Analytical Study of Tensegrity Lattices for Mass-Efficient Mechanical Energy Absorption

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Abstract
This paper studies tensegrity structures known as “D-bar” systems for applications as lightweight components for mechanical energy absorption. Aerospace structures such as planetary landers, designed to absorb energy from large impact loads while requiring minimal mass, would benefit from such components. Previous studies showed that D-bar systems support compressive loads with minimal mass compared to continuum options such as single columns. In this work, analytical equations for the mechanical (elastic) energy stored in D-bar systems of any complexity (a quantity proportional to the number of strings/bars in the system) are derived for the first time. The energy stored in D-bar systems is compared with that of bent buckled beams used in “flexible-bar tensegrity” concepts, which were proposed in the literature as energy absorption components for planetary landers. Comparisons are made between D-bar systems and bent buckled beams as isolated components subjected to a compressive load and as components of planetary landers. In all comparisons, the results show that D-bar systems of low complexity allow for higher energy storage and lower mass than bent buckled beams. Thus, it is concluded that D-bar systems can enhance the design of planetary landers and other applications that need lightweight mechanical energy absorption components.
Keywords
Tensegrity, energy absorption, compressive structures, self-similar structures, tensegrity lander, planetary lander

1 Introduction

Tensegrity systems are structurally stable networks of axially loaded one-dimensional members [1, 2, 3, 4, 5, 6]. These members are classified as strings/cables (subjected to tensile loads) and bars/struts (subjected to compressive loads). It is conventionally assumed that the members are connected through frictionless ball joints such that no moments are transmitted among them. The position of the joints and the connectivity and pre-stress of the members of a tensegrity system can be modified to alter its shape or stiffness [7, 8, 9, 10, 11, 12, 13, 14], or to achieve an optimal property such as minimum mass for a structure under a given set of loading conditions [1, 15, 16, 17, 18].

This work presents a study on lightweight components for mechanical energy absorption based on a special tensegrity topology known as D-bar system [1]. The study was motivated by aerospace structures such as planetary landers that need lightweight components to absorb energy from large impact loads (to protect their interior systems and payload) while requiring low mass. Recent examples of planetary lander designs include those proposed by SunSpiral and coworkers [19], who developed ball-like tensegrity robots for planetary exploration and landing that can safely absorb impact loads. Rimoli [20, 21] extended the approach of [19] by proposing bent buckled beams used in “flexible-bar tensegrity” concepts for energy absorption components in planetary landers. Here, conversely, the proposed approach is to store energy from external loads as mechanical (elastic) energy in the strings and bars of D-bar systems without triggering
local structural instabilities such as buckling and thereby allowing for more reliable structures. Other applications of D-bar tensegrity systems as lightweight energy absorption components include advanced materials formed by tensegrity lattices [22, 23, 24, 25, 26] and buildings with enhanced seismic-resistance and lightweight reinforcements [13, 27, 28].

As illustrated in Fig. 1, a D-bar system of complexity $1$ (a D-bar unit) is generated by replacing a compressive structural member with $2p$ bars emanating from its end-points towards the vertices of a centered $p$-sided polygon formed by strings ($p = 3$ in Fig. 1(b)). A D-bar system of complexity $q$ is generated by replacing each bar in a D-bar system of complexity $q - 1$ with a D-bar unit. It is worth mentioning that increasing the complexity of a D-bar system results in a higher number of joints and members which increases manufacturing costs and makes scalability more difficult. Therefore, D-bar systems of low complexity are more convenient for practical applications.

Previous studies showed that D-bar systems can support compressive loads with minimal mass considering buckling and material yielding constraints as compared to single columns [1]; however, their mechanical energy storage properties have not been explored thus far. It should be noted that the conventional D-bar structures studied in [1] include strings connecting the end-points of each D-bar unit in the D-bar systems, which are not present in this work (see Fig. 1(b)). These strings are necessary only to ensure self-equilibrium in the absence of an external compressive force or to actuate/deploy the D-bar systems. Such strings are omitted in this work as the D-bar systems considered herein are strictly designed to be in the presence of a compressive load, and our focus is solely on the compressive load-bearing and energy storage functionalities of D-bar systems. For applications of D-bar systems where external compressive loads are absent or actuation/deployment is needed, the strings connecting the end points of each D-bar unit must be included.

Analytical equations for the mechanical energy stored in D-bar systems of any complexity are derived in this paper for the first time. The energy stored in D-bar systems is compared with that of bent buckled beams (Fig. 1(a)) used in flexible-bar tensegrity concepts, which have been proposed in the literature as energy absorption components for planetary landers [20, 21]. Comparisons between the energy stored in D-bar systems
Figure 1. (a) A buckled beam subjected to a compressive force of magnitude $f$. (b) D-bar systems of different complexity $q$ subjected to a compressive force of magnitude $f$.

and bent beams at the single component level and as parts of a tensegrity lander assembly are presented.

The remainder of the paper is organized as follows: Section 2 summarizes the assumptions made in the analytical derivations, Section 3 presents the analysis of the energy stored and minimal mass of bent buckled beams subjected to compressive loads, Section 4 addresses the analysis of the energy stored and minimal mass of D-bar systems, and Section 5 presents the comparisons in energy stored and minimal mass between D-bar systems and bent buckled beams. Concluding remarks are provided in Section 6. The definition of each parameter employed in this study is provided in Appendix A.
2 Summary of Assumptions

The assumptions made in the analysis of bent buckled beams (Section 3) are as follows:

- Deflections are small and thus captured by Euler-Bernoulli beam theory
- The beam is long and slender such that Euler theory of buckling applies
- The beam is subjected to a single axial compressive load applied at its end-points (no moments or transverse loads are applied)
- The beam has uniform full circular cross-section and is comprised of a homogeneous and isotropic linear elastic material
- The beam is in a controlled post-buckled state under the applied force. Therefore, failure occurs when stress reaches the material yield stress at any point in the beam during post-buckling
- The maximum energy stored in the beam is that exhibited at the onset of failure (yielding at any point of the beam) during post-buckling

The assumptions made in the analysis of D-bar systems (Section 4) are as follows:

- The D-bar system is subjected to a single axial compressive load applied at its end points
- The strings and bars are connected through frictionless ball joints such that no moments are transmitted among them
- The strings and bars are subjected only to tensile and compressive axial loads, respectively
- The strings and bars have uniform circular cross-sections and are comprised of homogeneous and isotropic linear elastic materials
- All the strings are comprised of the same material. All the bars are comprised of the same material
- Mass of the joints is neglected
- Failure at any string occurs when the stress reaches its material yield stress
- Failure at any bar occurs when the compressive force reaches its critical buckling force
- Linearized (small) strain assumptions are used for the strings and bars as the strain is expected to remain within the small deformation domain
• The minimal mass of the D-bar system is determined by finding the minimum radii of strings and bars, which corresponds to the radii for which all the members are at the onset of failure under the applied force.

• The maximum energy stored in the D-bar system is that exhibited at the onset of failure in the strings and bars.

The assumptions made in the analysis of planetary landers with bent buckled beams and D-bar systems (Section 5.2) are as follows:

• The bent buckled beams and D-bar systems in the lander follow the aforesaid assumptions made in Sections 3 and 4.

• The geometry of the tensegrity lander developed in [19] is assumed.

• The static loads in the analysis are assumed to be the maximum (peak) loads on the structure during the landing operation.

3 Bent Beam Analysis

The goal in this section is to derive analytical formulas for the mechanical (elastic) energy stored ($V_B$) and the minimal mass ($m_B$) of a bent buckled beam subjected to an axial compressive force of magnitude $f$; refer to Fig. 1(a). Such an axial compressive force is assumed in order to simulate impact loading conditions. The results presented in this section are based on Euler-Bernoulli beam theory and Euler theory of buckling [29, 30]. Since the beam is subjected to a single axial compressive load, it is assumed that the beam is in a post-buckled state.

The beam has length $l$ and is initially aligned with the $x$-axis. It is subjected only to a compressive force of magnitude $f$ applied at its end points and aligned with the $x$-axis as indicated in Fig. 2. It is assumed that the beam is composed of a homogeneous and isotropic material with Young’s modulus $E_b$, yield stress $\sigma_b$, and mass density $\rho_b$. The beam has a uniform cross-section with minimum area moment of inertia $I$. It is assumed that the beam has a full circular cross-section of radius $r$.

To determine the mechanical energy stored and minimal mass of the bent beam in a post-buckled state, the formulas for its deflection $w$ and magnitude of the critical buckling load $f$ must be first provided. It is a known result from Euler theory of buckling [29, 30] that the critical buckling load of a beam is given as:
Figure 2. Geometry of a buckled beam subjected to an axial compressive load $f$. The deflection $w$ along the $y$-axis is indicated.

\[ f = \frac{n^2 \pi^3 E_b r^4}{4 l^2}, \quad n = 1, 2, 3, \ldots, \]  

(1)

where $n$ indicates the buckling mode. Furthermore, the deflection $w$ in its buckled configuration is given as:

\[ w = w_{\text{max}} \sin \left( \frac{n \pi x}{l} \right), \]  

(2)

where $w_{\text{max}}$ is the maximum deflection of the beam and $0 \leq x \leq l$.

To calculate the highest mechanical energy stored in the bent beam, it is assumed that the beam is loaded up to the onset of failure. In the controlled post-buckled state of the beam studied here, failure occurs when the stress anywhere in the bent beam reaches the material yield stress. Using the expression for deflection field $w$ in Eq. (2), formulas for $w_{\text{max}}$ at material yield and the associated axial strain field are derived.

Lemma 3.1. Consider a beam of length $l$ and deflection field $w$ given by Eq. (2). Suppose that the beam is composed of a homogeneous and isotropic material with Young’s modulus $E_b$ and yield stress $\sigma_b$ and has a uniform full circular cross-section of radius $r$. Then, the maximum allowable deflection of the beam $w_{\text{max}}$ that occurs at the onset of material yield is given by:

\[ w_{\text{max}} = \frac{\sigma_b}{r E_b} \left( \frac{l}{n \pi} \right)^2, \]  

(3)

and the axial strain of the beam at such a maximum deflection (i.e., the maximum allowable axial strain) is given as:

\[ \varepsilon = \frac{1}{r} \frac{\sigma_b}{E_b} \sin \left( \frac{n \pi x}{l} \right), \]  

(4)
where $0 \leq x \leq l$ and $-r \leq y \leq r$.

The proof is given in Appendix B.

The mechanical energy stored in the bent beam $V_B$ at post-buckling is obtained by integrating the strain energy density $\frac{1}{2}\sigma\varepsilon$ over the entire volume of the beam:

$$V_B = \int_{vol} \frac{1}{2}\sigma\varepsilon \, dv. \tag{5}$$

The following theorem provides the final expressions for $V_B$ and the minimal mass $m_b$ of the bent beam.

**Theorem 3.1.** Consider a beam of length $l$ subjected to a compressive force of magnitude $f$. Suppose that the beam is composed of a homogeneous and isotropic material having Young’s modulus $E_b$, yield stress $\sigma_b$, and mass density $\rho_b$. Furthermore, the maximum allowable strain of the beam at the onset of material yield is given by Eq. (4). Then, the mechanical energy stored in the bent beam $V_B$ is given as:

$$V_B = \frac{l^2\sigma_b^2}{8} \left( \frac{f}{\pi E_b^3} \right)^{\frac{1}{2}}, \tag{6}$$

and the minimal mass of the beam $m_B$ for a given beam length $l$, in terms of the applied compressive force $f$, is given as:

$$m_B = 2l^2 \rho_b \left( \frac{f}{\pi E_b} \right)^{\frac{1}{2}}. \tag{7}$$

**Proof.** For a material with Young’s modulus $E_b$, the stress $\sigma$ is given by $\sigma = E_b \varepsilon$. Substituting this into Eq. (5), the following is obtained:

$$V_B = \int_{vol} \frac{1}{2}E_b \varepsilon^2 \, dv. \tag{8}$$

The highest energy stored in the buckled beam $V_B$ is determined by substituting the maximum allowable strain from Eq. (4) into Eq. (8):

$$V_B = \int_{vol} \frac{1}{2}y^2 \frac{\sigma_b^2}{r^2 E_b} \sin^2 \left( \frac{n \pi x}{l} \right) \, dv. \tag{9}$$
The previous volume integral is evaluated as an integral over the length of the beam along the \( x \)-axis and an integral over the cross-section area of the beam:

\[
V_B = \frac{1}{2} \frac{\sigma_b^2}{r^2 E_b} \int_0^l \int_{\text{area}} y^2 \sin^2 \left( \frac{n\pi x}{l} \right) \, da \, dx = \frac{l I \sigma_b^2}{4r^2 E_b}, \tag{10}
\]

where \( I \) is the area moment of inertia. For a beam having a circular cross-section, \( I = \frac{\pi r^4}{4} \):

\[
V_B = \frac{\pi l r^2 \sigma_b^2}{16 E_b}. \tag{11}
\]

To determine an expression of \( V_B \) as a function of the magnitude of the compressive force \( f \), a formula for \( r \) as a function of \( f \) from Eq. (1) is first determined:

\[
r = \left( \frac{4l^2 f}{n^2 \pi^3 E_b} \right)^{\frac{1}{3}}, \tag{12}
\]

and then it is substituted into Eq. (11):

\[
V_B = \frac{l^2 \sigma_b^2}{8n} \left( \frac{f}{\pi E_b^3} \right)^{\frac{1}{2}}. \tag{13}
\]

From Eq. (13), it is clear that the first beam buckling mode \( (n = 1) \) results in the highest stored mechanical energy \( V_B \) in comparison to higher buckling modes \( (n > 1) \). Therefore, the first buckling mode is assumed for all subsequent calculations and the energy stored in the beam is given by Eq. (6). The mass of the beam \( m_B \) is obtained by multiplying \( \rho_b \) by its total volume:

\[
m_B = \rho_b \pi l r^2. \tag{14}
\]

The expression for the radius \( r \) as a function of \( f \) from Eq. (12) is substituted into Eq. (14) to obtain the formula for the minimal mass of the beam \( m_B \) in terms of \( f \) stated in Eq. (7).
4 D-bar Tensegrity System Analysis

The goal in this section is to derive analytical formulas for the mechanical (elastic) energy stored \( (V_D) \) and minimal mass \( (m_D) \) of a D-bar system subjected to an axial compressive force; refer to Fig. 1(b). The boundary condition assumed for the D-bar system is identical to that assumed for the bent beam in Section 3, where a single compressive force of magnitude \( f \) is applied at the end points of the D-bar. The strings and the bars of the D-bar system are composed of homogeneous, isotropic, linear elastic materials. The geometry of D-bar systems of complexity \( q = 1, 2 \) is illustrated in Fig. 3. The total length of the D-bar system is denoted by \( l \). The angle between the line connecting the end points of each D-bar unit and the line along any of its associated bars is denoted by \( \alpha \). The number of strings in each D-bar unit is denoted as \( p \), where \( p \geq 3 \). Figure 4 shows D-bar tensegrity systems of complexities \( q = 1, 2 \) with parameter \( p = 3, 4, 5 \).

\[
\begin{align*}
q = 1 & \quad l_1 & \quad l_{s1} & \quad \alpha \\
q = 2 & \quad l_{s2} & \quad l_2 & \quad \alpha
\end{align*}
\]

**Figure 3.** Geometric parameters of D-bar tensegrity systems of complexities \( q = 1, 2 \).

The length of all the bars in a D-bar system of complexity \( q \) is denoted by \( l_q \). The number of bars in a D-bar system is denoted by \( n_b \). It can be verified that:
The length of the strings introduced in each self-similar iteration is denoted as $l_{si}, i = 1, \ldots, q$. The number of strings introduced in each self-similar iteration is denoted as $n_{si}, i = 1, \ldots, q$. It can be shown that:

$$l_{si} = 2l_i \sin \left( \frac{\pi}{p} \right) \sin(\alpha) \quad i = 1, \ldots, q,$$

$$n_{si} = p(2p)^{i-1} = 2^{i-1}p^i \quad i = 1, \ldots, q. \quad (17)$$

The total number of strings, denoted by $n_s$, is given as follows:

$$n_s = \sum_{i=1}^{q} n_{si} = \sum_{i=1}^{q} 2^{i-1}p^i. \quad (20)$$
It is shown in Chapter 3 of [1] that the magnitude of the compressive force in all the bars of the D-bar system, denoted by $f_q$, is given as follows:

$$f_q = \frac{f}{(p \cos(\alpha))^q}, \quad (21)$$

and the magnitude of the tensile force in the strings introduced in each self-similar iteration, denoted by $t_i$, $i = 1, \ldots, q$, is given as:

$$t_i = \frac{f \tan(\alpha)}{p^i \sin \left( \frac{\pi}{p} \right) \cos^{i-1}(\alpha)} \quad i = 1, \ldots, q. \quad (22)$$

After presenting the geometry and force distribution of D-bar systems with arbitrary complexity, we then determine their energy stored and minimal mass under material yield and buckling constraints. To determine the highest energy stored in the D-bar system, we assume that all the strings and bars are loaded up to the onset of failure corresponding to reaching the material yield stress of the strings and the critical buckling force of the bars. Given these assumptions, the strain in the strings and bars is expected to remain within the small deformation domain. Therefore, linearized strain assumptions, such as those used in the beam calculations of Section 3, are also assumed here.

The aforesaid assumption that all the strings and bars are loaded up to the onset of failure is critical in this study. It allows us to determine the values for the minimum radii, and thereby the minimum mass, of the strings and bars*. The validity of such an assumption is enforced by calculating and employing the radii of the strings and bars corresponding to those for which the members are loaded exactly to the onset of failure (yield for strings and buckling for bars) under their calculated forces from Eqs. (22) and (21). Sizing the minimum radii of the strings and bars with respect to their member force is a feasible procedure in this study as the member forces in Eqs. (22) and (21) are independent from their cross-sectional area. Once the minimum radii of the strings and bars are determined (in terms of the applied force $f$, D-bar geometric parameters

*Using the radii for which the members are loaded exactly to the onset of failure corresponds in practice to using a factor of safety of one, which is the case when structures are designed for minimum mass (minimum resources). A larger factor of safety can be readily introduced in the calculations based on specific application requirements.
l, α, q and p, and material properties), the minimal mass and the energy stored of the D-bar system can be analytically assessed.

**Theorem 4.1.** Consider a D-bar system of complexity q, angle parameter α, length l, and p strings per D-bar unit subjected to a compressive force of magnitude f. Suppose that all the strings and bars are loaded up to the onset of failure corresponding to reaching the critical buckling force of the bars and the material yield stress of the strings. The material comprising the strings has Young’s modulus $E_s$ and yield stress $\sigma_s$, and the material comprising the bars has Young’s modulus $E_b$. Then, the total mechanical energy stored in the D-bar system $V_D$ is given as:

$$V_D = \frac{2^{q-2}}{p^2} \cos^{\frac{3q}{2}}(\alpha) \left( \frac{\pi f^3}{E_b} \right)^{\frac{1}{2}} + \frac{l f\sigma_s (\sec^{2q}(\alpha) - 1)}{2E_s}.$$  (23)

**Proof.** We start by determining the total mechanical energy stored in the bars. The magnitude of the compressive force $f_q$ of each bar in a D-bar system of complexity q provided in Eq. (21) can also be expressed in terms of their extensional stiffness $k_q$, current length $l_q$, and rest length $l_{q0}$:

$$f_q = k_q(l_q - l_{q0}),$$  (24)

where $k_q$ is given as:

$$k_q = \frac{E_b A_q}{l_q},$$  (25)

and $A_q$ is the cross-sectional area of the bars. Assuming failure by buckling at the bars of the D-bar system, the maximum value of force allowed for each bar corresponds to the critical buckling force, obtained from Euler theory of buckling [29, 30] (cf. Eq. (1)):

$$f_q = \frac{\pi^3 E_b r_q^4}{4l_q^2} = \frac{\pi E_b A_q^2}{4l_q^2}.$$  (26)

where $r_q$ is the radius of each bar (assumed to have circular cross-sections). Equation (26) is solved for $A_q$:

$$A_q = 2l_q \left( \frac{f_q}{\pi E_b} \right)^{\frac{1}{2}},$$  (27)
and the previous expression for $A_q$ is substituted in Eq. (25) to obtain an expression for $k_q$ in terms of the compressive force $f_q$:

$$k_q = 2 \left( \frac{f_q E_b}{\pi} \right)^{\frac{1}{2}}. \tag{28}$$

The elastic energy stored in each bar of a D-bar system of complexity $q$ is denoted as $V_q$ and is given as follows:

$$V_q = \frac{1}{2} k_q (l_q - l_{q0})^2. \tag{29}$$

We now proceed to determine an expression for $V_q$ in terms of $f$, $q$, $p$, $\alpha$, and material parameters. First, Eq. (24) is substituted into Eq. (29) and the following expression for $V_q$ is determined:

$$V_q = \frac{f_q^2}{2k_q}. \tag{30}$$

The expression for $k_q$ from Eq. (28) is substituted into Eq. (30):

$$V_q = \frac{1}{4} \left( \frac{\pi f_q^3}{E_b} \right)^{\frac{1}{2}}. \tag{31}$$

The expression for $f_q$ from Eq. (21) is substituted into Eq. (31) to obtain the mechanical energy stored of each bar in terms of $f$, $q$, $p$, $\alpha$, and material parameters:

$$V_q = \frac{1}{4} \left( \frac{\pi f_q^3}{E_b (p \cos(\alpha))^3 q} \right)^{\frac{1}{2}} = \frac{1}{4p^{\frac{3q}{2}} \left( \frac{\pi f_q^3}{E_b \cos^3 q(\alpha)} \right)^{\frac{1}{2}}}. \tag{32}$$

The total mechanical energy in the bars of the D-bar system is obtained by adding the energy stored in each bar:

$$V_b = n_b V_q = \frac{2^{q-2}}{p^{\frac{q}{2}}} \left( \frac{\pi f_q^3}{E_b \cos^3 q(\alpha)} \right)^{\frac{1}{2}}. \tag{33}$$

After determining the total mechanical energy in the bars of the D-bar system, we continue by determining the total mechanical energy stored in the strings. First, the magnitude of the tensile force $t_i$ of every elastic
string introduced in each self-similar iteration provided in Eq. (22) can also be expressed in terms of their extensional stiffness $k_{si}$, current length $l_{si}$, and rest length $l_{si0}$:

$$t_i = k_{si}(l_{si} - l_{si0}) \quad i = 1, \ldots, q, \quad (34)$$

where $k_{si}$ is given as:

$$k_{si} = \frac{E_s A_{si}}{l_{si}} \quad i = 1, \ldots, q, \quad (35)$$

and $A_{si}$ is the cross-sectional area of the strings. Assuming that failure occurs by material yielding at the strings and that the tensile force of the strings reaches the onset of yielding, $A_{si}$ is determined as follows:

$$t_i = \sigma_s A_{si} \quad \rightarrow \quad A_{si} = \frac{t_i}{\sigma_s}. \quad (36)$$

where $\sigma_s$ is the yield stress of the material comprising the strings.

Substituting the expression for $A_{si}$ from Eq. (36) into Eq. (35), the following is obtained:

$$k_{si} = \frac{E_s t_i}{\sigma_s l_{si}}. \quad (37)$$

The expressions for $t_i$ and $l_{si}$ from Eqs. (22) and (19), respectively, are substituted into Eq. (37) and the following is obtained:

$$k_{si} = \frac{E_s f \tan(\alpha)}{\sigma_s p^i \sin\left(\frac{\pi}{p}\right) \cos^{-1}(\alpha) l \sin\left(\frac{\pi}{p}\right) \tan(\alpha)} = \frac{2^{i-1} f E_s}{p^i \sin^2\left(\frac{\pi}{p}\right) l \sigma_s}. \quad (38)$$

The elastic energy stored in each string introduced at every self-similar iteration is denoted as $V_{si}$ and is given as follows:

$$V_{si} = \frac{1}{2} k_{si}(l_{si} - l_{si0})^2 \quad i = 1, \ldots, q. \quad (39)$$

We now proceed to determine an expression for $V_{si}$ in terms of $f$, $l$, $p$, $\alpha$, and material parameters. First, Eq. (34) is substituted into Eq. (39) and the following expression for $V_{si}$ is determined:
\[ V_{si} = \frac{t_i^2}{2k_{si}} \quad i = 1, \ldots, q, \quad (40) \]

and the expression for \( k_{si} \) from Eq. (38) is substituted into Eq. (40) to obtain the following:

\[ V_{si} = \frac{p_i \sin^2 \left( \frac{\pi}{p} \right) l \sigma_s t_i^2}{2 f E_s} \quad i = 1, \ldots, q. \quad (41) \]

By substitution of the expression for \( t_i \) from Eq. (22) into Eq. (41), the following expression for \( V_{si} \) in terms of \( f, l, p, \alpha, \) and material parameters is obtained:

\[ V_{si} = \frac{lf \sigma_s \tan^2(\alpha)}{(2p)^i E_s \cos^{2i-2}(\alpha)} \quad i = 1, \ldots, q. \quad (42) \]

The total mechanical energy stored in the strings of a D-bar system of complexity \( q \), denoted as \( V_s \), is determined as the sum of the energy stored in each of its strings as follows:

\[ V_s = \sum_{i=1}^{q} n_{si} V_{si}. \quad (43) \]

Substituting Eqs. (18) and (42) into the previous equation, the following is obtained:

\[ V_s = \sum_{i=1}^{q} p^{2i-1} \frac{lf \sigma_s \tan^2(\alpha)}{(2p)^i E_s \cos^{2i-2}(\alpha)} = \frac{lf \sigma_s \tan^2(\alpha)}{2E_s} \sum_{i=1}^{q} \frac{1}{\cos^{2i-2}(\alpha)}. \quad (44) \]

To simplify the formula for the energy stored in a D-bar system in Eq. (44), we consider the following trigonometric identity that can be verified by the reader:

\[ \tan^2(\alpha) \sum_{i=1}^{q} \frac{1}{\cos^{2i-2}(\alpha)} = \sec^2q(\alpha) - 1. \quad (45) \]

Substituting the previous identity into Eq. (44), we obtain the following formula for the mechanical energy stored in the strings of the D-bar system:
\[ V_s = \frac{lf \sigma_s (\sec^2 q(\alpha) - 1)}{2E_s}. \] (46)

The total mechanical energy in the D-bar system, denoted as \( V_D \), is obtained by adding the energy stored in the bars \( V_b \) and the energy stored in the strings \( V_s \):

\[ V_D = V_b + V_s. \] (47)

By substitution of \( V_b \) from Eq. (33) and \( V_s \) from Eq. (46) into Eq. (47), the final expression for \( V_D \) stated in Eq. (23) is obtained.

We now proceed to provide a formula for the minimal mass \( m_D \) of the D-bar system in terms of the magnitude of the applied compressive force \( f \).

**Theorem 4.2.** Consider a D-bar system of length \( l \), complexity \( q \), angle parameter \( \alpha \), and \( p \) strings per D-bar unit subjected to a compressive force of magnitude \( f \). Suppose that the material comprising the bars has Young’s modulus \( E_b \) and mass density \( \rho_b \) while the material comprising the strings has yield stress \( \sigma_s \) and mass density \( \rho_s \). Then, the minimal mass of the D-bar system \( m_D \) is given as:

\[ m_D = \frac{\frac{q^2}{2} \rho_b l^2}{2^{q-1} \cos \frac{q}{2}(\alpha)} \left( \frac{f}{\pi E_b} \right)^{\frac{1}{2}} + \frac{lf \rho_s (\sec^2 q(\alpha) - 1)}{\sigma_s}. \] (48)

The proof is given in Appendix C. Chapter 3 of [1] has a formula for mass \( m_D \) that includes additional strings connecting the end nodes of each D-bar unit, which are not required in a minimal mass design for a D-bar. From Eqs. (23) and (48), it is clear that a D-bar system stores more energy and requires less mass for lower values of the number of strings in each D-bar unit \( (p) \). Therefore, the minimum value of \( p \), corresponding to 3, is assumed for the remainder of the paper.

5 Comparison of Energy Stored and Mass of a D-bar System and a Bent Beam

5.1 Single Component Assessment

To compare D-bar systems with bent buckled beams at the single component level, the ratio of energy stored in a D-bar to that of a bent
beam \((V_D/V_B)\) and the ratio of the mass of a D-bar to that of a bent beam \((m_D/m_B)\) are determined. The compressive force \(f\) and length \(l\) of the bent buckled beam and the D-bar system are assumed equal. The ratio \(V_D/V_B\) is determined by dividing \(V_D\) from Eq. (23) by \(V_B\) from Eq. (6):

\[
\frac{V_D}{V_B} = \frac{2^{q+1}}{3^q \cos \frac{3q}{2} (\alpha)} \left( \frac{\pi E_b}{\sigma_b^2} \right) \left( \frac{f^{\frac{1}{2}}}{l} \right)^2 + \frac{4 \sigma_s (\sec^{2q}(\alpha) - 1) (\pi E_b^3)^{\frac{1}{2}}}{E_s \sigma_b^2} \left( \frac{f^{\frac{1}{2}}}{l} \right)
\]  

(49)

For given values of \(\alpha\), \(f\), \(l\), and material parameters, the ratio \(V_D/V_B\) in Eq. (49) monotonically increases as \(q\) increases. Moreover, the limit of \(V_D/V_B\) as \(q\) approaches infinity is infinity. Therefore, there always exist D-bar systems that store more energy than a buckled beam (i.e., for which \(V_D/V_B > 1\)). The minimum value of \(q\) for such D-bar systems can be determined by increasing \(q\) until the value of \(V_D/V_B\) becomes greater than 1. In order to further comparatively assess the feasibility of D-bar systems against buckled beams, their mass ratio must also be evaluated. The ratio \(m_D/m_B\) is determined by dividing \(m_D\) from Eq. (48) by \(m_B\) from Eq. (7):

\[
\frac{m_D}{m_B} = \frac{3^q \cos \frac{5q}{2} (\alpha)}{2^q} + \frac{\rho_s (\sec^{2q}(\alpha) - 1) (\pi E_b)^{\frac{1}{2}}}{2 \sigma_s \rho_b} \left( \frac{f^{\frac{1}{2}}}{l} \right)
\]  

(50)

It is noted that the length \(l\) and applied force magnitude \(f\) appear only in the parameter \(f^{\frac{1}{2}}/l\) in Eqs. (49) and (50). This parameter can be used as a scaling factor to evaluate D-bar energy storage components across scales.

We now proceed to provide quantitative comparisons between D-bar systems and bent buckled beams using the stored energy and mass ratios provided in Eqs. (49) and (50). Conventional material parameters of aluminum are assumed for the beam, bars, and strings \((E_b = E_s = 60\ \text{GPa}, \sigma_b = \sigma_s = 110\ \text{MPa}, \rho_b = \rho_s = 2700\ \text{kg/m}^3)\). These values are used to generate the contour plots of \(V_D/V_B\) shown in Fig. 5 for \(f^{\frac{1}{2}}/l = 100\ \text{N}^{\frac{1}{2}}/\text{m}, 150\ \text{N}^{\frac{1}{2}}/\text{m},\) and \(200\ \text{N}^{\frac{1}{2}}/\text{m}\). The axes of the contour plots correspond to the D-bar system complexity \(q\) and angle \(\alpha\) (refer to Fig. 3). As indicated in Fig. 5, as the value of \(f^{\frac{1}{2}}/l\) increases, there exist D-bar systems of lower complexity \(q\) and lower \(\alpha\) that store more energy than that of bent buckled beams (i.e., for which \(V_D/V_B > 1\)). D-bars of low \(q\) are desirable as they require fewer members/joints and are easier to manufacture while D-bars of low \(\alpha\) occupy smaller volumes. It is
also noted that even though the results show that increasing complexity would generate D-bar systems of higher energy storing capabilities, manufacturing and scaling issues would prevent the synthesis of D-bar systems of very high complexity.

Figure 5. Contours of ratio of mechanical energy stored for the D-bar system to the bent beam \( \frac{V_D}{V_B} \). The shaded areas show the regions for which \( \frac{V_D}{V_B} > 1 \). Material parameters of aluminum are assumed for the beam, bars, and strings. D-bar structures for the red dots corresponding to complexities \( q = 1, 2, 3 \) and \( \alpha = 7.5^\circ \) are shown in Fig. 8.

Contours of the ratio for the mass of the D-bar system to the bent beam \( \frac{m_D}{m_B} \) are shown in Fig. 6. The design space of D-bar systems that have lower mass than bent beams (i.e., for which \( \frac{m_D}{m_B} < 1 \)) is shown in the shaded regions. It is observed that there are D-bar systems of any of the shown complexities for which \( \frac{m_D}{m_B} < 1 \) provided that \( \alpha \) is lower than approximately 17°.
Figure 6. Contours of ratio of mass for the D-bar system to the bent beam ($m_D/m_B$). The shaded areas show the regions for which $m_D/m_B < 1$. Material parameters of aluminum are assumed for the beam, bars, and strings. D-bar structures for the red dots corresponding to complexities $q = 1, 2, 3$ and $\alpha = 7.5^\circ$ are shown in Fig. 8.

A favorable D-bar system stores more energy while requiring less mass to take the same compressive load than a bent buckled beam. The design space of D-bar systems that meet such requirements corresponds to the intersection of the regions where $V_D/V_B > 1$ and $m_D/m_B < 1$. Based on Figs. 5 and 6, Fig. 7 shows the regions where D-bar systems are favorable in terms of having lower mass and higher energy storage. The trends in Fig. 7 show that D-bar systems with lower values of $q$ and $\alpha$ have better performance than bent beams for higher values of $f^2/l$, which correspond to higher values of compressive force $f$ and/or lower values of length $l$. This is observed as the design region for which $V_D/V_B > 1$ and $m_D/m_B < 1$ expands towards lower values of $q$ and $\alpha$ as $f^2/l$ is increased in Fig. 7. These trends indicate that D-bar systems are more favorable as mass-efficient energy absorption components in systems subjected to high...
impact loads (large values of $f$) and having limited volumes (low values of $l$).

A single beam (in its initial configuration) and D-bar systems of complexities $q = 1, 2, 3$ indicated with red dots in Figs. 5–7 are illustrated in Fig. 8. The calculated radii of the beam (Eq. (12)), bars (Eq. (77)), and strings (Eq. (82)) are considered in such figures. It is worth noting that the radii of the bars of the D-bar system decrease by increasing the complexity $q$. No intersection between any bar and string was observed for all complexities in Fig. 8.

Figure 7. Regions showing the design space of D-bar systems for which $V_D/V_B > 1$ and $m_D/m_B < 1$. 
Figure 8. Schematics of a beam (initial configuration prior to buckling) and D-bars of different complexities. The axes units are in meters. Values of $\alpha = 7.5^\circ$, $l = 0.5$ m, $f^{1/2}/l = 150$ N$^{1/2}$/m, and material parameters of aluminum for the beam, bars, and strings are assumed. The calculated radii of the beam, bars, and strings are considered in the schematics.
5.2 Assessment as Parts of a Larger Tensegrity Structure

Having compared the energy stored and mass for single D-bar systems and bent beams, this section considers an example of a planetary lander to compare D-bar systems and bent buckled beams as parts of a larger tensegrity structure. The geometry and boundary conditions of the tensegrity lander are first described. Then, the equilibrium equations used to calculate the forces in each member are provided. Finally, the energy stored and the total mass of a lander with bent beams are compared against those of a lander with D-bar systems replacing the beams.

The geometry of the tensegrity lander developed in [19] for planetary exploration is considered for this assessment. The lander has 6 compressive members of length $L$ and 24 tensile members of length $(3/8)^2L$. As illustrated in Fig. 9(a), vertical compressive forces are applied to the top three nodes of the structure and the bottom three nodes are kept fixed to simulate the impact conditions. The total force applied to the top surface is denoted as $F$.

To calculate the forces in each member, the equilibrium equations for tensegrity systems developed in [31] are used. First, let $n_i \in \mathbb{R}^3$ be the position vector of the $i$th node and $N$ be the matrix containing the node position vectors $N = [n_1 \ n_2 \ \cdots]$. Also, let $w_i \in \mathbb{R}^3$ be the vector of external forces applied at the $i$th node and $W$ be the matrix containing such vectors $W = [w_1 \ w_2 \ \cdots]$. The magnitude of the tensile force per unit length at the $i$th string is denoted $\gamma_i$ and the magnitude of the compressive force per unit length at the $j$th bar is denoted $\lambda_j$. Define the vectors $\gamma = [\gamma_1 \ \gamma_2 \ \cdots]^T$ and $\lambda = [\lambda_1 \ \lambda_2 \ \cdots]^T$. Denote $\hat{\gamma}$ and $\hat{\lambda}$ as the square matrices whose diagonal components correspond to the elements of $\gamma$ and $\lambda$. The equilibrium equations are then written as follows [31]:

$$N(C_s^T \hat{\gamma} C_s - C_b^T \hat{\lambda} C_b) = W,$$

where $C_b$ and $C_s$ are connectivity matrices for bars and strings, respectively [32]. The node positions for the geometry shown in Fig. 9 are given by:
Figure 9. (a) Geometry and boundary conditions of a tensegrity lander [19]. The strings are shown in red and have length \((\frac{L}{3})^\frac{1}{2}\) \(L\) and the beams are shown in blue and have length \(L\). The yellow triangles indicate the nodes that are fixed. (b) Node labels (see Eq. (52)). (c) Tensegrity lander obtained by replacing the beams with D-bar systems of complexity \(q = 1\). (d) Tensegrity lander obtained by replacing the beams with D-bar systems of complexity \(q = 2\).

\[
N = \begin{bmatrix}
\frac{L}{4} & \frac{L}{4} & -\frac{L}{4} & 0 & 0 & 0 & 0 & \frac{L}{2} & -\frac{L}{2} & -\frac{L}{2} \\
0 & \frac{L}{4} & -\frac{L}{4} & \frac{L}{4} & 7L & -3L & \sqrt{5}L & 0 & 0 & \frac{L}{2} \\
0 & 0 & \frac{L}{4} & -\frac{L}{4} & \frac{L}{4} & 2L & \sqrt{5}L & 0 & \sqrt{5}L & 0 \\
0 & 2L & 0 & \frac{L}{4} & \frac{L}{4} & \frac{L}{4} & \frac{L}{4} & \frac{L}{4} & \frac{L}{4} & \frac{L}{4} \\
\end{bmatrix},
\]

and the external forces are given by:

\[
w_2 = w_4 = w_5 = \begin{bmatrix} 0 & 0 & -\frac{F}{3} \end{bmatrix}^T.
\]
The reaction forces $w_1$, $w_3$, and $w_8$ at the fixed nodes can be calculated in a subsequent step after determining the loading of each tensile and compressive member. Equation (51) can also be written as:

$$\begin{bmatrix} (C_s^T \otimes I_3) \hat{S} & -(C_b^T \otimes I_3) \hat{B} \end{bmatrix} \begin{bmatrix} \gamma \\ \lambda \end{bmatrix} = w,$$

where $[b_1 \ b_2 \ \cdots] = NC_b^T$ and $[s_1 \ s_2 \ \cdots] = NC_s^T$ are respectively the vectors along the length of each bar and each string, $\hat{B} = \text{b.d.}(b_1, b_2, \cdots)$, $\hat{S} = \text{b.d.}(s_1, s_2, \cdots)$, $w^T = [w_1^T \ w_2^T \ \cdots]$, b.d. is the block diagonal operator, and $\otimes$ is the Kronecker product.

The rows of Eq. (54) associated with the fixed nodes (nodes 1, 3, and 8 in this example) are removed. Subsequently, such an equation is solved numerically to find $\gamma$ and $\lambda$ such that the total sum of forces in all members is minimized with a constraint of $\gamma$ and $\lambda$ being positive, i.e., all the strings in the structure are in tension and all the bars are in compression. We calculate the compressive forces in the bars and the tensile forces in the strings under different landing configurations that correspond to different boundary conditions, i.e., as if the structure touched the ground at nodes $\{1,3,8\}$ (shown in Fig. 9), $\{3,7,10\}$, $\{2,4,5\}$, $\{2,6,11\}$, etc. Among different boundary conditions, the maximum load in each bar and each string is employed to design each individual element. From the symmetry of the structure, all the bars will have the same maximum compressive load, denoted as $f$, and all the strings will have the same maximum tensile load, denoted by $t$, among all the different landing configurations.

**Proposition 5.1.** Consider the tensegrity lander whose geometry is illustrated in Fig. 9. The compressive members of the lander take a maximum load $f$ and are comprised of a material with Young’s modulus $E_b$, and mass density $\rho_b$. The strings take a maximum tensile load $t$ and are comprised of a material with Young’s modulus $E_s$, yield stress $\sigma_s$, and mass density $\rho_s$. In a minimum mass design, if the compressive members of the lander are beams that undergo buckling, its stored energy $V_{SB}$ and mass $m_{SB}$ are given as:

$$V_{SB} = (24) \left( \frac{3}{8} \right)^{\frac{1}{2}} \frac{\sigma_s L t}{2E_s} + (6) \frac{L^2 \sigma_b^2}{8} \left( \frac{f}{\pi E_b^3} \right)^{\frac{1}{2}},$$

where $\rho = \rho_b = \rho_s$. The design mass is then

$$m_{SB} = (6) \frac{1}{2} \frac{L^2 \sigma_b^2}{8} \left( \frac{f}{\pi E_b^3} \right)^{\frac{1}{2}}.$$
\[ m_{SB} = (24) \left( \frac{3}{8} \right)^{\frac{1}{2}} \frac{\rho_s L t}{\sigma_s} + (6) 2L^2 \rho_b \left( \frac{f}{\pi E_b} \right)^{\frac{1}{2}} . \]  

(56)

Furthermore, if the compressive members of the lander are D-bar systems, its stored energy \( V_{SD} \) and mass \( m_{SD} \) are given as:

\[
\begin{align*}
V_{SD} &= (24) \left( \frac{3}{8} \right)^{\frac{1}{2}} \frac{\sigma_s L t}{2E_s} \\
&+ (6) \left( \frac{2^{q-2}}{3^q \cos^{\frac{3q}{2}}(\alpha)} \left( \frac{\pi f^3}{E_b} \right)^{\frac{1}{2}} + \frac{L f \sigma_s (\sec^{2q}(\alpha) - 1)}{2E_s} \right), \\

m_{SD} &= (24) \left( \frac{3}{8} \right)^{\frac{1}{2}} \frac{\rho_s L t}{\sigma_s} \\
&+ (6) \left( \frac{3^q \rho_b L^2}{2^{q-1} \cos^{\frac{5q}{2}}(\alpha)} \left( \frac{f}{\pi E_b} \right)^{\frac{1}{2}} + \frac{L f \rho_s (\sec^{2q}(\alpha) - 1)}{\sigma_s} \right).
\end{align*}
\]

(57)

(58)

**Proof.** The energy stored in each of the strings is determined using the procedure followed in Eqs. (34)–(41). This energy stored is denoted \( V_S \) and is given as:

\[ V_S = \left( \frac{3}{8} \right)^{\frac{1}{2}} \frac{\sigma_s L t}{2E_s} \]  

(59)

The mass of each string is denoted \( m_S \) and is found using the approach described in Eqs. (79) and (80). The expression for \( m_S \) is given as follows:

\[ m_S = \left( \frac{3}{8} \right)^{\frac{1}{2}} \frac{\rho_s L t}{\sigma_s} \]  

(60)

If the compressive members are beams that undergo buckling, their energy stored \( (V_B) \) and mass \( (m_B) \) as functions of their length \( L \) and compressive force \( f \) are given by Eqs. (6) and (7), respectively. Conversely, if the compressive members are D-bar systems, their energy...
stored \(V_D\) and mass \(m_D\) as functions of \(L\) and \(f\) are given by Eqs. (23) and (48), respectively.

The energy stored \(V_{SB}\) and mass \(m_{SB}\) of a lander with beams that undergo buckling are given as:

\[
V_{SB} = 24V_S + 6V_B, \tag{61}
\]

\[
m_{SB} = 24m_S + 6m_B. \tag{62}
\]

Substituting Eqs. (59) and (6) into Eq. (61), the expression for \(V_{SB}\) of Eq. (55) is obtained. Substituting Eqs. (60) and (7) into Eq. (62), the expression for \(m_{SB}\) of Eq. (56) is obtained.

Similarly, the energy stored \(V_{SD}\) and mass \(m_{SD}\) of a lander with D-bar systems as its compressive members are given as:

\[
V_{SD} = 24V_S + 6V_D, \tag{63}
\]

\[
m_{SD} = 24m_S + 6m_D. \tag{64}
\]

Substituting Eqs. (59) and (23) into Eq. (63), the expression for \(V_{SD}\) of Eq. (57) is obtained. Substituting Eqs. (60) and (48) into Eq. (64), the expression for \(m_{SD}\) of Eq. (58) is obtained. This concludes the proof. \(\square\)

Having determined the mechanical energy stored and the total mass of tensegrity landers whose compressive members are beams that undergo buckling or D-bar systems, the two designs are now compared. Conventional material parameters of aluminum are assumed for the beams, bars, and strings are assumed \((E_b = E_s = 60 \text{ GPa}, \sigma_b = \sigma_s = 110 \text{ MPa}, \rho_b = \rho_s = 2700 \text{ kg/m}^3)\). Values of \(F = 15000 \text{ N}\) and \(L = 1 \text{ m}\) are also assumed. These values of \(F\) and \(L\) resulted in maximum member forces \(f = 22360 \text{ N}\) and \(t = 9128 \text{ N}\). Contours for the ratios of mechanical energy stored and total mass for the two designs are provided in Fig. 10. In the \(m_{SD}/m_{SB}\) contour, it is observed that landers with D-bar systems have lower mass than landers with bent beams for the entire studied domain of \(q\), provided that \(\alpha\) is below approximately 17°. The design region of landers with D-bar systems that store more energy \((V_{SD}/V_{SB} > 1)\) and require less mass \((m_{SD}/m_{SB} < 1)\) than landers with bent buckled beams is also shown in Fig. 10. Such a design region spans the entire studied domain.
domain of $q$, with lower values of $\alpha$ feasible by increasing $q$. At $q = 3$, any value of $\alpha$ below approximately $17^\circ$ generates a favorable D-bar design in terms of energy stored and mass. Overall, the results in Fig. 10 indicate that designs of tensegrity landers can be significantly enhanced if beams that undergo buckling during vehicle impact are replaced with D-bar systems of low complexities.

![Figure 10](image.png)

**Figure 10.** Contours of ratios of mechanical energy stored ($V_{SD}/V_{SB}$) and total mass ($m_{SD}/m_{SB}$) for tensegrity landers whose compressive members are D-bar systems or beams that undergo buckling. The design region with $V_{SD}/V_{SB} > 1$ and $m_{SD}/m_{SB} < 1$ is also shown. Material parameters of aluminum are assumed for the beams, bars, and strings.

### 6 Summary and Concluding Remarks

This paper presented an analytical study of D-bar tensegrity systems for applications as lightweight components for mechanical energy absorption. This work was motivated by aerospace structures such as planetary landers that necessitate these lightweight components to absorb energy from large impact loads (to protect their interior systems and payload) while...
requiring minimal mass. Recent works proposed bent buckled beams used in “flexible-bar tensegrity” concepts as energy absorption components in planetary landers. Here, conversely, the approach was to absorb energy from external loads as mechanical (elastic) energy in the strings and bars of D-bar systems without triggering local instabilities such as buckling (thereby enabling more reliable structures).

Previous studies have demonstrated that D-bar systems support compressive loads with minimal mass compared to continuum structures such as prismatic columns. This work adds to the body of knowledge of tensegrity structures by developing analytical formulas to describe the energy absorption properties of D-bar systems. The analytical equations are applicable to D-bar systems of any complexity, number of strings in each D-bar unit, elastic material properties, system length, and applied compressive force.

The energy stored in D-bar systems was compared with that of bent buckled beams. The comparisons were made between D-bar systems and bent buckled beams as isolated components subjected to a compressive load and components of planetary landers. The following was concluded from the comparative studies:

- In both comparisons (component-level and as parts of a larger tensegrity structure), the results showed that D-bar systems of low complexity allow for higher energy storage and lower mass than bent buckled beams
- D-bar systems of lower complexities had better performance (in terms of energy stored and mass) than bent beams for higher compressive force and lower system length. These trends indicate that D-bar systems are more favorable as mass-efficient energy absorption components in structures subjected to high impact loads and placed in small volumes (such as aerospace systems)

Therefore, it is finally concluded that D-bar systems can enhance the design of planetary landers and other applications that require lightweight mechanical energy absorption components.

A Nomenclature

A.1 Scalars
A_q \quad \text{Cross-sectional area of each bar of a D-bar system of complexity } q, \text{ m}^2

A_{si} \quad \text{Cross-sectional area of each string introduced at the } i^{th} \text{ iteration of a D-bar system, m}^2

E_b \quad \text{Young’s modulus of the beam and the bars, N/m}^2

E_s \quad \text{Young’s modulus of the strings, N/m}^2

F \quad \text{Total force applied at the top surface of the lander, N}

f \quad \text{Magnitude of compressive force, N}

f_q \quad \text{Magnitude of the compressive force at each bar of a D-bar system of complexity } q, \text{ N}

I \quad \text{Area moment of inertia, m}^4

k_q \quad \text{Extensional stiffness of each bar of a D-bar system of complexity } q, \text{ N/m}

k_{si} \quad \text{Extensional stiffness of each string introduced at the } i^{th} \text{ iteration of a D-bar system, N/m}

L \quad \text{Length of all the beams and D-bar systems in a lander, m}

l \quad \text{Total length of the beam and the D-bar system, m}

l_q \quad \text{Length of each bar of a D-bar system of complexity } q, \text{ m}

l_{q0} \quad \text{Rest length of each bar of a D-bar system of complexity } q, \text{ m}

l_{si} \quad \text{Length of each string introduced at the } i^{th} \text{ iteration of a D-bar system, m}

l_{si0} \quad \text{Rest length of each string introduced at the } i^{th} \text{ iteration of a D-bar system, m}

m_b \quad \text{Total mass of the bars in a D-bar system, kg}

m_B \quad \text{Mass of the bent beam, kg}

m_D \quad \text{Mass of a D-bar system, kg}

m_{SB} \quad \text{Mass of the lander with bent beams, kg}

m_{SD} \quad \text{Mass of the lander with D-bar systems, kg}

m_q \quad \text{Mass of each bar of a D-bar system of complexity } q, \text{ kg}

m_s \quad \text{Total mass of the strings in a D-bar system, kg}

m_{si} \quad \text{Mass of each string introduced at the } i^{th} \text{ iteration of a D-bar system, kg}

n \quad \text{Buckling mode}

n_b \quad \text{Number of bars in a D-bar system}

n_s \quad \text{Number of strings in a D-bar system}

n_{si} \quad \text{Number of strings introduced at the } i^{th} \text{ iteration of a D-bar system}
\( p \) Number of strings in each D-bar unit
\( q \) Complexity of a D-bar system
\( r \) Radius of the cross-section of the beam, m
\( r_{si} \) Radius of the cross-section of each string introduced at the \( i^{th} \) iteration of a D-bar system, m
\( r_q \) Radius of the cross-section of each bar of a D-bar system of complexity \( q \), m
\( t \) Magnitude of the tensile force at each string in the lander which is not a part of a D-bar system, N
\( t_i \) Magnitude of the tensile force at each string introduced at the \( i^{th} \) iteration of a D-bar system, N
\( V_b \) Total elastic energy stored in the bars of a D-bar system, J
\( V_B \) Elastic energy stored in the bent beam, J
\( V_D \) Elastic energy stored in a D-bar system, J
\( V_{SB} \) Elastic energy stored in the lander with bent beams, J
\( V_{SD} \) Elastic energy stored in the lander with D-bar systems, J
\( V_q \) Elastic energy stored in each bar of a D-bar system, J
\( V_s \) Total elastic energy stored in the strings of a D-bar system, J
\( V_{si} \) Elastic energy stored in each string introduced at the \( i^{th} \) iteration of a D-bar system, J
\( w \) Deflection of the beam, m
\( w_{\text{max}} \) Maximum deflection of the beam, m
\( \alpha \) Angle parameter of a D-bar system, radians
\( \gamma_i \) Tensile force per unit length at the \( i^{th} \) string of a lander, N/m
\( \varepsilon \) Axial strain
\( \varepsilon_b \) Axial strain when yield stress is reached
\( \kappa \) Curvature of the beam, 1/m
\( \lambda_i \) Compressive force per unit length at the \( i^{th} \) beam or D-bar system of a lander, N/m
\( \rho_b \) Mass density of the beam and the bars, kg/m\(^3\)
\( \rho_s \) Mass density of the strings, kg/m\(^3\)
\( \sigma_b \) Yield stress of the beam and the bars, N/m\(^2\)
\( \sigma_s \) Yield stress of the strings, N/m\(^2\)

### A.2 Vectors and Matrices

\( B \) Bar matrix
$b_i$ Vector along the length of the $i^{th}$ bar
$C_b$ Bar connectivity matrix
$C_s$ String connectivity matrix
$I_3$ $3 \times 3$ identity matrix
$n_i$ Position vector of the $i^{th}$ node
$N$ Node matrix
$S$ String matrix
$s_i$ Vector along the length of the $i^{th}$ string
$w$ Vector containing the force vectors $w_i$ applied at all the nodes
$w_i$ Force vector applied at the $i^{th}$ node
$W$ Applied force matrix
$\gamma$ Vector of tensile forces per unit length of all the strings
$\lambda$ Vector of compressive forces per unit length of all the bars

### B Proof of Lemma 3.1

Let us consider a section of a beam such as that illustrated in Fig. 11. The axial strain $\varepsilon$ at any location in the beam is given as follows:

$$\varepsilon = \varepsilon_0 - y \kappa = \varepsilon_0 - y \frac{d^2 w}{dx^2}. \quad (65)$$

where $-r \leq y \leq r$ and $\varepsilon_0$ is the strain at $y = 0$. Here, we assume that $\varepsilon_0 = 0$ (i.e., that the bar is undergoing bending deformation only). By making this assumption, $\varepsilon$ is given as:

$$\varepsilon = -y \frac{d^2 w}{dx^2}. \quad (66)$$

**Figure 11.** (a) Section of a beam aligned with the $x$-axis. (b) Circular beam cross-section of radius $r$. 

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The first and second derivatives of deflection $w$ from Eq. (2) with respect to the axial coordinate $x$ are given as follows:

$$\frac{dw}{dx} = w_{\text{max}} \frac{n\pi}{l} \cos\left(\frac{n\pi x}{l}\right), \quad \frac{d^2w}{dx^2} = -w_{\text{max}} \left(\frac{n\pi}{l}\right)^2 \sin\left(\frac{n\pi x}{l}\right).$$  \hspace{1cm} (67)

By substitution of Eq. (67) into Eq. (66), the following expression of the axial strain $\varepsilon$ is obtained:

$$\varepsilon = y w_{\text{max}} \left(\frac{n\pi}{l}\right)^2 \sin\left(\frac{n\pi x}{l}\right).$$  \hspace{1cm} (68)

For the linear elastic material comprising the beam, $\varepsilon$ is related to the axial stress $\sigma$ via Hooke’s law:

$$\varepsilon = \frac{\sigma}{E_b}.$$  \hspace{1cm} (69)

As stated in Section 3, at post-buckling it is assumed that failure of the beam to support the applied force occurs when material yielding starts. This failure condition is met when the stress at any point in the beam reaches the material yield stress, denoted by $\sigma_b$. At such a stress, the strain of the material at failure $\varepsilon_b$ is determined via Eq. (69):

$$\varepsilon_b = \frac{\sigma_b}{E_b}.$$  \hspace{1cm} (70)

From Eq. (68), the maximum absolute value of strain in the beam occurs at:

$$y = y_{\text{max}}, \quad \sin\left(\frac{n\pi x}{l}\right) = \pm 1,$$  \hspace{1cm} (71)

where $y_{\text{max}}$ is the largest value (in magnitude) of the off-axis coordinate $y$ in the cross-section of the beam. Assuming that the beam has a circular cross-section of radius $r$, then $y_{\text{max}} = \pm r$. At post-buckling failure due to material yield, the maximum absolute value of strain corresponds to $\varepsilon_b$. By substituting Eqs. (70) and (71) into Eq. (68), the following expression is obtained:

$$\varepsilon_b = \frac{\sigma_b}{E_b} = r w_{\text{max}} \left(\frac{n\pi}{l}\right)^2.$$  \hspace{1cm} (72)
The previous equation is rearranged to obtain the expression for $w_{\text{max}}$ as a function of the material yield stress $\sigma_b$ provided in Eq. (3).

The expression for $w_{\text{max}}$ from Eq. (3) is substituted in Eq. (68) to obtain the maximum allowable strain field in the bent beam:

$$
\varepsilon = y \frac{\sigma_b}{r E_b} \left( \frac{n \pi}{l} \right)^{-2} \left( \frac{n \pi}{l} \right)^2 \sin \left( \frac{n \pi x}{l} \right) = y \frac{\sigma_b}{r E_b} \sin \left( \frac{n \pi x}{l} \right). 
$$

(73)

cf. Eq. (4). This concludes the proof.

C Proof of Theorem 4.2

Let us consider a D-bar system of length $l$, complexity $q$, angle parameter $\alpha$, and $p$ strings per D-bar unit subjected to a compressive force of magnitude $f$. The material comprising the bars has Young’s modulus $E_b$ and mass density $\rho_b$ while the material comprising the strings has yield stress $\sigma_s$ and mass density $\rho_s$. The mass of each bar, denoted by $m_q$, is given as:

$$
m_q = \pi \rho_b l_q r_q^2.
$$

(74)

From Eqs. (26) and (74), it follows that:

$$
m_q = 2 \rho_b l_q^2 \left( \frac{f_q}{\pi E_b} \right) \frac{1}{2}.
$$

(75)

By substitution of the expressions for $f_q$ and $l_q$ from Eqs. (21) and (15), respectively, the following equation for the mass of each bar is obtained:

$$
m_q = \frac{\rho_b l^2}{2^{2q-1} p^2 \cos^{3 \frac{3}{2}} (\alpha)} \left( \frac{f}{\pi E_b} \right)^{\frac{1}{2}}.
$$

(76)

It can also be verified from Eqs. (74) and (76) that the radius $r_q$ of each bar is given as:

$$
r_q = \left( \frac{l}{2^{2q-1} p^2 \cos^{3 \frac{3}{2}} (\alpha)} \right)^{\frac{1}{2}} \left( \frac{f}{\pi^3 E_b} \right)^{\frac{1}{4}}.
$$

(77)

We now add the mass of each bar in the D-bar system to obtain the total mass of the bars, denoted by $m_b$: 

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\[ m_b = n_b m_q = (2p)^q \frac{\rho_b l^2}{2^{2q-1} p^q \cos^{\frac{5q}{2}}(\alpha)} \left( \frac{f}{\pi E_b} \right)^{\frac{1}{2}} = \frac{p^q \rho_b l^2}{2^{q-1} \cos^{\frac{5q}{2}}(\alpha)} \left( \frac{f}{\pi E_b} \right)^{\frac{1}{2}}. \]

The mass of each string introduced at the \( i \)th self-similar iteration is denoted by \( m_{si}, i = 1, \ldots, q \), and is determined as follows:

\[ m_{si} = \rho_s l_{si} A_{si} \quad i = 1, \ldots, q. \quad (79) \]

An expression for \( A_{si} \) determined by assuming failure due to material yielding at the strings is provided in Eq. (36). We substitute such an expression into Eq. (79) and obtain the following function for \( m_{si} \):

\[ m_{si} = \frac{\rho_s l_{si} t_i}{\sigma_s} \quad i = 1, \ldots, q. \quad (80) \]

The expressions for \( t_i \) and \( l_{si} \) from Eqs. (22) and (19), respectively, are then substituted into the previous equation:

\[ m_{si} = \frac{lf \rho_s \tan^2(\alpha)}{2^{i-1} p^i \sigma_s \cos^{2i-2}(\alpha)} \quad i = 1, \ldots, q. \quad (81) \]

It can also be verified that the radius \( r_{si} \) of each string introduced at the \( i \)th self-similar iteration is given as:

\[ r_{si} = \left( \frac{f \tan(\alpha)}{\pi p^i \sin \left( \frac{\pi}{p} \right) \sigma_s \cos^{i-1}(\alpha)} \right)^{\frac{1}{2}} \quad i = 1, \ldots, q. \quad (82) \]

We now add the mass of all the strings in the D-bar system to obtain the total mass of the strings, denoted by \( m_s \):

\[ m_s = \sum_{i=1}^{q} n_{si} m_{si} = \sum_{i=1}^{q} 2^{i-1} p^i \frac{lf \rho_s \tan^2(\alpha)}{2^{i-1} p^i \sigma_s \cos^{2i-2}(\alpha)}, \quad (83) \]

\[ m_s = \frac{lf \rho_s \tan^2(\alpha)}{\sigma_s} \sum_{i=1}^{q} \frac{1}{\cos^{2i-2}(\alpha)}. \quad (84) \]
Using the identity in Eq. (45), the following expression for the total mass of the strings in a D-bar system is determined:

\[ m_s = \frac{l f \rho_s (\sec^2 q(\alpha) - 1)}{\sigma_s}. \]  

(85)

The total mass of the D-bar system, \( m_D \), is obtained by adding the total mass of the bars and the total mass of the strings:

\[ m_D = m_b + m_s. \]  

(86)

The final expression for \( m_D \) stated in Eq. (48) is obtained by substituting \( m_b \) and \( m_s \) from Eqs. (78) and (85), respectively, into Eq. (86). This concludes the proof. \( \square \)

**Declaration of Conflicting Interests**

The Authors declare that there is no conflict of interest.

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**References**


