up to Wormholes!
An Even Bigger Challenge
An Even Bigger Challenge

Talking at Rob’s 80th Birthday Fest.
An Even Bigger Challenge

Talking at Rob’s 80th Birthday Fest.

HAPPY 70th BIRTHDAY, ROB