Getting to the heart of smooth 4-manifolds

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Joint work with Ron Fintushel
Examples of smooth 4-manifolds

\[ S^4 = \text{unit sphere in } \mathbb{R}^4 \]

\[ S^2 \times S^2 \]

\[ \mathbb{CP}^2 = \text{complex lines through origin in } \mathbb{C}^3 \]

\[ K3 \text{ surface} = \text{quartic in } \mathbb{CP}^3 = \{ z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0 \} \]

\[ S^4 \#_k \mathbb{CP}^2 \#_l \overline{\mathbb{CP}^2} \#_s (S^2 \times S^2) \#_t K3 \]
Classify $\pi_1 = 0$ smooth 4-manifolds

A possible scheme

A collection of well-understood 4-manifolds

A collection of surgery operations

All $\pi_1 = 0$ smooth 4-manifolds

Invariants to distinguish up to finite discrepancy
Classify $\pi_1 = 0$ smooth 4-manifolds ????

A possible scheme

\[
S^4 \#_k \mathbb{CP}^2 \#_\ell \overline{\mathbb{CP}^2} \#_s (S^2 \times S^2) \#_t K3
\]

\[\downarrow\]

A collection of surgery operations

\[\downarrow\]

All $\pi_1 = 0$ smooth 4-manifolds

\[\uparrow\]

Invariants to distinguish up to finite discrepancy
Classify $\pi_1 = 0$ smooth 4-manifolds

A possible scheme

$S^4 \#_k \mathbb{CP}^2 \#_\ell \overline{\mathbb{CP}^2} \#_s (S^2 \times S^2) \#_t K3$

$\Downarrow$

Surgery on null-homologous tori

$\Downarrow$

All $\pi_1 = 0$ smooth 4-manifolds

$\Uparrow$

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A possible scheme

\[ S^4 \#_k \mathbb{CP}^2 \#_\ell \overline{\mathbb{CP}^2} \#_s (S^2 \times S^2) \#_t K3 \]

\[ \Downarrow \]

Surgery on null-homologous tori

\[ \Downarrow \]

All $\pi_1 = 0$ smooth 4-manifolds

\[ \Uparrow \]

Seiberg-Witten Invariants
Wild Conjecture

Every topological 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every $n$-manifold has only finitely many distinct smooth $n$-manifolds which are homeomorphic to it.

Main goal: Discuss recent techniques developed to study this conjecture and to validate the proposed classification scheme.
Classify $\pi_1 = 0$ smooth 4-manifolds

A reasonable scheme

$$S^4 \#_k \mathbb{C}P^2 \#_\ell \overline{\mathbb{C}P^2} \#_s (S^2 \times S^2) \#_t K3$$

$\Downarrow$

Surgery on null-homologous tori

$\Downarrow$

All $\pi_1 = 0$ smooth 4-manifolds

$\Uparrow$

Seiberg-Witten Invariants
Basic facts about 4-manifolds

Invariants

- Euler characteristic: $e(X) = \sum_{i=0}^{4} (-1)^i \text{rk}(H^i(M; \mathbb{Z}))$

- Intersection form: $H^2(X; \mathbb{Z}) \otimes H^2(X; \mathbb{Z}) \to \mathbb{Z};$

$$\alpha \cdot \beta = (\alpha \cup \beta)[X]$$

is an integral, symmetric, unimodular, bilinear form.

Signature of $X = \sigma(X) = \text{Signature of intersection form} = b^+ - b^-$

Type: Even if $\alpha \cdot \alpha$ even for all $\alpha$; otherwise Odd

- (Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has $X \times S^1$ smoothable.

- (Donaldson, 1982) Two simply connected smooth 4-manifolds are homeomorphic iff they have the same $e$, $\sigma$, and type.
Basic facts about 4-manifolds

Invariants

- Euler characteristic: $e(X) = \sum_{i=0}^{4}(-1)^i \text{rk}(H^i(M; \mathbb{Z}))$
  
  
  $e(S^4) = 2$, $e(\mathbb{C}P^2) = 3$, $e(S^2 \times S^2) = 4$, $e(K3) = 24$, $e(X \# Y) = e(X) + e(Y) - 2$, $e(-X) = e(X)$

- Signature of $X = \sigma(X) = \text{Signature of intersection form} = b^+ - b^-$
  
  $\sigma(S^4) = 0$, $\sigma(\mathbb{C}P^2) = 1$, $\sigma(S^2 \times S^2) = 0$, $\sigma(K3) = -16$, $\sigma(X \# Y) = \sigma(X) + \sigma(Y)$, $\sigma(-X) = -\sigma(X)$

- type($S^4$) = even, type($\mathbb{C}P^2$) = odd, type($S^2 \times S^2$) = even, type($K3$) = even, type($X \# Y$) = odd if one of $X$ or $Y$ is odd, even otherwise, type($-X$) = type($X$)
What do we know about smooth 4-manifolds?

Wild Conjecture

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

Reminder: for \( n > 4 \), every \( n \)-manifold has only finitely many distinct smooth \( n \)-manifolds which are homeomorphic to it.

▶ Need more invariants: Donaldson, Seiberg-Witten Invariants

\[
SW : \{ \text{characteristic elements of } H_2(X; \mathbb{Z}) \} \rightarrow \mathbb{Z}
\]

\( \alpha \) characteristic \( \iff \alpha \cdot x = x \cdot x(2) \)

▶ \( SW(\beta) \neq 0 \) for only finitely many \( \beta \): called basic classes.

▶ For each surface \( \Sigma \subset X \) with \( g(\Sigma) > 0 \) and \( \Sigma \cdot \Sigma \geq 0 \)

\[
2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|
\]

for every basic class \( \beta \). (adjunction inequality[Kronheimer-Mrowka])

Basic classes = smooth analogue of the canonical class of a complex surface

▶ \( SW(X \# Y) = 0 \) if \( b^+(X) \geq 1 \)

▶ \( SW(\kappa) = \pm 1 \), \( \kappa \) the first Chern class of a symplectic manifold [Taubes].
Oriented minimal \((\pi_1 = 0)\) 4-manifolds with \(\mathcal{S}\mathcal{W} \neq 0\)

Geography

\[
c = 3\sigma + 2e
\]

\[
\chi_h = \frac{\sigma + e}{4}
\]

All lattice points have \(\infty\) smooth structures except possibly near \(c = 9\chi_h\) and on \(\chi_h = 1\).

For \(n > 4\) TOP n-manifolds have finitely many smooth structures.
Every 4-manifold has zero or infinitely many distinct smooth structures

A VERY useful technique
▶ One way to try to prove this conjecture is to find a “dial” to change the smooth structure at will.
▶ An effective dial: Surgery on null-homologous tori

\( T \): any self-intersection 0 torus \( \subset X \), Tubular nbd \( N_T \cong T^2 \times D^2 \).

Surgery on \( T \): \( X \setminus N_T \cup_\varphi T^2 \times D^2 \), \( \varphi : \partial(T^2 \times D^2) \to \partial(X \setminus N_T) \)

\( \varphi(pt \times \partial D^2) = \) surgery curve

Result determined by \( \varphi_*[pt \times \partial D^2] \in H_1(\partial(X \setminus N_T)) = \mathbb{Z}^3 \)

Choose basis \( \{\alpha, \beta, [\partial D^2]\} \) for \( H_1(\partial N_T) \) where \( \{\alpha, \beta\} \) are pushoffs of a basis for \( H_1(T) \).

\( \varphi_*[pt \times \partial D^2] = p\alpha + q\beta + r[\partial D^2] \)

Write \( X \setminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r) \)

This operation does not change \( e(X) \) or \( \sigma(X) \)

Note: \( X_T(0, 0, 1) = X \)

Need formula for the Seiberg-Witten invariant of \( X_T(p, q, r) \) to determine when the smooth structure changes:

Due to Morgan, Mrowka, and Szabo.
The Morgan, Mrowka, Szabo Formula

\[ SW_{X_{T(p,q,r)}} = pSW_{X_{T(1,0,0)}} + qSW_{X_{T(0,1,0)}} + rSW_{X} \]

- When a core torus is nullhomologous.
- When a core torus is essential, but there is a square 0 torus that intersects it algebraically nontrivially.
Old Application: Knot Surgery

$K$: Knot in $S^3$, $T$: square 0 essential torus in $X$

$X_K = X \setminus N_T \cup S^1 \times (S^3 \setminus N_K)$

Note: $S^1 \times (S^3 \setminus N_K)$ has the homology of $T^2 \times D^2$.

Facts about knot surgery

- If $X$ and $X \setminus T$ both simply connected; so is $X_K$
  (So $X_K$ homeo to $X$)
- If $K$ is fibered and $X$ and $T$ both symplectic; so is $X_K$.
- $SW_{X_K} = SW_X \cdot \Delta_K(t^2)$

Conclusions

- If $X$, $X \setminus T$, simply connected and $SW_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to $X$.
- If in addition $X$, $T$ symplectic, $K$ fibered, then there is an infinite family of distinct symplectic manifolds all homeomorphic to $X$.

e.g. $X = K3$, $SW_X = 1$, $SW_{X_K} = \Delta_K(t^2)$
Knot surgery and nullhomologous tori

Knot surgery on torus $T$ in 4-manifold $X$ with knot $K$:

$$X_K = X \#_{T=S^1 \times m} S^1 x m$$

$\Lambda = S^1 \times \lambda =$ nullhomologous torus — Used to change crossings:

Now apply Morgan-Mrowka-Szabó formula + tricks

- **Weakness of construction:** Requires $T$ to be homologically essential
  
  Open conjecture: If $\chi_h(X) > 1$, $SW_X \neq 0$, then $X$ contains a homologically essential torus $T$ with trivial normal bundle.

- If $X$ homeomorphic to $\mathbb{C}P^2$ blown up at 8 or fewer points, then $X$ contains no such torus - so what can we do for these small manifolds?
Oriented minimal $\pi_1 = 0$ 4-manifolds with $SW \neq 0$

**Geography**

\[
c = 3\text{sign} + 2e
\]

\[
\chi_h = \frac{\text{sign} + e}{4}
\]

All lattice points have $\infty$ smooth structures except possibly near $c = 9\chi_h$ and on $\chi_h = 1$

For $n > 4$ TOP $n$-manifolds have finitely many smooth structures

\[
c = 9\chi_h
\]

symplectic with one $SW$ basic class

\[
\chi_h - 3 \leq c \leq 2\chi_h - 6
\]

\[
c = 2\chi_h - 6
\]

symplectic with $(\chi_h - c - 2)$ $SW$ basic classes

\[
0 \leq c \leq (\chi_h - 3)
\]

\[
c = \chi_h - 3
\]

\[
c = 8\chi_h
\]

sign $= 0$

\[
c > 9\chi_h
\]

sign $> 0$

\[
sign < 0
\]

surfaces of general type

\[
2\chi_h - 6 \leq c \leq 9\chi_h
\]

Elliptic Surfaces $E(n)$

\[
c < 0
\]

$c = 3\text{sign} + 2e$

$\chi_h = \frac{\text{sign} + e}{4}$

$\chi_h = \frac{\text{sign} + e}{4}$
Reverse Engineering

- Difficult to find useful nullhomologous tori like \( \Lambda \) used in knot surgery.

Recall: 
\[
SW_{X_{T(p,q,r)}} = pSW_{X_{T(1,0,0)}} + qSW_{X_{T(0,1,0)}} + rSW_{X}
\]

Note: 
\[
b_1(X_{T(1,0,0)}) = b_1(X_{T(0,1,0)}) = b_1(X) + 1
\]

IDEA: First construct \( X_{T(1,0,0)} \) so that \( SW_{X_{T(1,0,0)}} \neq 0 \) and then surger to reduce \( b_1 \).

Dual situations for surgery on \( T \)

a. \( T \) primitive, \( \alpha \subset T \) essential in \( X \setminus T \).
   \[
   \Rightarrow \quad T_{(1,0,r)} \text{ nullhomologous in } X_T(1,0,r).
   \]

b. \( T \) nullhomologous, \( \alpha \) bounds in \( X \setminus N_T \)
   \[
   \Rightarrow \quad (1,0,0) \ (i.e. \ nullhomologous) \text{ surgery on } T \text{ gives (a)}.
   \]
Reverse Engineering

- Difficult to find useful nullhomologous tori like $\Lambda$ used in knot surgery.

Recall: $SW_{X_T(p,q,r)} = pSW_{X_T(1,0,0)} + qSW_{X_T(0,1,0)} + rSW_X$

Note: $b_1(X_T(1,0,0)) = b_1(X_T(0,1,0)) = b_1(X) + 1$

IDEA: First construct $X_{T(1,0,0)}$ so that $SW_{X_T(1,0,0)} \neq 0$ and then surger to reduce $b_1$.

- Procedure to insure the existence of effective null-homologous tori

1. Find model manifold $M$ with same Euler number and signature as desired manifold, but with $b_1 \neq 0$ and with $SW \neq 0$.

2. Find $b_1$ disjoint essential tori in $M$ containing generators of $H_1$. Surger to get manifold $X$ with $H_1 = 0$. Want result of each surgery to have $SW \neq 0$ (except perhaps the very last).

3. $X$ will contain a “useful” nullhomologous torus.
Luttinger Surgery

- For model manifolds with $H_1 \neq 0$: nature hands you symplectic manifolds.
- We seek tori that will kill $b_1$. Nature hands you Lagrangian tori.

$X$: symplectic manifold \hspace{1cm} T: Lagrangian torus in $X$

Preferred framing for $T$: Lagrangian framing w.r.t. which all pushoffs of $T$ remain Lagrangian

$(1/n)$-surgeries w.r.t. this framing are again symplectic (Luttinger; Auroux, Donaldson, Katzarkov)

If $S^1_\beta = \text{Lagrangian pushoff}$, $X_T(0, \pm 1, 0)$: symplectic mfd

$$\implies \text{X}_T(0, \pm 1, 0) \text{ has } SW \neq 0$$

Recall: $SW_{X_T(p, q, r)} = pSW_{X_T(1, 0, 0)} + qSW_{X_T(0, 1, 0)} + rSW_X$

Then have infinitely many $H_1 = 0$ manifolds - keep fingers crossed $\pi_1 = 0$. 

Reverse Engineering in Action

Infinite families of fake $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$

Need Model Manifolds for $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$, $k \geq 2$

i.e. symplectic manifolds $X_k$ with same $e$ and $\sigma$ as $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$, and $b_1 \geq 1$ disjoint lagrangian tori carrying basis for $H_1$.

- Surger lagrangian tori to decrease $b_1$.
- Resulting manifold has $H_1 = 0$ - but with a dial.
- Get infinite family of distinct manifolds all homology equivalent to $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$
- Keep fingers crossed that result has $\pi_1 = 0$, so all homeomorphic to $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$
Model Manifolds for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$

Basic Pieces: $X_0, X_1, X_2, X_3, X_4$

$X_r \#_2 X_s$ is a model for $\mathbb{CP}^2 \# (r + s + 1) \overline{\mathbb{CP}^2}$

$X_0$: $\Sigma_2 \subset T^2 \times \Sigma_2$ representing $(0, 1)$

$X_1$: $\Sigma_2 \subset T^2 \times T^2 \# \overline{\mathbb{CP}^2}$ representing $(2, 1) - 2e$

$X_2$: $\Sigma_2 \subset T^2 \times T^2 \# 2 \overline{\mathbb{CP}^2}$ representing $(1, 1) - e_1 - e_2$

$X_3$: $\Sigma_2 \subset S^2 \times T^2 \# 3 \overline{\mathbb{CP}^2}$ representing $(1, 3) - 2e_1 - e_2 - e_3$

$X_4$: $\Sigma_2 \subset S^2 \times T^2 \# 4 \overline{\mathbb{CP}^2}$ representing $(1, 2) - e_1 - e_2 - e_3 - e_4$

Exception: $X_0 \#_2 X_0 = \Sigma_2 \times \Sigma_2$ is a model for $S^2 \times S^2$

Enough lagrangian tori to kill $H_1$; The art is to find tori and show result has $\pi_1 = 0$

- First successful implementation of this strategy for $\mathbb{CP}^2 \# 3 \overline{\mathbb{CP}^2}$ (i.e. find tori, show surgery on model manifold results in $\pi_1 = 0$) obtained by Baldridge-Kirk; Akhmedov-Park
- Full implementation (i.e. infinite families) for $\mathbb{CP}^2 \# 3 \overline{\mathbb{CP}^2}$: Fintushel-Park-Stern using the 2-fold symmetric product $\text{Sym}^2(\Sigma_3)$ as model.
- Full implementation (i.e. infinite families) for $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $k \geq 4$ by Baldridge-Kirk, Akhmedov-Park, Fintushel-Stern, Akhmedov-Baykur-Baldridge-Kirk-Park, Akhmedov-Baykur-Park.
Oriented minimal $\pi_1 = 0$ $4$-manifolds with $SW \neq 0$

Geography

$c = 3\text{sign} + 2e$

$\chi_h = \frac{\text{sign} + e}{4}$

All lattice points have $\infty$ smooth structures except possibly near $c = 9\chi$ and on $\chi_h = 1$

For $n > 4$ TOP $n$-manifolds have finitely many smooth structures

$2\chi - 6 \leq c \leq 9\chi$

surfaces of general type

$c = 2\chi - 6$

symplectic with one $SW$ basic class

$\chi_h - 3 \leq c \leq 2\chi - 6$

$c = \chi_h - 3$

symplectic with $(\chi_h - c - 2) SW$ basic classes

$0 \leq c \leq (\chi_h - 3)$

$c > 9\chi$ ??

sign $> 0$

$c = 9\chi$

sign $= 0$

$c = 8\chi$

sign $< 0$

$c < 0$ ??

$\mathbb{C}P^2$

$S^2 \times S^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

$\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$

$3 \leq k \leq 8$

$S^4$

Elliptic Surfaces $E(n)$

$c < 0$ ??

$\mathbb{C}P^2# \mathbb{C}P^2$

$3 \leq k \leq 8$
Next Challenges

- Model for $\mathbb{CP}^2$; topological construction of the Mumford plane.
- What about $S^2 \times S^2; \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}; \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$? ($\pi_1$ issues)
- Are the fake $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$ obtained by surgery on null-homologous torus in the standard $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$?

RECENT WORK OF F-S: YES
\[ S^4 \#_k \mathbb{C}P^2 \#_{\ell} \overline{\mathbb{C}P^2} \#_s (S^2 \times S^2) \#_t K3 \]

\[ \Downarrow \]

Surgery on (null-homologous) tori

\[ \Downarrow \]

All \( \pi_1 = 0 \) smooth 4-manifolds

\[ \Uparrow \]

Seiberg-Witten Invariants
The Dream

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on null-homologous tori.

▸ Then, euler characteristic, signature, and type will classify smooth 4-manifolds up to surgery on null-homologous tori.

▸ In other words, algebraic topology will classify smooth 4-manifolds up to
The Dream

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on null-homologous tori.

▶ Then, euler characteristic, signature, and type will classify smooth 4-manifolds up to surgery on null-homologous tori.

▶ In other words, algebraic topology will classify smooth 4-manifolds up to Wormholes!