Robustness of graph/hypergraph properties

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May 1, 2020

Introduction

Recall that a k-uniform hypergraph (or a k-graph for short) $G = (V, E)$ consists of a set V of vertices, and a set E of edges, where $E \subseteq \binom{V}{k}$. The case $k = 2$ is referred to as a graph.

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A typical result in graph/hypergraph theory has the following flavor:

Theorem

Let $\mathcal{P}$ be a k-graph property. Then, every k-graph G that satisfies certain conditions has the property $\mathcal{P}$.

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Examples of interest:

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Once such a result is established, it is natural to ask how strongly does G possess $\mathcal{P}$? In other words, we want to determine the robustness of G with respect to $\mathcal{P}$.

Measures of robustness

We illustrate various measures that can be used to study the robustness of graph properties. For example, suppose that G is any graph that contains a Hamilton cycle. Then, one could ask:

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- etc.

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- Its local resilience w.r.t $\mathcal{P}$ is the max α for which every subgraph H of G for which $d_H(v) \geq (1-\alpha)d_G(v)$ for all v satisfies $\mathcal{P}$.

Models of interest

Among other popular models, one could study "robustness" w.r.t to certain properties of the following model:

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- Random/deterministic directed/oriented graphs

Many graphs – history

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- Sárközy, Selkow and Szemerédi 2003 showed $h(G) \geq c^n n!$ for some small c, and conjectured that $c \approx \frac{1}{2}$.
- Cuckler and Kahn 2009 proved the stronger result that for every Dirac graph G on n vertices one has

$$h(G) \geq \left(\frac{\delta(G)}{e}\right)^n (1 - o(1))^n .$$

Many graphs – (relatively) new approach

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The permanent of an $n \times n$ matrix A is defined as $per(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A_{i\sigma(i)}$.
A 0-1 matrix A is called r-regular if it contains exactly r 1's in every row and every column.

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By combining the above estimates we obtain that for an r-regular 0/1 matrix we have that $per(A) = (1 - o(1))^n (r/e)^n$.

Many graphs – permanent cont'd

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A is the adjacency matrix of G, then $per(A)=$'the number of 2-factors of G'.

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A is the adjacency matrix of G, then $per(A) =$'the number of 2-factors of G'. To illustrate our technique we sketch the proof of

the following statement, which gives a lower bound on the number of Hamilton cycles in a dense graph G in terms of $\mathrm{reg}(G)$, where $\mathrm{reg}(G)$ is the maximal even r for which G contains an r-factor.

Theorem (F., Krivelevich and Sudakov 2017)

Let G be a Dirac graph on n vertices. Then the number of Hamilton cycles in G is at least $(1 - o(1))^n \left(\frac{\mathrm{reg}(G)}{e}\right)^n$.

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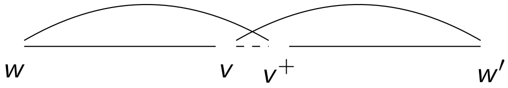
- By the above permanent estimate, one can show that H has at least $(r/e)^n(1 - o(1))^n$ 2-factors with at most $s = n^{1/2+o(1)}$ cycles in each.
- For every such factor F, we can turn F into a Hamilton cycle of G by adding and removing $O(s)$ edges.



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- All in all, we obtain at least $(r/e)^n(1-o(1))^n$ Hamilton cycles.

Many graphs – other models

A tournament is an oriented complete graph. It is easy to prove by induction that every tournament has a Hamilton path. Moreover, if the tournament is regular, i.e., if all vertices have the same in/outdegrees, then it also has a Hamilton cycle.

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Cuckler 2007 proved, using the above mentioned random walk approach, that for every regular tournament T we have $h(T) \geq \frac{n!}{(2+o(1))^n}$.

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Using our permanent based approach we obtained

Theorem (F., Krievlevich and Sudakov '17)

Let G be an oriented graph with all in/out degrees in $cn \pm o(n)$ for some constant $c > 3/8$. Then $h(G) \geq \left(\frac{(c+o(1))n}{e}\right)^n$.

Many graphs – binomial model

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For the number of Hamilton cycles in $G_{n,p}$, observe that, by Markov's it is always at most $\omega(1)(n-1)!p^n$. It turns out that the lower bound is not that simple, and following results by Cooper and Frieze 1989, and Janson 1994, Glebov and Krivelevich 2013 obtained such a bound that holds for all p for which $G_{n,p}$ is typically Hamiltonian.

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One of the first important insights, due to McDiarmid 1980, is that one can use a simple coupling argument to compare certain probabilities between $D_{n,p}$ and $G_{n,p}$. In particular, he showed that $\Pr[D_{n,p} \text{ is hamiltonian}] \geq \Pr[G_{n,p} \text{ is hamiltonian}]$.

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However, this is not optimal, and Frieze 1988 designed an algorithm to show Hamiltonicity at the optimal $p \geq (\log n + \omega(1))/n$.

McDiarmid's argument

In case that you are not familiar with such arguments, here we sketch McDiarmid's proof:

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- Observe that $\Gamma_0 = G_{n,p}$ and $\Gamma_N = D_{n,p}$. Therefore, it is enough to show $\Pr[\Gamma_{i+1} \text{ hamiltonian}] \geq \Pr[\Gamma_i \text{ hamiltonian}]$.

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- To see this, expose all arcs but those obtained from e_i. If Γ_{i-1} is Hamiltonian – done. If not hamiltonian even after adding e_i – done. If needs e_i, then at least one orientation is good and the probability to see it is at least p and we're done

Many graphs – binomial

For the counting part we again have an upper bound by Markov's. For the lower bound, improving upon F., Kronenberg and Long 2017 ($p \geq \log^2 /n$), F. and Long 2019 ($p \geq \log\log n \cdot \log n/n$), F. Kwan and Sudakov 2019 have finally proved an asymptotically tight lower bound that also works in a 'hitting time' version. Here we sketch a proof for $p = \omega(\log n/n)/$

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Proof sketch (F., Kwan and Sudakov) The model $D_{n,p}$ can be coupled as follows:

- Let $p_1 \approx p$ and $p_2 \ll p$ be such that $1 - p = (1 - p_1)(1 - p_2)$.
- Expose a random bipartite graph B_{n,n,p_1}.
- Let $V_1 \cup V_2$ be its parts, and take a random labeling of $V_2 = \{u_1, \ldots, u_n\}$.
- Take D_{n,p_2}.
- Define $D := D_{n,p_2} \cup \{\overrightarrow{ij} \mid v_i u_j \in B_{n,n,p_1}\}$.

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- Since the expected number of cycles in a random permutation is around $\log n$, by Markov's we know that 'most' permutations are mapped with at most (say) $\log^2 n$ many cycles.
- Finally, given any such 1-factor, we show that whp, using the (random) edges of D_{n,p_2}, we can complete it into a Hamilton cycle.

This completes the proof.

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Theorem (F., Krivelevich and Sudakov 2017)

Every n-vertex Dirac graph contains at least $(1 - o(1))reg(G)/2$ edge-disjoint Hamilton cycles.

For directed graphs, a remarkable result of Kühn and Osthus 2013, settling an old conjecture by Kelly, states that every r-regular oriented graph G has a Hamilton decomposition, provided that $r \geq 3n/8 + \varepsilon n$.

Edge disjoint copies

For non-regular, we conjecture:

Conjecture (F., Long and Sudakov 2017)

Let G be an oriented graph on n vertices with $\delta^{\pm}(G) \geq 3n/8$.
Then G contains $\mathrm{reg}(G)$ edge-disjoint Hamilton cycles.

Edge disjoint copies

For non-regular, we conjecture:

Conjecture (F., Long and Sudakov 2017)

Let G be an oriented graph on n vertices with $\delta^{\pm}(G) \geq 3n/8$. Then G contains $\mathrm{reg}(G)$ edge-disjoint Hamilton cycles.

Theorem (F., Long and Sudakov 2017)

If $\delta^{\pm}(G) \geq cn$ for some constant $c > 3/8$ then G contains $(1 - o(1))reg(G)$ edge-disjoint Hamilton cycles.

The proof also gives a rather short alternative proof of an approximate version of Kelly's conjecture.

Edge disjoint copies

- F., Kronenberg and Long 2017 studied counting and packing Hamilton cycles problems in random and pseudorandom (di)graphs.
- F. and Vu 2016 packing of perfect matchings in random hypergraphs (and introduced an 'online sprinkling technique').
- F. and Samotij 2018 packing bounded-degree trees in random graphs.
- F. and Jain 2019 spectral expanders have 1-factorization.
- F., Jain and Sudakov 2019 regular Dirac graphs have the 'correct' number of 1-factorizations.
- F. and Kwan 2020+ a random Steiner Triple System contains an approximate decomposition into perfect matchings.
- And more..

Edge disjoint copies – the packing problem

In a greater generality, one could state the following problem:

The Packing Problem

- Let $H_1, \ldots, H_N$ be a family of (hyper)graphs,

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Question: Is it possible to find edge-disjoint copies of the H_is in a typical n-vertex (hyper)graph with $(1 + o(1))m$ edges?

Edge disjoint copies – online sprinkling

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- Each H_i is a rainbow Hamilton cycle (F. and Mousset '17).
- All the H_is are trees (F. and Samotij '19).

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Then, $H^{(k)}_{n,p}$ has the same distribution as $H = H_1 \cup \ldots \cup H_\ell$, where $H_i = H^{(k)}_{n,p_i}$ for each i.

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- Clearly, if $\omega(e) \leq p$ for each k-tuple e, then the resulting structure can be coupled with a subgraph of $H \sim \mathcal{H}^k(n, p)$.

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BUT very recently we proved

Theorem (F. and Kwan 2020+)

For $p \geq n^{-k/2+o(1)}$, a typical $H_{n,p}^{(k)}$ is $\mu_d(k)$-resilient, where $\mu_d(k)$ is the optimal (and yet, unknown) ratio $\delta_d(k,n)/\binom{n}{k}$ that enforces a perfect matching.

Maker-Breaker games

An $(\mathcal{F}, (a, b))$-game is define as follows:

- V board and $\mathcal{F} \subseteq 2^V$ winning sets.
- Two players, Maker and Breaker, alternating turns in occupying a and b elements, respectively.
- Maker wins if and only if she claims all the elements of a winning set.

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That is, if we have a (weak) local resilience result for $H_{n,p}^{(k)}$, it can be immediately translated into a Maker-Breaker result!

End of time :-)

If you are interested at hearing more, don't hesitate to contact me!
asaff@uci.edu

Thank you for listening!

Questions?