"Leibniz versus Kant on Euclid's Axiom of Parallels"

ABSTRACT: It is well known that geometrical research on Euclid's axiom of parallels led at the end of the nineteenth and beginning of the twentieth century to a fierce philosophical debate about the tenability of Leibnizian and Kantian philosophies of mathematics. What is less known is that a similar debate about the standing of Euclid's axiom in Leibniz's and Kant's philosophies broke out already in Kant's lifetime. This debate reached its high point in the Kant-Eberhard controversy, with the mathematician Abrahan Kästner joining Eberhard in arguing that Euclid's axiom could be demonstrated only in a Leibnizian way from improved definitions, while the mathematician and Kantian disciple Johann Schultz argued that the axiom could be demonstrated only with an improved way of constructing the size of angles in pure intuition. This controversy, which drew on earlier work on the theory of parallels from Leibniz and Wolff, centered around the nature of space and infinity, inciting Kant (in his short essay replying to Kästner) to draw his important distinction between geometrical and metaphysical space.

The most notorious of Euclid's axioms is his axiom of parallels, which states

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.[1] [figure 1]

Geometers already in the ancient world felt that the axiom, lacking the self-evidence of Euclid's other axioms, ought to be proven, and attempts to prove it continued to appear until it was finally shown in the 1870s to be indemonstrable from Euclid's other axioms. This geometrical research on Euclid's axiom of parallels led at the end of the nineteenth and beginning of the twentieth century to a fierce philosophical debate about the tenability of Kantian philosophy of mathematics. In particular, many philosophers –

[1] Euclid, *The Thirteen Books of the Elements*, vol.1, trans and ed. Heath, 202.

starting with Russell and Couturat – came to believe that the consistency of Non-Euclidean geometries confirmed Leibniz's claim that the propositions of geometry are derivable from logic and definitions, and so are not synthetic as Kant claimed. Ernst Cassirer expressed this common point of view nicely:

> For Kant … sensibility has ceased to be a mere means of representation, as in Leibniz, and has become an independent ground of knowledge: intuition has now achieved a grounding, legitimizing value. … In this methodological divergence it seems clear that modern mathematics has followed the road taken by Leibniz rather than that suggested by Kant. This has followed particularly from the discovery of non-Euclidean geometry. The new problems growing out of this discovery have turned mathematics more and more into a hypothetico-deductive system, whose truth value is grounded purely in its inner logical coherence and consistency, and not in any material, intuitive statements.[2]

This debate is historically important and by now well-known. It is surprisingly almost completely unknown, however, that a similar debate about the standing of Euclid's axiom in Leibniz's and Kant's philosophies broke out already in Kant's lifetime. Indeed, this debate drew in many of the leading German philosophers and mathematicians and played a significant role in the attacks and counterattacks of the rival Kantian and Leibnizian philosophers in the decades following the 1781 publication of Kant's *Critique of Pure Reason*. The goal of this paper is to tell this still unknown story.[3]

The controversy among geometers over the theory of parallels was of particular significance to Leibnizians and Kantians for a number of reasons. First, Leibniz argued that in principle every geometrical axiom could be proven from definitions by logical principles alone. The ability of geometers to come up with these proofs could then act as

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[2] Cassirer, *Philosophy of Symbolic Forms*, vol.3, 363-4, echoing Russell, *Principles of Mathematics*, §5. Gottfried Martin (*Kant's Metaphysics and Theory of Science*, 16-9) has argued that the same history argues for Kant over Leibniz.

[3] The only work I know that discusses this history is Webb, "Hintikka on Aristotelean Constructions, Kantian Intuitions, and Peircean Theorems," 230-2, 242. On Webb's reading, see note 92 below.

a test of Leibniz's philosophical claim. As Kant put it, replying to the Neo-Leibnizian philosophy of his critic J.A. Eberhard:

> Euclid himself is supposed to "have amongst his axioms propositions which actually require a demonstration, but which nevertheless are presented without demonstration." […] If only this philosophy, which is so obliging in providing demonstrations, would also be obliging enough to produce an example from Euclid where he presents a proposition which is mathematically demonstrable as an axiom.[4]

Given the centuries of effort put into proving it, Euclid's axiom of parallels would then be a particularly apt place for Leibnizians to start their program, and (if only the correct definitions and proof could be found) a nice confirmation of Leibniz's philosophy of geometry. Indeed, as we'll see, Leibniz did precisely this, proposing a new definition of parallel lines and a new proof of Euclid's axiom.

Unfortunately, Leibniz's approach to the theory of parallels (though taken up early in the eighteenth century by Christian Wolff in his hugely influential geometrical works) was widely regarded in the late eighteenth century as a failure, and a collection of alternative proofs sprung up in its place. This led to a second way in which the theory of parallels touched on Leibnizian and Kantian philosophy. Many of these new proofs appealed to novel axioms that seemed no better than the axiom they were meant to replace. Here are some examples:

1. A line everywhere equidistant to a straight line is itself straight (Wolff).

2. For any triangle, there is a similar triangle larger than it (Kästner).

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[4] *On a Discovery*, Ak 8:196 (trans. in Allison, *The Kant-Eberhard Controversy*). Kant is quoting Eberhard, *Philosophisches Magazin* [hereafter, *PM*] 1 (1789), 162.

Citations of Kant's works are according to the German Academy ("Ak") edition pagination: *Gesammelte Schriften*, edited by the Königlich Preußischen Akademie der Wissenschaften, later the Deutschen Akademie der Wissenschaften zu Berlin. I also cite Kant's reflexions [Refl] by number. For the *Critique of Pure Reason*, I cite the original page numbers in the first ("A") or second ("B") edition.

All translations are my own, unless a translation is listed among the works cited.

3. Space has no absolute measure of magnitude (Lambert).[5]

4. Two lines that have the same situation relative to some third line have the same situation absolutely (Karsten).[6]

5. The magnitude of an angle can be measured by the infinite portion of the plane contained within the arms of the angle (Schultz).

Would a proof from one of these principles count as a genuine proof of the parallel axiom? This was surely a philosophical problem whose solution depended on addressing the prior philosophical problem What is an axiom? (And a related question, Do the "discursive" fundamental principles of philosophy differ in kind from the axioms of geometry?)

This issue was expressed nicely by the Wolffian philosopher and mathematician Karsten, in the process of defending his proof against objections from Kant's student Johann Schultz:

> It is easy to say that the truth of an axiom [Grundsatz] must be already obvious [einleuchtend] for itself to anyone who merely understands the words. […] Some difficulties of their own come about when positing a new axiom that has not been used, if one wishes to take it to be obvious for itself in the proper sense. … Customarily, one demands that it must be contained in an already accepted definition, or can be derived from it using an easy inference: but what if the concept that is at issue in the axiom is a very general and entirely simple concept, of which one can give no proper definition?[7]

Again, a series of opposing philosophical explanations were given for what this "obviousness" or "self evidence" or "immediate certainty" would require. For Wolff, a self evident proposition must be a corollary of a real definition; for Lambert, it must

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[5] Lambert, *Theorie der Parallellinien* (1786, written 1766), §79; *Anlage zur Architectonic.* vol.1 (1771), §79, axiom 2.
[6] Karsten, *Versuch einer völlig berichtigten Theorie der Parallelen* (1778), 15 (axiom 6). I discuss Wolff's, Kästner's, and Schultz's proposals below.
[7] Karsten, "Über die Parallellinien" (1786), §12.

contain only simple concepts standing in simple relations; for Kästner, it must contain only "common" notions – concepts available to pre-scientific thinking; for Schultz, it must be constructible in pure intuition and not provable merely conceptually.[8]

The debate over what an axiom is, however, quickly led into a third philosophical issue that divided Leibnizians and Kantians: the rival ways of contrasting sensibility and understanding. The significance of the theory of parallels for this fundamental philosophical issue was forcefully expressed by Eberhard, in the closing pages of his essay "Of the concepts of space and time in relation to the certainty of human knowledge," which appeared in the anti-Kantian journal *Philosophisches Magazin*, which he edited. Having argued that Euclid's axiom of parallels could only be proved conceptually from a definition (as Leibniz maintained), and could not be proved intuitively (as Kant would require), Eberhard drew the general conclusion that the truth of geometrical axioms could only be grounded in the understanding alone, not in sensibility. He concluded:

> If these arguments are incontestable, then the content of dogmatic philosophy is justified. For if the grounds of truth in geometry are not the sensible [Bildlich] and subjective [aspects] of its concepts, but rather the objective, then there are true synthetic judgments whose grounds of truth lie not in sensible marks. Synthetic judgments of the pure understanding are possible, and they can be apodictically certain. True things are knowable, things that are not appearances, but are things in themselves, even knowable by a finite understanding with respect to their general determinations. There are possible synthetic general rational judgments about them, that is, those whose predicates are attributes of the subject. These can be apodictically certain, because the grounds of the truth and apodictic certainty of geometrical axioms themselves are not in the sensible and subjective but in the nonsensible and objective.[9]

[8] I discuss each of these proposals individually below.
[9] *PM* 2 (1790), 91.

Given the philosophical significance of the debate over Euclid's axiom, it is not surprising that work on the theory of parallels in Germany reached a peak in the closing decades of the 18th century, with six works appearing in 1786 alone.[10]

This story will be told in seven sections. Leibniz, convinced that Euclid's axiom – like all truths of reason – could be proved, attempted a new (but fallacious) proof using a new definition of <straight line> and a new definition of <parallel lines> (§I). Though this proof was unpublished and unknown, a similar proof was popularized by Wolff in his mathematical works, defining parallel lines as equidistant straight lines and employing (without explicit recognition) the axiom that a line equidistant from a coplanar straight is itself straight (§II). In §III, I explain that the errors in Wolff's approach were famously discovered by Abraham Kästner in 1758. Kästner, a committed Wolffian, believed that an acceptable proof would require a new definition of <straight line>, though he knew of no acceptable candidate; he also rejected alternative proofs, like Wallis's, on the grounds that they did not contain only "common" notions.

§IV ("Kant on axioms") begins the discussion of Kant, who – though he wrote a series of remarks on the definition of <parallel lines> – never explicitly discusses the axiom in extant writings. I argue that Kant, systematically distinguishing axioms and postulates, would have considered Euclid's principle a candidate axiom (not a postulate), and would have held it to be a truth in need of demonstration (not a proper axiom). What then would a demonstration that passes Kantian muster look like? Kant's student and expositor Johann Schultz argued (surprisingly) that Euclid's axiom could be proven in the proper geometrical sense only if there are pure intuitions of constructed infinitary geometrical figures (§V). Both Kästner and Eberhard wrote papers in volume 2 of

[10] Engel and Stäckel, *Die Theorie der Parallellinien von Euklid bis auf Gauss* (1895), 300.

Eberhard's pro-Leibnizian *Philosophisches Magazin* attacking Schultz's geometry of the infinite (§VI), arguing that the impossibility of intuitions of infinitary figures shows that there can be no pure intuition of space (as Kant claimed). Eberhard added that a Kantian proof of Euclid's axiom would require such impossible intuitions, and so Kantian philosophy of mathematics would leave the problems in the theory of parallels insoluble. As I argue in §VII, this debate drove Kant to write his essay "Über Kästners Abhandlungen," where he argues that there can be a pure intuition of infinite "metaphysical" space, even though infinite constructed figures (like those Schultz employs in his proofs of Euclid's axiom) are impossible. Unfortunately, this leaves Kant, like Leibnizm without a philosophically and mathematically acceptable proof of Euclid's axiom.

I. Leibniz

Leibniz believed that there are two kinds of truths: truths of reason (or necessary truths) are distinguished from truths of fact (or contingent truths) in being reducible to identities in a finite number of steps. All necessary truths, then, are either explicit identities or provable from identities using only the principles of contradiction and identity. Truths of fact, on the other hand, could be reduced to identities only through an infinite analysis and thus could be proved by finite creatures like us only by appealing to the principle of sufficient reason.[11] Since the truths of geometry are truths of reason, each of them should be provable in a finite number of steps from definitions using only the principles of

[11] *New Essays on Human Understanding*, IV.ii.1; "On Contingency" (in *Philosophical Essays,* ed. by Ariew and Garber), 28-9; "On Freedom" (in Ariew and Garber), 96; *Discourse on Metaphysics* (in Ariew and Garber), 46; *Monadology*, §33 (in Ariew and Garber).

contradiction and identity. Thus, what are often called "axioms" in geometry are not actually axioms, but require proof.

> You will find a hundred passages in which scholastic philosophers have said that [axioms] are evident *ex terminis* – from the terms – as soon as they are understood. That is, they were satisfied that that the 'force' of their convincingness is grounded in the understanding of the terms, i.e. in the connections of the associated ideas. But the geometers have gone further still: they have often undertaken to demonstrate such propositions. […] Anyway, I have for a long time been urging, both publicly and in private, the importance of demonstrating all the secondary axioms that we ordinarily use, by bringing them back to axioms that are primary, i.e. immediate and indemonstrable; they are the ones I have been calling 'identities.' (*New Essays*, IV.vii.1)

Euclid's axioms, on Leibniz's view, are then not genuine (or "primary" axioms), and need to be demonstrated.

Secondary axioms, such as Euclid's, nevertheless can have a special role in geometry inasmuch as they are evident by means of images or sense experience. Of course, images and sense experiences are for Leibniz merely confused (and so unanalyzed) ideas, and so neither are they the source of geometrical truths nor can they play a role in demonstrating these truths. So it is philosophically important to realize that these secondary axioms can be proven, and it is mathematically important to find these proofs. In *New Essays* (written in 1704), Leibniz illustrates his point by considering Euclid's axioms of straight lines, which Leibniz argues were forced on Euclid because his definition of <straight line> (as a "line that lies evenly with the points on itself"[12]) is faulty.

> Euclid, for instance, includes in his axioms what amounts to the statement that two straight lines can meet only once. We can't on the basis of our sense-experience imagine two straight lines meeting more than once, but that is not the right foundation for a science. […] Images of this sort are merely confused ideas; someone who knows about straight lines only from his images won't be able to demonstrate anything about straight lines. […] Euclid had no distinctly expressed

[12] *Elements*, Definition 4 (165); Leibniz, "In Euclidis πρωτα," 185.

> idea of a straight line, i.e. no definition of it (for the one he offers provisionally is unclear, and useless to him in his demonstrations), so he had to resort to two axioms that served him in place of a definition and that he uses in his demonstrations: Two straight lines don't have any parts in common, [and] Two straight lines don't enclose a space.[13]

Indeed, Leibniz himself tried to prove these two axioms (in works left unpublished until the nineteenth century) based on an improved definition of <straight line>.[14]

Leibniz believes that the sought for proofs of Euclid's axioms will require new definitions of many geometrical terms. These definitions will have to be of a special kind to meet the constraints that Leibniz's conception of demonstration puts on proper mathematical definitions. Since a demonstration cannot proceed from contradictory premises, and since a demonstration of a necessary truth bottoms out in definitions alone, the definitions themselves must secure their own possibility. A "real definition" for Leibniz is a definition that "shows that the thing being defined is possible."[15] Thus, reasoning from definitions cannot be done safely "unless we know first that they are real definitions, that is, they include no contradictions."[16] A special kind of real definition is a causal (or genetic) definition, which "contains the possible generation of a thing."[17]

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[13] *New Essays*, IV.xii.6. These two axioms do not in fact appear in Euclid's *Elements*, though Leibniz (following Clavius's edition) believed that these were Euclid's tenth and fourteenth axioms (see "In Euclidis," 207).

[14] See, for instance, the proof that no two straight lines enclose a space from "In Euclidis," 209, using his definition that a straight line divides the plane into two congruent parts. Leibniz's various proofs of these axioms are discussed by De Risi (*Geometry and Monadology*, 248-54).

[15] *New Essays,* III.iii.18. On real definitions, see also *New Essays,* III.iii.15, 19; "Meditations on Knowledge, Truth, and Ideas" (in Ariew and Garber), 25-6; "On Universal Synthesis" (*Philosophical Papers and Letters*, ed. by Loemker), 230-1; Letter to Tschirnhaus (in Loemker), 194; *Discourse on Metaphysics*, §24.

[16] "Meditations," 25.

[17] *Discourse on Metaphysics*, §24. See also *New Essays*, III.iii.18; "Meditations," 26; "Universal Synthesis," 230.

Leibniz therefore requires that mathematical definitions be genetic and thus "explain the method of production" of the thing being defined.[18]

Leibniz sees special problems for Euclid's treatment of parallels, since Euclid's axiom is not demonstrated and (as Leibniz argues in *New Essays*) his definition is not a real definition.

> [T]he real definition displays the possibility of the definiendum and the nominal does not. For instance, [Euclid's] definition of two parallel straight lines as 'lines in the same plane which do not meet even if extended to infinity' is only nominal, for one could at first question whether that is possible.

Leibniz proposes instead an alternative definition, which he claimed is genuinely real.

> But once we understand that we can draw a straight line in a plane, parallel to a given straight line, by ensuring that the point of the stylus drawing the parallel line remains at the same distance from the given line, we can see at once that the thing is possible, and why the lines have the property of never meeting, which is their nominal definition.[19]

Leibniz's point (and the point was not new with Leibniz but was well known already in the 16th century) is that Euclid's definition of <parallel lines> as coplanar non-intersecting straight lines does not itself make patent that such lines are possible, and says nothing about *how* (and *if*) such lines could be produced at all. Leibniz's alternative is to characterize a parallel to a given straight line as the trace left by the endpoint of a perpendicular line segment of fixed length as it slides along the given line (see figure 2).

However, it does not seem to follow immediately from the description of this procedure that the line traced out will itself be *straight*. Proving this is particularly pressing, since Clavius (whose 1607 edition of Euclid Leibniz uses) shows that Euclid's axiom of parallels can in fact be proven from the proposition that the line traced out by a

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[18] "Universal Synthesis," 231.
[19] Leibniz, *New Essays*, III.iii.18; cf. "In Euclidis," 201.

perpendicular line segment of fixed length on a given straight is itself straight.[20] (I'll call this assumption "Clavius's Axiom.") Leibniz therefore attempts to prove this assumption, in his 1712[21] essay "In Euclidis πρωτα," which is a long line-by-line commentary on the definitions, postulates, and axioms in Clavius's edition of the *Elements*. Leibniz's proof depends on new definitions of <straight line> and <parallel lines>. A straight line is "a section of the plane that has the same situation on both sides [utrinque se habens eodem modo]"[22] – that is, it divides the plane into two congruent parts. Parallel lines are "straight lines that everywhere have the same situations with respect to one another [rectae, quae se invicem ubique habent eodem modo]" – that is, the relative position of a point B of the second line with respect to points $A_1, \ldots, A_n$ of the first line will be the same as the relative position of a different point B' with respect to some other set of points $A'_1, \ldots, A'_n$ of the first line. For instance, take a triangle A_1A_2B, with the points A_1 and A_2 on the straight line *a* (see figure 3). Construct a congruent triangle $A'_1A'_2B'$, with the segment $A'_1A'_2$ on the line *a* and of the same length as A_1A_2, with B and B' falling on the same side of *a*. Repeat this procedure, and if the line *b* composed of all such points B, B', B", … is straight, then it will be parallel to *a*. An even simpler procedure is to erect equal perpendiculars AB, A'B', A"B" on the points of *a*; if the line *b* composed of these B, B', B" is straight, it will be parallel to *a* (figure 2).

One approach to proving the straightness of *b* exploits the definition of <straight line>. Leibniz argues that since the points on the line *b* always maintain a constant

[20] Clavius, Cristoph. *Euclidis Elementorum Libri XV*, 89-98. A proof is also contained in Saccheri (*Euclides Vindicatus*), Part II (215-31). Leibniz ("In Euclidis," 204), Def XXXIV, §15, comments that Clavius proved Euclid's Axiom from this assumption, but that he can prove Clavius's assumption from the definition of <straight line>.
[21] This date has been proposed (and convincingly argued for) by De Risi, 117-8.
[22] "In Euclidis," Def IV, §4.

situation with respect to a straight line, *b* must also be straight: "we may infer from the fact that [the line *b*] always has the same situation with respect to that which is everywhere uniform [namely, the straight line *a*], that it is also everywhere uniform" (§7, cf. §4). Leibniz's argument, however, uses an unexplained notion of "uniform." Indeed, if we use the more precise notion of uniformity that Leibniz prefers – that a straight line cuts the plane into congruent parts – then all we can infer from the straightness of *a* is that a different line *c* constructed in the same way as *b* but on the other side of *a* would be straight if *b* is (see figure 4).[23] Clavius's axiom remains unproven.

II. Wolff

Unfortunately, Leibniz's writings on the theory of parallels were unknown until well after his death in 1716: the essay "In Euclidis πρωτα," which contains his proofs of the parallel axiom, was not published until 1858, and *New Essays*, which contains Leibniz's remarks on the proper definition of <parallel lines>, did not appear until 1765. Nevertheless, the Leibnizian approach to the theory of parallels had an enormous impact on geometers in eighteenth century Germany – largely through the influence of Christian Wolff. Though Wolff never saw Leibniz's geometrical writings, and though the Leibniz-Wolff correspondence contains no mention of the theory of parallels,[24] Wolff did accept and promote Leibnizian views on geometrical axioms and definitions, and Wolff adopted a definition of <parallel lines> similar to Leibniz's. Indeed, as Lambert pointed out in

[23] Leibniz elsewhere (§15) tries to execute the proof using the fact that the constructed perpendicular A'B' is straight, from which it follows that the two half planes to the right and left of A'B' are congruent (figure 2). But, again, the straightness of *b* does not follow from the straightness of *a*: instead we can only infer that the half line B'B" has the same curvature as B'B, not that both half lines are straight.

Alternately Leibniz, instead of proving Clavius's Axiom, tries to prove the rectangularity of a Saccheri quadrilateral directly from the principle of sufficient reason (§12-4). On this proof from the PSR, see De Risi, 258-60.

[24] See *Briefwechsel zwischen Leibniz und Christian Wolff*, ed. Gerhardt.

1766, the debate over the theory of parallels in Germany was largely shaped by Wolff, who "was in a period of forty or more years the leader of the flock with respect to the geometrical works that appeared" in Germany (*Theorie*, §4), and whose definition of parallel lines soon replaced Euclid's as the standard definition.[25]

Wolff, like Leibniz, believed that all axioms must be reducible to identities. For him, an axiom is a theoretical proposition that is an immediate consequence of a definition.[26] This is what it means for an axiom to be "self-evident" ["vor sich selbst klar"] (*German Logic*, Ch.6, §2), "knowable per se", or "manifest from its terms" (*EM*, §31). Like Leibniz, Wolff argues that the definitions that ground all demonstrations must be real. In fact, Wolff opens the discussion of mathematical methodology in his *Elementa Matheseos Universalis* by citing Leibniz's "Meditations on Knowledge, Truth, and Ideas," defining real definitions as "a distinct notion of the genesis of a thing, that is, of the way in which it can come to be" (§18)[27], and arguing that it must be demonstrated that mathematical definitions are possible ("which geometers present concerning figures, when they relate their constructions") (§24). Like Leibniz, Wolff argues that Euclid failed to demonstrate his axioms because he had clear but not distinct conceptions of his terms: "even *Euclid***,** who surpassed others in demonstrating, included among the axioms demonstrable propositions, since he was not able to define equality, congruence, straight

[25] Schultz, *Entdeckte*, 7: "because every attempt [to prove Euclid's axiom] was fruitless, one began to believe that one could help oneself by changing the definition of parallel lines and consider them no longer as Euclid did as non-concurrent lines, but rather as lines that are everywhere equidistant from one another, and since the well known [work] of Wolff assumes this definition, this definition became dominant almost everywhere."
[26] *Elementa Matheseos Universae* [hereafter, *EM*], §30; *Die Vernünfftige Gedancken von den Kräfften des menschlichen Verstandes* [hereafter, *German Logic*], Ch.3, §XIII and Ch.6, §1
[27] See also *Philosophia Rationalis sive Logica*, §191; *Der Anfangs-Gründe aller mathematischen Wissenschaften* [hereafter, *AG*], Intro, 4.

line, and other notions of things" (*EM* §32).[28] Again, like Leibniz, Wolff argues that Euclid's proofs (though employing clear but confused concepts) were not fallacious, since his axioms were sufficient to prove what needed proving and were "made certain through experience" (*EM* §11).

Given this deep methodological agreement between Leibniz and Wolff, it is not surprising that Wolff defines parallels in a way similar to Leibniz in his 1710 *Anfangs-Gründe*[29]:

> If two lines AB and CD always have the same distance from each other, then they are parallel lines.[30]

Wolff never proves or even mentions Euclid's axiom, but instead uses his definition to prove directly the propositions that Euclid could only prove using his axiom. Now, though Wolff's definition is similar to Leibniz's, it is inferior to it in two respects. First, Wolff does not require explicitly, as Leibniz does, that parallel lines be straight lines. In fact, Wolff is quite insensitive to the problem of the straightness of parallels: he infers the reality of his definition from the fact CD can be generated by sliding AC along AB,[31] and then goes on to assume that CD is straight without ever explicitly proving or even mentioning Clavius's Axiom.[32] As Schultz puts the criticism:

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[28] For similar explicit criticisms of Euclid, see also *EM* §11, *German Logic* I, §18. In the preface to *EM*, Geometriae, Wolff also claims that Euclid lacked a definition of <similar>.

[29] I do not want to imply that Wolff learned of this approach from Leibniz, or even that Wolff knew that Leibniz himself preferred this approach. Most likely, Wolff's ideas came from the same source as Leibniz's: Clavius, whose edition of Euclid Wolff studied as a boy (*Christian Wolffs eigene Lebenschreibung*, ed. Wuttke, 118).

[30] *AG*, Geometrie, Definition 14 (125). Cf. also *EM* (first published 1713), Geometriae, Definition 44, §81 (128).

[31] In fact, Wolff argues in *EM,* Geometriae, §258 that parallels can be constructed by attaching a fixed open compass at a fixed angle to a ruler and sliding the ruler along a straight line; he agues that they can also be constructed by erecting perpendiculars AC and BD on AB and joining CD. But these constructions, though both possible, are not constructions of the concept <equidistant straight lines> (and in fact, are not even constructions of the same lines) unless Clavius's Axiom is true – and this has not been proven.

[32] Wolff assumes the straightness of the line CD parallel to a straight line AB in *EM* §230.

> [Wolff] gets into still greater trouble. Because here arises the question whether parallel straight lines would be possible at all, or whether instead a line everywhere equidistant from a straight line would be a curved line, and therefore the entire concept of everywhere equidistance straight lines is an empty figment [leeres Hirngespinnst]. (*Entdeckte*, 7)

Wolff's definition differs from Leibniz's in another way: he defines parallels as everywhere having equal *distance*, not as everywhere having the same situation. Leibniz notes that his definition in terms of situation (which, we saw, amounts to congruence) would imply that parallels are equidistant straight lines if we had a distinct notion of distance, but declines to use the definition in terms of distance because he does not have a definition of the minimal curve from a straight line to a straight line.[33] Wolff, on the other hand, is less circumspect. He defines parallel lines in §81 without saying in general what the distance between two lines should mean. And this proves fatal to Wolff, because in his proof of a central theorem in the theory of parallels (*Elements* I.27-8) Wolff falls into error by unconsciously exploiting an ambiguity in his notion of distance.[34]

III. Kästner

Through Wolff's influence, this approach to the theory of parallels dominated Germany for half a century, being reproduced in numerous other works published after Wolff's.[35] This consensus, however, was overturned starting in 1758, when the mathematician Abraham Kästner very publicly called into question Wolff's (and indeed every other known) treatment of Euclid's axiom.

> Geometry would not remain a model of the most perfect science [Lerhart] if one were to allow oneself to accept in it without proof propositions whose truth is not

[33] "In Euclidis," Def XXXIV, §9.
[34] The fallacy is in *EM* §255. On the point, see my paper "Kant on Parallel Lines."
[35] For instance, Segner's 1739 *Elementa Arithmeticae et Geometriae* (famously cited by Kant at B15) essentially follows Wolff's approach (Def XIV, 88-89; Prop XII, 100-1).

> apparent [augenscheinlich]. For this reason I have gone as far in my demonstrating [erweisen] as can be demanded. [...] The difficulty which one finds in the doctrine of parallel lines, has concerned me for many years already. I believed that it was fully solved by Hausen's *Elementa matheseos* [1734]. [But it was pointed out to me] that there is a place where Hausen draws an inference that does not follow. I soon discovered this mistake myself and committed myself from that time on to overcome the difficulty myself or to find a writer who had overcome it, but I failed in both purposes, even though I collected almost a small library of single writings or elements of geometry, where this object had been considered especially.[36]

Indeed, Kästner's small library formed the basis for his student Georg Simon Klügel's dissertation, completed five years later, that catalogued twenty-eight attempted proofs of the axiom of parallels and showed each to be fallacious.[37]

Kästner and his successors were then forced to decide how to proceed. One alternative was to look for a new analysis, distinct from both Wolff's and Euclid's, of <parallel lines> -- an alternative adopted by the Leibnizian mathematician W.J.G. Karsten in a series of works starting in 1778.[38] Kästner saw no hope in this route, and reverted to Euclid's definition of <parallel lines> as non-concurrent co-planar straights.[39] A second alternative was to give up on the Wolffian view of axioms – an alternative adopted by Lambert and later by Kantian mathematicians like Schultz. Kästner rejected this route as well, remaining a committed Wolffian until his death in 1800. Indeed, in the same 1758 preface, he praised Wolff, writing

> Germany will name the Freiherr von Wolff still with the highest respect when the name of most of his critics will remain only in the inventory of insects that the diligence of German literature collects. It has him very much to thank for the

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[36] Kästner, *Anfangsgründe der Mathematik,* part I [*AG*], preface (no page nos.)

[37] Klügel, *Conatuum praecipuorum theoriam parallelarum demonstrandi recensio*, described in Kästner, *Anfangsgründe* (2nd ed., preface) and "Was heißt in Euclids Geometrie möglich?" (396). I have unfortunately been unable to find a copy of Hausen's *Elementa* or Klügel's dissertation.

[38] Karsten argued that an analysis of <parallel lines> reveals that coplanar lines are non-concurrent iff they have the same situation [Lage] (*Versuch*, 13). Unfortunately, space constraints prevent me from discussing Karsten's proposals further.

[39] Kästner, *AG*, Geometrie, Def 9.

> proliferation of reason and of mathematics, which constitutes such a great part of reason.[40]

Kästner's commitment to Wolffianism ran deep. Like Wolff, he believed that the mathematical method can be employed in other sciences (23, 7) and described as "laughable" (32) the view (defended by Kant himself a few years later in 1763) that philosophy and mathematics have distinct methods. Like Wolff, he argued that mathematical definitions have to be real definitions (26), that axioms could be proven from definitions (27), and that every mathematical proof could be put into syllogistic form (30). Wolff was correct, he argued, to demand that Euclid's axioms be proven and to try to redefine the fundamental concepts of Euclid's geometry in the hope of finding these proofs.

Kästner therefore recommended a third alternative – to look for a new definition of <straight line>. Kästner presented his argument for this approach in his essay "Über den mathematischen Begriff des Raums," written in 1790 in Eberhard's anti-Kantian *Philosophisches Magazin*. He began by repeating the standard Leibnizian view that "secondary axioms" like Euclid's have a special status inasmuch as their truth is patent from a merely clear grasp of their component concepts (even if their proof would require a still undiscovered distinct grasp of these concepts) (*PM* 2, 421-3). Even without a satisfactory definition of <straight line>, it is evident that no two straight lines can enclose a space. Why then does Euclid's axiom of parallels lack this evidence? It must be, he reasons, that a clear but not distinct concept of <straight line> is not sufficient to make the truth of the parallel axiom evident, even though it is sufficient for Euclid's other

[40] Kästner expressed a similar sentiment in his letter to Kant, Ak 11:213, eliciting from Kant a sarcastic response (Ak 22:545).

axioms. Consider again figure 1. Why is it that the straight line BB" must intersect the straight line AA" when a curved line would not?

> One can instead of the acute straight line [BB"] represent a curved line whose asymptote would be the perpendicular [AA"], e.g. an arm of a hyperbola. In such a curved line one can proceed as far as one wants and never intersect the perpendicular. Why should something take place in the case of a straight acute line that does not occur when one replaces it with a curved line? (416)

Kästner reasons that the parallel axioms lacks evidence because (having only a clear but not distinct concept of <straight line>) we lack a clear conception of the mark that distinguishes straight from curved lines. This provides an explanation for why Euclid's axiom of parallels seems less evident than Euclid's other axioms. (Kästner thus fills in an explanatory lacuna in Leibniz's and Wolff's diagnosis of Euclid's theory of parallels: even if we grant the general point that the axiom of parallels – like all Euclid's "secondary" axioms – requires a demonstration, we still want a story about what makes the axiom of parallels *uniquely* problematic.) And it provides a plan for finding a proof: isolate the correct definition of <straight line>, and the proof of Euclid's axiom will follow.

Though this third alternative allows Kästner to hold onto the fundamentals of Leibniz's and Wolff's philosophy of mathematics, it does require him to reject the three rival definitions of <straight line> then in common use – including Wolff's own. Euclid defined a straight line as a "line that lies evenly with the points on itself," which Kästner interpreted to mean a "line all of whose points lie in the same direction." Though he thinks no candidate proposed since Euclid has done better, he still thinks that this is not a proper definition. "No one will first become acquainted with the straight line from this definition" (*AG* 169): <direction> is not intelligible independently of <straight line>, since to proceed from A in the direction of B is just to travel along the straight line AB.

He also rejects Wolff's well known definition of a straight lines as "a line all of whose parts are similar to the whole" (Wolff, *AG*, Geometrie, Def.4; *Elementa*, Geometriae, §17, Def.7).[41] Kästner argues that this definition is circular, since we could not know what it means for two lines to be similar without knowing what it means for two lines to have the same direction (Kästner, *AG,*170; *PM* 2, 397, 423). A third definition – a straight lines is the shortest distance between two points – was attributed to Archimedes,[42] and was rejected on the grounds that it is in fact a theorem, provable from Euclid's other axioms (*PM* 2, 424; cf. *AG* 188).

With no distinct definition of <straight line>, Kästner argues that a satisfying proof of Euclid's axiom does not yet exist. In his own *Anfangsgründe*, he employs a variant on Wallis's axiom – that for a given triangle, there are other triangles similar to but larger than it.[43] Still, though, Kästner does not believe that this is any more a legitimate secondary axiom than Euclid's is. Kästner holds that what he calls "learned" (or "scientific") mathematics is a development from "common" (or "natural") mathematics (*PM* 2, 420-1). Common mathematics is possessed by every human being innately and is applied to particular objects often without notice (*AG* 5, 20, 27). Kästner argues that genuine secondary axioms must be drawn from common mathematics (this is how he interprets Euclid's phrase "common notions"), and thus knowable without any

[41] On Wolff's definition of similarity and its use in his geometry, see Sutherland "Philosophy, Geometry, and Logic in Leibniz, Wolff, and the Early Kant."
[42] See Leibniz, *New Essays,* IV.xii.6. Leibniz's preferred definition from "In Euclidis" (a section that cuts the plane into congruent parts) was unknown to Kästner.
[43] *AG*, Satz 11, Zusätze 5ff. Kästner first proves that if the length of A'B' in figure 1 is gradually decreased, B' will eventually reach a point where BB" intersects AA". As A'B' is lengthened that intersection point will move further and further out (and the triangles formed from A'B', B'B", and A'A" will get larger and larger), and Euclid's axiom will be true if the intersection point never disappears no matter how large A'B' gets. He motivates this claim by showing that there could not be a "last" point position for B' where the two lines intersect.

explicit training and employed implicitly by every human. Every candidate he knows of (including Wallis' axiom[44]) fails this test.

IV. Kant on axioms

Kant was surely not ignorant of these debates. According to Warda, Kant himself owned both Kästner's *Anfangsgründe* and Hausen's *Elementa*.[45] In fact, Kant thought very highly of Kästner's mathematical abilities, expressing in a letter to Kästner his "unbounded respect for the Nestor of all philosophical mathematicians in Germany."[46] Kant also owned Wolff's *Anfangsgründe* and *Elementa*, was well acquainted with Leibniz's *New Essays*, and (as will become clearer in the sequel) actively followed both Schultz's treatment of parallels and the discussions of the theory of parallels in the *Philosophisches Magazin*. Most telling, though, is a series of notes (Refl 5-10, Ak 14:23-52) from the critical period that discuss the theory of parallels in detail. In these notes Kant argues against both Wolff's definition of <parallel lines> and Euclid's own – though he unfortunately says nothing about Euclid's axiom itself except to criticize Wolff's fallacious proof of it.[47]

Because Kant never says anything directly about Euclid's axiom, his readers (including even his contemporaries and closest students) were forced to infer the authentic Kantian view of Euclid's axiom from features of his philosophy of geometry. And for Kant's philosophy, just as for Leibniz's or Kästner's, determining the status of

[44] His argument seems to be that Wallis's axiom needs to be motivated by appealing to mathematical facts like the ones mentioned in the previous note, and that these facts would not be available already to the common human understanding.
[45] Warda, *Immanuel Kants Bücher* (1922), 38-9.
[46] Ak 11: 186, 5 August 1790.
[47] On these notes, see [[reference removed to maintain author anonymity]]

Euclid's axiom requires first determining what precisely an axiom is for Kant. We will answer that question in this section, and in sections V-VII we will look at how his students and critics answered the questions Is Euclid's axiom a genuine axiom on Kant's view? and What would an adequate replacement be like?

I'll begin my discussion with what I take to be a very common, but still mistaken, approach to the question. Euclid's axiom is listed as the fifth "postulate" in modern editions of the *Elements*, and Kant has a distinctive and satisfying explanation of geometrical postulates. For Kant, mathematics is knowledge from construction of concepts. Construction is an activity: the "(spontaneous) production of a corresponding intuition" (*On a Discovery*, Ak 8:192). Mathematics therefore requires the possibility of certain spontaneous acts. The possibility of such an act is guaranteed by a "postulate": "a practical, immediately certain proposition or a fundamental proposition which determines a possible action of which it is presupposed that the manner of executing it is immediately certain."[48]

Kant has an elegant explanation for why postulates are immediately certain. To think the postulate, one must of course possess the concepts contained in it; but the procedure described by the postulate *is* itself the means by which the concepts in question are first generated.

> Now in mathematics a postulate is the practical proposition that contains nothing except the synthesis through which we first give ourselves an object and generate its concept, e.g., to describe a circle with a given line from a given point on a plane [Euclid's third postulate]; and a proposition of this sort cannot be proved,

[48] Kant, *Jäsche Logic* [hereafter, *JL*] (trans. by Young in *Lectures on Logic*), §38. On postulates, see also Refl 3133, Ak 16:673; Heschel Logic 87 (trans. by Young in *Lectures on Logic*, 381); "Über Kästner's Abhandlungen," Ak 20:410-1; Letter to Schultz, 25 Nov 1788, Ak 10:556. On Kant's view of mathematical postulates and their role in his philosophy, see especially Alison Laywine, "Problems and Postulates: Kant on Reason and Understanding" and "Kant and Lambert on Geometrical Postulates and the Reform of Metaphysics."

> since the procedure that it demands is precisely that through which we first generate the concept of such a figure. (A234/ B287)

On Kant's view of mathematical concepts (as *made* not *given*), we cannot possess the concept <circle> without having its definition. But its definition, being genetic, enables me to describe circles in pure intuition *a priori* and *in concreto*.[49] So it impossible that I should have the concept <circle> and not know that circles can be described with a given line from a given point. Geometrical postulates are then virtually interchangeable with genetic definitions:

> The possibility of a circle is ... *given* in the definition of the circle, since the circle is actually constructed by means of the definition, that is, it is exhibited in intuition [...] For I may always draw a circle free hand on the board and put a point in it, and I can demonstrate all the properties of a circle just as well on it, presupposing the (so-called nominal) definition, which is in fact a real definition, even if this circle is not at all like one drawn by rotating a straight line attached to a point. I assume that the points of the circumference are equidistant from the center point. The proposition "to inscribe a circle" is *a practical corollary of the definition* (or so-called postulate), which could not be demanded at all if the possibility – yes, the very sort of possibility of the figure – were not already given in the definition. (Letter to Herz, 26 May 1789; Ak 11:53, emph. added)

Kant illustrates his view that postulates are practical corollaries of definitions with Euclid's third postulate, "to describe a circle with any center and radius." His view seems to accord equally well with Euclid's first two postulates, "to draw a straight line from any point to any point" and "to produce a finite straight line continuously in a straight line."[50] But the fifth postulate, the parallel postulate, does not seem to fit this

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[49] On mathematical definitions: A727-32/B755-60; *JL* §106. See also Dunlop, "Kant and Strawson on the Content of Geometrical Concepts."

[50] Though it *seems* as if Kant's account of postulates fits well with Euclid's first and second postulates, Kant in fact gets himself into serious difficulties here. Kant's account of postulates would have it that the first and second postulates are practical corollaries of a real definition, presumably of the concept <straight line>. However, Kant (repeating Kästner's criticism of Euclid's definition; see p.18 above) seems to deny that this concept is definable (Letter to Herz, 26 May 1789, Ak 11:53), and even classifies it as "given," not "made" (Blomberg Logic, Ak 24:268). Schultz, trying to ward off claims from Leibnizians that geometrical truths are analytic, also argues that the concept <straight line> is indefinable (*Prüfung*, vol.1, 58; *Anfangsgründe*, 253-4). As Hans Freudenthal points out, Maimon recognized that Kant's insistence that the

model at all. Michael Friedman writes:

> Geometry ... operates with an initial set of specifically geometrical functions [viz, the operations of extending a line, connecting two points, and describing a circle from a given line segment] ... To do geometry, therefore, ... [we] need to be "given" certain initial operations: that is, intuition assures us of the existence and uniqueness of the values of these operations for any given arguments. Thus the axioms of Euclidean geometry tell us, for example, "that between any two points there is only one straight line, from a given point on a plane surface a circle can be described with a given straight line" (Ak 2:402). Serious complications stand in the way of the full realization of this attractive picture... Euclid's Postulate 5, the Parallel Postulate, does not have the same status as the other Postulates: it does not simply 'present' us with an elementary constructive function which can then be iterated.[51]

If this diagnosis were correct, Friedman's observation would allow him to locate a place in the theory of parallels where it fails to satisfy Kant's characterization of the proper mathematical method.[52] This would be gratifying because it would show that Kant's philosophy of geometry breaks down at the very point where it should – in the most problematic area of traditional Euclidean geometry. However, I believe that this reading makes Kant look unreflective even by the standards of eighteenth century philosophers. A moment's reflection shows that Kant's characterization of the mathematical postulates does not fit Euclid's fifth, and this is something that Kant could have – and *should* have – noticed. Fortunately, though, Friedman's diagnosis begins to look oversimplified when

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concept is indefinable poses serious difficulties for the constructability of straight lines (Freudenthal, "Definition and Construction: Maimon's Philosophy of Geometry," 29ff.; cf. Maimon, *Essay on Transcendental Philosophy*, 423 (page 215 in Midgley et al translation)).

[51] Friedman, *Kant and the Exact Sciences*, 88.

[52] Alison Laywine "Kant and Lambert on Geometrical Postulates," 119, argues that the parallel postulate does fit Kant's description: "the fifth postulate *does* follow the pattern of the first three. For it too is constructive. It tells us that we can construct two lines that will meet if extended far enough – on the condition that, if we let a third line fall on both of them, the angles so formed on one side are less than two right angles." But on Kant's view the postulate must be a practical corollary of a definition. It cannot be the practical corollary of <parallel line>, since – as was pointed out by Lambert (*Theorie*, §8) and many others -- Euclid I.31 tells us how to construct parallels and its proof is independent of the axiom of parallels. It cannot be the practical corollary of <intersecting lines>, since postulates 1 and 2 already tell us how to construct intersecting lines by choosing a point on a given line, a point not on that line, and connecting them. Most telling, though, is the historical evidence (which I will canvass below): not a single geometrical writer of 18th century Germany thought that Euclid's axiom of parallels had the practical character of a postulate. Kant was surely not the lone exception.

we notice that in the passage quoted Friedman uses the words "axiom" and "postulate" interchangeably. Now, at least since Frege[53] mathematicians have not distinguished between axioms and postulates, and have used the words as synonyms. But Kant surely meant to distinguish them, as will become clear when we look at the history of geometrical work in the early modern period.

Recall that Euclid distinguishes between "common notions" (principles like "equals added to equals are equals") and "postulates." Already in the ancient world, the common notions were called "axioms,"[54] and early modern geometric texts (and editions of Euclid) followed suit in distinguishing between axioms and postulates. In addition, Euclid's list of axioms frequently was expanded to include new axioms such as "No two straight lines enclose a space" – which is one of Kant's favorite examples of an axiom (see A163/B204 et al).[55] Furthermore, it was a very common practice to relabel Euclid's Parallel "Postulate" as an "axiom."[56]

Deciding whether the Euclid's fifth postulate is a genuine postulate depends obviously on deciding what a postulate is. In the mathematics text that Kant taught in his mathematics courses,[57] Wolff's *Anfangsgründe aller mathematischen Wissenschaften*, Wolff distinguishes between "postulata," which are indemonstrable *practical*

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[53] Frege, "Logic in Mathematics," *Posthumous Writings,* 206-7.

[54] "Axiom" is preferred by Aristotle over "common notion," and Proclus, with some apology, refers to Euclid's common notions as axioms. See Heath, 221-2. (Heath also comments – correctly, in my view – that those writers who, like Kant, characterized postulates as *practical* propositions ought to relabel Euclid's fifth as an axiom: see p.123.)

[55] This axiom, though it is not included in Euclid's *Elements*, was given as an axiom (though not a postulate) in every geometrical work that we know Kant owned. See note 63 for further references.

[56] For the history and references, see Shabel, *Mathematics in Kant's Critical Philosophy*, 44-5; Dunlop, "Why Euclid's Geometry Brooked No Doubt," 35-6.

[57] Kant taught mathematics at Königsberg from 1750s until 1764, sometimes using Wolff's shorter *Auszug aus den Anfangsgründe* instead of the longer *Anfangsgründe*. See Martin, *Arithmetic and Combinatorics*, xxi-ii, 143-4.

propositions, and "axiomata," which are indemonstrable *theoretical* propositions.[58] In the critical period Kant departed from Wolff in considering an axiom to be a "fundamental proposition *that can be exhibited in intuition*" (*JL,* §35), thus distinguishing axioms from analytic truths (such as "the whole is greater than its part" (B16-7, Ak 8:196)) and conceptual, philosophical principles (which Kant calls at *JL* §35 "acroams").[59] But it is clear from Kant's characterization of a postulate that he retained Wolff's view that postulates are *practical* fundamental propositions.

It is true that Kant's characterization of an axiom does not say explicitly that axioms are *theoretical* – and thus leaves open the possibility that postulates are a specific kind of axiom – but there is overwhelming evidence that Kant thought that axioms and postulates are distinct kinds of indemonstrables. First, all of the texts Kant used in his mathematics and logic lectures distinguished the class of axioms, which were theoretical propositions, from a distinct class of postulates, which were practical.[60] Second, all of the works that Kant owned (or that we know that Kant read) do so as well: a list that includes works by Baumgarten, Karsten, Kästner, Lambert, and Segner.[61] Third, two of Kant's students, Schultz and Kiesewetter, wrote mathematical texts self-consciously presented

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[58] Wolff, *AG* §30. Wolff draws the distinction in the same way in all of his logical and mathematical works: see *Logica*, Pars II; §§267-9; *EM,* §30; *German Logic*, chapter 3, §XIII. In fact, Wolff defends his definition of "postulate" on the grounds that it agrees with Euclid's use (!) and gives an intrinsic distinction between axioms and postulates: *Logica*, Pars II; §§267-9.

There was debate already in the ancient world about the distinction between postulates and axioms (see Bonola, *Non-Euclidean Geometry*, 18-9).

[59] On axioms, see also A732/B760; Refl 3135, Ak 16:673.

[60] See the references to Wolff's mathematical works in the previous note; Meier, *Auszug aus der Vernunftlehre* (1752), §315, Ak 16:668.

[61] Baumgarten, *Logica*, §169. Karsten, *Mathesis theoretica elementaris* (1760), 2; Kästner, *Anfangsgründe* (1758), 14 (§28), 176 (cf. "Was heißt in Euclids Geometrie möglich?," §1ff. and "Über den geometrischen Axiomen," §1ff.); Lambert, *Architectonic*, §12-3 (cf. also 13 Nov 1765 Letter to Kant, Ak 10:52); Segner, *Elementa*, 5-6, 93. Baumgarten, Karsten, and Kästner appear in Warda's list of Kant's library; Lambert and Segner's books are referred to explicitly in Kant's writings (Refl 4893, Ak 18:21; B15). Lambert (*Abhandlung vom Criterium Veritatis*, §48) suggests that the consensus view among German logicians was that a postulate is a "practical proposition whose truth is granted as soon as one understands the words."

according to Kantian principles; they also distinguish practical postulates from theoretical axioms.[62] Fourth, as far as I can tell, all of the examples of axioms that Kant gives in his writings (published and unpublished) are theoretical, whereas all examples of postulates are practical.[63] Fifth, Kant argued against Schultz that 7+5=12 is not an axiom, but it is a postulate.[64] Though this argument is obscure, it surely makes clear that for Kant postulates and axioms are distinct.

Kant's student Kiesewetter,[65] who wrote both a logic textbook (*Grundriss einer allgemeinen Logik nach Kantischen Grundsätze*, 1791) and a mathematics textbook (*Die ersten Anfangsgründe der reinen Mathematik,* 1799) according to Kantian principles,

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[62] Schultz, *Prüfung*, vol.1, 65, 67; Kiesewetter, *Anfangsgründe*, xxi, 215ff.

[63] In his writings, Kant gives two examples of postulates, and four examples of axioms.

Postulate 1: To draw a straight line. (Ak 20:410-11; Heschel Logic 87 (Young, *Lectures on Logic*, 381))
Postulate 2: To describe a circle from a given line from a given point on a plane. (Ak 11:53; Ak 2:402; A234/B287; Ak 11:43)

Axiom 1: Only one straight line lies between two points; with two straight lines no space can be enclosed. (Ak 2:281, Ak 2:402, A24; A47/B65; A163/B204; A239-40/B299, A300/B356; Heschel Logic 87 (Young, *Lectures on Logic*, 381))
Axiom 2: The straight line between two points is the shortest. (B16)
Axiom 3: Space has only three dimensions. (Ak 2:402, B41, A239-40/B299)
Axiom 4: Three points always lie in a plane. (A732/B760)

Kant's two postulates appear in Euclid, with Kant (as was commonly done in 18th century Germany) condensing Euclid's first two into one. None of Kant's axioms appear in Euclid. However, each of these postulates and axioms (labeled, always in agreement with Kant's practice, as a "postulate" or "axiom") appeared in at least one contemporary geometrical work that Kant knew (though, interestingly, no single work has all of Kant's examples). Again, in each of these writers, the lists of axioms are distinct from the lists of postulates:

Postulate 1: Wolff (*Elementa*), Segner, Karsten, Kästner, Lambert (*Architectonic*), Schultz (*Prüfung*, vol.1, *Anfangsgründe*), Kiesewetter.
Postulate 2: Segner, Karsten, Kästner, Schultz (*Prüfung*, vol.1, *Anfangsgründe*), Kiesewetter.

Axiom 1: Wolff (*Anfangsgründe*), Segner, Karsten, Kästner, Lambert (*Architectonic*), Schultz (*Prüfung*, vol.1, *Anfangsgründe*), Kiesewetter.
Axiom 2: Segner, Schultz (*Prüfung*, vol.1).
Axiom 3: Lambert (*Architectonic*), Schultz (*Anfangsgründe*), Kiesewetter.
Axiom 4: Kästner, Schultz (*Anfangsgründe*), Kiesewetter.

[64] 25 Nov 1787, Ak 10:555-6. Note that Kant's argument exploits the *practical* character of 7+5=12 (the judgment "seems indeed to be a merely theoretical judgment"; but in fact it is "the expression of a problem," that "designates a kind of synthesis").

[65] On Kiesewetter, see Förster, *Kant's Final Synthesis*, 3, 48-53.

wrote to Kant while composing his mathematics text and asked Kant for a definition of "Postulat" that would clearly distinguish postulates from "axioms" (Grundsätze).[66] We do not have Kant's reply, but there is very good reason to believe that Kiesewetter's way of drawing the distinction in his text accords with Kant's own view.

> *Postulate*. A practical axiom is called a *postulate* or a *postulating proposition*. An axiom properly speaking [das eigentliche Axiom] and a postulate agree in that in neither case are they derived from a different proposition, and they only differ from one another in that an axiom relates to the knowledge of an object, augments the concept of it, (is theoretical), while a postulate adds nothing to the knowledge of the object, does not augment its concept, but rather concerns only the construction, the intuitive exhibition [Darstellung], of the object. The postulate postulates [fordert] the possibility of the action of the imagination to bring about the object, which one had already known with apodictic certainty to be possible. So it is, for example, a postulate to draw a straight line between two points. The possibility of the straight line is given through its concept, [and] it is now postulated [gefordert] that one could exhibit it in intuition. One can see right away that through the drawing of the straight line its concept is not augmented in the least, but rather the issue is merely to ascribe [unterlegen] to the concept an object of intuition.[67]

Kiesewetter goes on to argue that postulates are essential elements of mathematics, playing a role that, though distinct from the role of axioms, is just as necessary. He writes: "Now since all of mathematics rests on construction, postulates constitute a foundation stone of the structure of mathematics." When Kiesewetter claims that postulates do not "augment our knowledge of a concept," he does not mean that they are analytic. Rather, he is arguing – as Kant had in A234/B287 – that to be certain of the postulate "to draw a straight line between any two points," it is sufficient to know the concept <straight line>.[68] It is impossible that a person could possess the concept

[66] Ak 12:267.
[67] Kiesewetter, *Anfangsgründe*, xxi.
[68] Kiesewetter, when he is contrasting geometrical "Grundsätze" and "Postulate" in the quoted passage, is closely mimicking Kant's wording when he contrasts the "postulates [Postulate] of empirical thinking," which correspond to the categories of modality, with the other "principles [Grundsätze] of pure understanding," which correspond to the other categories:

<straight line> and not know that it is possible to draw a straight line between any two points. So, knowing the truth of the postulate does not add anything to our knowledge of the concept beyond what was already known in the definition. Put another way, the postulate does not add any marks to the concept <straight line> that were not already contained in the definition of the straight line, but rather secures that the definition is real ("the possibility of the straight line is given through its concept") and not merely nominal. That the definiens of the definition is true of the definiendum is an analytic truth; that the definition is real, however, is a synthetic truth.[69]

Axioms, on the other hand, *do* "augment our knowledge of a concept." We can spell Kiesewetter's reasoning out with an example. To know the truth of the axiom "no two lines enclose a space," one needs first to construct two line segments – using Euclid's first two postulates – and bring the end points of the line segments to coincide with one another. (In this way, postulates are *more fundamental* than axioms, since knowing the truth of an axiom requires first constructing concepts in intuition, and this ability is secured by postulates.) Once this construction has been carried out, it is immediately evident that the two line segments must entirely coincide and that the axiom is true.[70] As

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> The categories of modality have this peculiarity: as a determination of the object *they do not augment the concept to which they are ascribed in the least*, but rather express only the relation to the faculty of cognition. *If the concept of the thing is already complete, I can still ask about this object rather it is merely possible*, or also actual, or, if it is the latter, whether it is also necessary? No further determinations of the object itself are hereby thought; rather, it is only asked: how is the object itself (together with all its determinations) related to the understanding and its empirical use, to the empirical power of judgment, and to reason (in application to experience)? (A219/B266, emph. added).

[69] Again, compare Kant's claim that the "postulates of empirical thinking" are "subjectively synthetic," despite the fact that that they do not "in the least augment the concept of which they are asserted," since they "do not assert of a concept anything other than the action of the cognitive faculty from which it is generated" (A233-4/B286-7). That real definitions are synthetic is suggested by Refl 2994, Ak 16:606-7. Schultz argues explicitly that real definitions (and so, he argues, postulates) are synthetic: *Prüfung* vol.1, 65-7; vol.2, 82-3.

[70] Note that this construction is not a demonstration of the axiom, since I am not proving the axiom on the basis of some other judgment. Rather, carrying out this construction is part of what it is to judge that "no

Kant put it: “by means of the construction of concepts in the intuition of the object [one] can connect the predicates of the [axiom] a priori and immediately” (A732/B760). Possessing the concept <straight line> is sufficient for knowing the truth of Euclid's first two postulates, and these two postulates together make it possible for me to carry out the construction of two line segments whose end points coincide. Carrying out this construction is sufficient for making the truth of the axiom patent. But it is important to note that the particular construction needed for the axiom is not identical to (and in fact more complicated than) the constructions expressed in the postulates. For this reason, it is possible to possess the concept <straight line>, and not know that “no two lines enclose a space”; when one learns that no two straight lines enclose a space, one has “augmented” the concept <straight line>.

The sure conclusion, then, is that Kant distinguishes practical indemonstrable propositions (such as Euclid’s postulates 1-3) from theoretical indemonstrable propositions (such as “no two lines enclose a space”), and gives an explanation of the immediate certainty for axioms that differs from the explanation given for postulates. Kant's account of axioms and postulates in this way bears an interesting relation to Wolff's own. He retains Wolff's view that the two kinds of principles differ as practical and theoretical, and he retains the Wolffian idea that postulates are immediate corollaries of definitions. However, he denies Wolff's view that axioms are also corollaries of definitions,[71] and he likewise insists – against Wolff – that postulates are synthetic, not analytic, since they are corollaries of *real* definitions, which are themselves synthetic.

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two lines can enclose a space." Thus, all that is required to know the axiom is to think it. (Of course, in a weaker sense it is possible to "think" the axiom without knowing it is true, as a philosopher might think an axiom without employing intuition: see A716/B744.)

[71] On Wolff's view, both axioms and postulates are immediate inferences from a definition. See Wolff, *AG*, §§28-31, 16-7; *EM*, §30-1, 9-10; *German Logic*, III.xiii.

Most significantly, postulates come to play a more central role in Kant's philosophy than they did in Wolff's, since mathematics for Kant is knowledge from the construction of concepts, and constructions are licensed by postulates.[72]

Given how poorly Euclid's fifth postulate fits Kant's account of postulates, it is clear that he would have considered it (if it were indemonstrable!) to be an axiom, not a postulate. In fact, historical evidence bears this out convincingly. Again, all of the books that Kant owned, and all of the works that we have firm evidence that Kant knew, classify it as a candidate axiom, not a postulate.[73] (This includes the works of Kant's students Schultz and Kiesewetter.) A particularly telling example is provided by Kästner, who – in addition to being a creative mathematician – was a historian of mathematics. In his *Geschichte der Mathematik* he reports, as if it were a very odd thing, that some Arab translations of Euclid have the axiom of parallels listed with the postulates, not the axioms.[74]

Unfortunately, though, this leaves us with no clear understanding of how Kant would have viewed the Euclidean theory of parallels. Given Kant's view of postulates,

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[72] Particularly telling is a passage in Schultz's review of vol.2 of *PM*, which Schultz wrote under Kant's supervision. Schultz (Ak 20:406) quotes Eberhard (*PM* 2, 153), who writes that "mathematics, critical idealism says, has other principles, other first truths that are not definitions and cannot be proved from definitions, it has axioms (*and postulates*)" [my emphasis]. The parenthetical phrase does not appear in Eberhard's text and was added by Schultz; evidently, Schultz thought it important to emphasize that the Kantian philosophy **does** carefully distinguish axioms and postulates. That philosophers of mathematics (and, indeed, metaphysicians) should put more weight on the distinction between axioms and postulates than Wolff did was argued very forcefully by Lambert (see, e.g., *Architectonic*, §12-4). Allison Laywine has argued persuasively that Kant agreed with Lambert (against Wolff) on the significance of postulates: "Kant and Lambert on Geometrical Postulates and the Reform of Metaphysics."

[73] Lambert, *Theorie*, §1. Karsten, *Lehrbegriff der gesamten Mathematik* (1767), §82. Kästner, *AG,* 195 (§12); "Über den mathematischen Begriff des Raums," 414ff. (§30ff). Schultz, *Entdeckte*, 5; *Prüfung,* vol.2, 93. Kiesewetter, *AG*, §68. (Wolff, Segner, and Karsten (in his *Mathesis theoretica elementa* (1760), §91) do not call the axiom of parallels either an "axiom" or a "postulate," but a "theorem," since they each try to demonstrate it from the other axioms.)

[74] Kästner, *Geschichte der Mathematik* (1796), vol.1, 268; he makes the same point in the note on Euclid's axiom in his *AG*, 195, adding that he learned of these manuscripts from David Gregory's (1703) edition of the *Elements*. (It would be interesting to know at what time(s) Euclid's postulate began being called an axiom, though I don't know the story myself.)

there is a more or less useable test for candidate postulates: is the postulate a practical corollary of a primitive definition? does it assert the possibility of a constructive procedure that would not otherwise be licensed? But to test whether Euclid's parallel axiom is a genuine axiom, the only guidance we get is: first construct the concepts contained in the axiom by drawing two lines cut by a third such that the interior angles are less than two rights, and then see whether one then "can connect the predicate [<intersecting>] a priori and immediately." What is the test for immediateness? This is a difficult question, both in itself and for Kantians, and Kant never says anything explicitly about Euclid's axiom in his extant writings.

V. Schultz

On one common view of Kant's philosophy of mathematics, it would be manifest what Kant's attitude toward Euclid's axiom would be. Kant, enshrining pure intuition as an independent source of mathematical cognition, in essence gave authority to our sense of what is obvious, and so the axiom of parallels – which appeared obviously true to everyone before the nineteenth century – would have to be a legitimate axiom in Kant's eyes. This unflattering reading, though it came to prominence in the twentieth century,[75] has a long history, beginning with Kant's neo-Leibnizian critics. Leibniz himself, addressing Locke, wrote:

> [I]f anyone believes that his imagination presents him with connections between distinct ideas, then he is inadequately informed as to the source of truths, and will count as immediate a great many propositions which really are demonstrable from prior ones. (*New Essays*, IV.xii.6)

Leibniz's example of a demonstrable proposition is the axiom that two straight lines can

[75] See my paper, "Kant on Parallel Lines" for references and discussion.

meet only once, which Leibniz claimed Euclid mistakenly took to be indemonstrable. Both Kästner and Eberhard, quoting these words in the *Philosophisces Magazin*, argued that Kant had fallen precisely into the error Leibniz warned against.[76] In thinking that intuition can ground the necessity and generality of geometrical truths, Kant relies on the faulty certainty provided by indeterminate and imprecise images and therefore is in danger of accepting without demonstration axioms that can and should be demonstrated.

I believe, however, that there is every reason to believe that Kant would have rejected Euclid's axiom as legitimate. Of course, like all of his contemporaries, he believed that Euclid's axiom is true; indeed, he uses the axiom explicitly in his construction of parallel transport in the Phenomenology chapter of *The Metaphysical Foundations of Natural Science*.[77] But the controverted question with Euclid's axiom was not its truth, but whether that truth was immediately certain. Indeed, it was the consensus view in 18th century Germany – and indeed in all of the geometers whose works we can be confident that Kant knew – that Euclid's axiom required demonstration. The axiom was rejected in all of the books in Kant's library that discuss the theory of parallels, including works by Wolff, Hausen, Kästner, and Karsten.[78] It was rejected in all of the books that we can by other means be certain that Kant knew – including works by Leibniz, Segner, and Lambert. It was also rejected by Kant's students Schultz and Kiesewetter. Kiesewetter, for instance, writes: "Euclid made this proposition an axiom

[76] Eberhard, "Über die apodiktische Gewisheit," 161. A similar criticism is given, again referring to this passage in Leibniz's *New Essays*, by Kästner, "Über den geometrischen Axiomen," 429.

[77] Ak 4:492-3. (I owe Marius Stan for this reference.) Kant also uses the parallel axiom in his example demonstration of Euclid I.32 (A716-7/B744-5). Kästner, in his thorough *Geschichte der Mathematik,* vol.1 (268), claims to know of no one who had denied the truth of the axiom.

[78] I discussed the views of Leibniz, Wolff, Kästner, Hausen, and Segner above; Schultz and Kiesewetter are discussed in the sequel. According to Warda, Kant also owned Karsten's 1760 *Mathesis theoretica elementaris.* In this early work (unlike the later works discussed in notes 6, 38), Karsten fallaciously infers Euclid's axiom from the (correct) theorem that if the angle B'BA in figure 2 is acute, then there will always be another perpendicular A'B' to the right of AB that is shorter than AB (§91, 31-5).

[Grundsatz], which it obviously is not, but one has sought in vain up till now for an acceptable proof for it, although nobody doubts its truth."[79]

Indeed, Schultz gave a principled reason for rejecting Euclid's axiom. In addition to being Kant's most trusted disciple, expositor, and defender, Schultz was an active mathematician and wrote two treatises specifically on the theory of parallels (*Endeckte Theorie der Parallelen* (1784) and *Darstellung der vollkommenen Evidenz und Schärfe seiner Theorie der Parallelen* (1786)), which Schultz claimed gave the only possible treatment consistent with Kant's philosophy of mathematics. (Schultz also defended his theory against Eberhard and Kästner's objections in vol.2 of his *Prüfung der Kantischen Critik der reinen Vernunft* (1792)) – a work Kant himself called the most faithful interpretation of his philosophy.[80]) Schultz argued that the parallel axiom is a legitimate axiom only if it so evident that it requires no proof or so simple that it could not be proved from anything simpler. A comparison of the axiom with Euclid's others shows right way that it lacks the requisite evidence. With regard to simplicity, the mere fact that Euclid proves the converse of the parallel axiom – I.27, that, if a straight line falling on two straight lines make the interior angles on the same side equal to two right angles, the two straight lines will not intersect (see figure 1) – shows that the axiom cannot be too simple to be proved. After all, the converse of a proposition is no simpler than the proposition itself.[81]

These objections are old – going back at least to Proclus – and have seemed

[79] Kiesewetter, *AG*, §68. Schultz agreed that the axiom is true, but requires demonstration: "one sees no ground to doubt [the parallel axiom's] truth, and it would therefore be contrary to reason not to believe it; still though [if the axiom were not proved] geometry would at least be reduced to either a merely empirical knowledge, or it would in the end be nothing more than reasonable belief, but not, as it should be, the ideal of a pure rational science" (*Entdeckte*, 3).
[80] Ak 12:367-8.
[81] *Enteckte* 5, 14; *Prüfung* vol.2, 92.

compelling to thinkers of very different philosophical views. In Kant's lifetime, the argument against the simplicity of Euclid's axiom was given a particularly sharp and compelling form by Lambert. Lambert argued against Wolff that not every combination of concepts could be proven to be non-contradictory from definitions and the principles of identity and contradiction. In particular, if a proposition contains only simple concepts standing in simple relations, then no analysis of its components could show it to be possible, since *ex hypothesi* it admits of no analysis at all. Lambert therefore claims it is the function of axioms and postulates to secure or preclude certain combinations of simple concepts.[82] It follows that an axiom must contain only simple concepts and simple relations, and thus that a proposition cannot be turned from a theorem into an axiom by simply rearranging these simples. Lambert therefore concludes that the provability of I.27 rules out the parallel axiom from being a genuine axiom (*Theorie,* §9). Kant, whose philosophy does not thematize the distinction between the simple and compound as Lambert does, cannot give such a compelling and satisfying elaboration of this ancient objection. But this does not mean that Kantians like Schultz cannot still recognize its force. For this reason, Schultz, after citing Eberhard's objection that I cited in the first paragraph of this section, denies that Kantian philosophy requires geometers to take as axioms whatever seems obvious to them, and he can point to himself as an example of a Kantian philosopher who does not (*Prüfung*, vol.2, 76).

Although everyone agrees, Schultz claims, that Euclid's axiom lacks proper geometrical evidence, still there has been widespread confusion over what geometrical evidence consists in. "In my view," he argues, "the true ground of this evidence is none

[82] On Lambert's views of axioms and simples, see [[reference removed to maintain author anonymity]]. This view of axioms motivates Lambert's proof of Euclid's axiom from the axiom that *space has no absolute measure of magnitude*, which (he claims) contains only simple concepts and relations.

other than what Kant developed thoroughly in his *Critique of Pure Reason*" (*Entdeckte*, 27). It is because, he claims, geometers have not been looking to ground the evidence of the parallel axiom in intuition and construction that the search for a proof has foundered for so long. For this reason he rejects proofs of the axiom (like Leibniz's, Wolff's, and Karsten's) that try to derive it in a purely "philosophical" way from a new analysis of <parallel lines>.

> The theory of concurrent or nonconcurrent [i.e. parallel] lines asks after that position that they have against a third line that cuts them, or after the angle that they make with it. This is clearly a predicate of a completely different kind, which does not lie in the concept of the concurrency of two lines. Therefore is it strictly impossible that through the mere analysis of the concept of concurrency or non-concurrency a new predicate is determined, in such a way that the extended theory can be proven… [A] proof through merely logical analysis of the concept would be merely *discursive*, and so would lack intuition and thus also evidence, which is what constitutes the special character of a geometrical proof, …which infers not from concepts but rather must *construct* the concept, i.e. exhibit it in concreto in an intuition a priori. (*Entdeckte*, 42-3)

Kästner's sought-for proof, which would proceed from a new definition of <straight line>, would fall into the same error. In fact, Schultz argues, <straight line> is in principle indefinable (*Prüfung* vol.1, 58): even Kästner grants that <direction> presupposes <straight line>, and the immediate representation of infinite given space that is presupposed by every geometrical definition already contains all possible directions. And, even if <straight line> were definable, there is no reason to believe that it would make Euclid's axioms provable. After all, Schultz argues, Kästner also grants that we have a distinct concept of <circle>, but this does not make Euclid's third postulate – to draw a circle from a given point and distance – any less synthetic and indemonstrable.[83]

Schultz motivates his own method in the following way. Consider two angles ∠DAB and ∠FBC (figure 5). If they are equal, then we can know without using the

[83] *Prüfung* vol.2, 80-2; cf. *Prüfung* vol.1, 68-9.

axiom of parallels that (by I.27) they will not intersect. Now consider a larger angle ∠EBC. If we move ∠EBC and superimpose it on ∠DAB, then the line DA will intersect FB at the vertex A, and (as ∠EBC is slid along AC) the intersection point will move along DA towards D. But without presupposing the possibility of parallel transport, we cannot know whether the intersection point will vanish as ∠EBC is slid further along AD. This illustrates, Schultz claims, that we need a way of intuitively representing the mutual relations of two angles when they are separated from one another that does not simply presuppose the parallel axiom (*Entdeckte*, 35). Schultz's surprising proposal is to represent the magnitude of the angles by the size of the infinite surface contained between the arms of the angle when the arms are extended out infinitely far.[84] These infinite surfaces obey the following axioms, which Schultz claims are evident to immediate intuition (61):

1. Every angle determines an infinite angle surface.
2. Equal angles have equal angle surfaces.
3. Greater angles have greater angle surfaces.
4. Magnitudes that cover one another are equal and similar.

By 2, we can infer that the angle surface of ∠FBC = the angle surface of ∠DAB (even though the former is a proper part of the latter), and by 3 we can infer that the angle surface of ∠EBC > the angle surface of ∠DAB (=∠FBC). The truth of Euclid's parallel axiom follows immediately. For if EB did not intersect DA, then the angle surface of ∠EBC would be a proper part of the angle surface of ∠DAB. But this would violate the analytic truth that

[84] For a discussion of Schultz's theory, see Schubring, Gert, "Ansätze zur Begründung theoretischer Terme in der Mathematik: Die Theorie des Unendlichen bei Johann Schultz (1739-1805)."

5. The part cannot be greater than the whole.

Euclid's axiom now follows.[85]

Schultz draws some fascinating conclusions using this method. First he argues (contrary to the Euclidean common notion that the whole is greater than the part) that the whole can be equal to its proper part if the part and the whole are both infinite and the remaining part is "completely null with respect to it, or infinitely small" (76). For example, the angle surface of ∠DAB = the angle surface of ∠FBC since the remaining part, the "parallel strip" contained between the non-intersecting lines DA and FB, is infinitesimal with respect to the angle surface of ∠FBC (61). This is because no matter how many copies of this parallel strip are pasted together, it will have only a finite width and so will never cover the angle surface of ∠FBC. The angle surface of ∠FBC, on the other hand, is not equal to the angle surface of ∠EBC, since the remaining part – the angle surface of ∠EBF – is finite with respect to it, with the ratio of the two infinite surfaces = the ratio of the sizes of the corresponding angles. This leads Schultz into a second surprising conclusion, that "there are uncountably many degrees of infinite magnitudes" (60). The angle surface of ∠EBC is greater than that of ∠FBC, even though both are infinite. More pointedly, the parallel strip bounded by DA and FB is itself infinite with respect to any finite closed planar figure, though infinitesimal with respect to the angle surfaces. Schultz concludes that there are then "orders" [Ordnungen] of infinite magnitudes (93).

The fallacies in this method are patent to anyone who has studied the basics of Cantorian set theory, let alone measure theory. For instance, if we measure relative size

[85] This is theorem #7 (*Entdeckte,* 74). Schultz actually has three ways of proving Euclid's axiom, all of which use infinitary angle surfaces and the axioms I've listed.

by seeing what covers what when figures are superimposed, then simple Cantorian arguments will show that the angle surface of ∠EBC = the angle surface of ∠FBC, and indeed that both are equal to the parallel strip after all: Schultz cannot consistently maintain both 3 and 4.[86] What's more, the whole method seems to presuppose the parallel axiom, since the assumption that the parallel strip is infinitesimal with respect to the angle surfaces is well motivated only if the distance between DA and FB remains finite no matter how far they are extended, and this is false in hyperbolic geometry. However, even if we put these mathematical fallacies aside, there are still deep philosophical issues with this approach, especially for a Kantian: Can we extend methods of reasoning with finite figures to infinitary figures? Indeed, does it even make sense to talk of geometrical figures that are actually infinitely large? Still more, could there be intuitions of such figures?

VI. *Philosophisches Magazin*

While Schultz was arguing that Kant's philosophy provided the only possible solution to the problems with Euclid's axiom of parallels, the Neo-Leibnizian philosopher J.A. Eberhard was arguing that in fact these problems would be intractable if Kantian philosophy were true. As is well-known, Eberhard's attacks on Kant in the four volumes of the *Philosophisches Magazin* – which in particular criticized Kant's philosophy of mathematics, questioned whether the critical philosophy really presented an original advance over Leibniz, and disputed the tenability of the analytic/synthetic distinction – were especially irksome to Kant. Kant responded to the first volume of the *Magazin* with

[86] This is made more patent in Schultz's third version of the argument, where he allows infinite surfaces to be cut into parts and rearranged (*Entdeckte* 98, Theorem 10).

his 1790 book *On a Discovery*, and he deputized Schultz to write a reply to the second volume, providing him with suggestions and a short essay "Über Kästners Abhandlungen" for Schultz to incorporate into his review.

A feature of Eberhard's criticism in volume 1 that enraged Kant was his equation of intuitions [Anchauungen] with images [Bilder].[87] Eberhard makes the same move in volume 2 (82, 151, 160), recasting Kant's distinction between the intuition and the concept of space as the distinction between an "imageable" [bildlich] concept and an "unimageable" [unbildlich] concept of space (84). This reduces Kant's doctrine of pure intuition of space to an absurdity:

> The imagination can only represent single things; the limits of its space belongs to the individuality of a spatial thing. Experience agrees fully with this remark. I cannot make an image [Bild] of a single infinite dimension, I cannot make an image of an infinite line; still less can I make an image of multiple infinite dimensions, of an infinite surface, of an infinite body. (85)

Eberhard concluded that there could not be an intuition of space as a whole, and the Leibnizian view of space (as a concept abstracted from the relations of bodies) is the only possible alternative.

More significantly for our purposes, however, Eberhard also used the impossibility of an intuition of an infinite geometrical object to show that a Kantian could never provide a proof of Euclid's axiom. Eberhard argues, quoting Leibniz's *New Essays*, that though the ultimate ground of "secondary" axioms like Euclid's is in the definitions together with the principles of identity and contradiction, nevertheless the axiom can be made evident through the imagination, since an image is a confused representation of what would be made distinct with a proper definition (154, 165, 182). The impossibility

[87] See, for instance, *On a Discovery* Ak 8:222 ("where have I ever regarded the intuitions of space and time, in which all images are first of all possible, as themselves images…?"), and the discussion in Allison's "Introduction," 30-1, 83-4.

of infinitary images then explains why the axiom of parallels lacks not only a proof but also evidence. Compare Euclid's axiom *Between any two points there is only one line* with *Two lines, which make with a third two angles that are less than two right angles, if they are produced to infinity, intersect one another.*

> Why does the first of these propositions have its evidence and why is it lacking in the latter? [...] – For this reason, for in the subject of the first is contained the concept of a finite space, a space determinate with respect to its limits, while in the subject of the second the concept of an infinite space, indeterminate with respect to its limits; of the first the imagination can make an image in which the understanding can represent the general, but of the second it cannot; for the imagination can no more represent something indeterminate, e.g. a line whose length is without limits, than it can be sketched for the senses. (89-90)

This diagnosis explains not only why Euclid's axiom is uniquely problematic, but it indicates what kind of proof must be found: a purely conceptual proof, because – though an intuition of something infinite is impossible – it is nevertheless possible to have a *concept* of what is infinite.

In *On a Discovery* Kant criticized Eberhard for claiming that every geometrical axiom is provable in principle, despite the fact that centuries of effort had failed to provide proofs for Euclid's axiom (Ak 8:196, quoted on p.3). This failure makes it very doubtful that the sought for proof of Euclid's axiom could proceed analytically from new definitions. Eberhard admits that in geometry "the image is not capable of being analyzed by a finite mind" (155). He continues:

> Admittedly the human understanding needs other principles besides the definitions, and these contain intuitions a priori, or general imageable marks. This is a peculiarity of geometry. But these imageable concepts in themselves – although not for a human mind – must be knowable through their grounds, and these grounds must be at the same time the grounds of the necessary truth of their axioms. (169)

Eberhard has now backed himself into a corner. Leibniz argued that necessary truths differed from contingent truths in being derivable from the principles of identity and contradiction in a finite number of steps. Eberhard cannot claim that the necessary truth of geometrical axioms would be evident only to an infinite understanding without giving up on the Leibnizian view altogether.[88]

In an unexpected way, Eberhard's diagnosis of what ails Euclid agrees with Schultz's: both think that Euclid's axiom lacks evidence because we do not have a method for representing in intuition infinite portions of the plane, and both think that a properly Kantian proof would require such an infinitary intuition. It is just that Schultz thinks such an intuition is possible, and Eberhard does not. Both Schultz and Eberhard, then, articulate a philosophical analysis of Euclid's axiom at variance with Kästner's – a fact not lost on Kästner himself. In fact, Kästner, who was invited by Eberhard to defend the Leibnizian view of mathematics against Kant in a series of essays in vol.2 of his *Magazin*, used the occasion to criticize both Eberhard's and Schultz's arguments. He points out that Euclid showed in a completely evident ways without using the parallel axiom that if the angle *u* in Figure 1 is a right angle then AA' and BB' will never intersect (this is *Elements* I.27), and will then bound an unlimited portion of the plane between these two lines. Thus, "the failure of evidence [of the parallel axiom] does not depend on the fact that one has no imageable concept of infinite space" (*PM* 2, 415).

Besides undermining its motivation, Kästner was keen to refute the very possibility of Schultz's treatment of the theory of parallels. This intent was certainly evident both to Schultz and to Kant (who mentions Kästner's criticism of Schultz in a letter, Ak 11:184). Schultz's method requires infinite figures. A figure, however, is "a

[88] For another criticism of Eberhard's claim, see Schultz, Ak 20:407.

space that has boundaries; therefore [an infinite figure] is an infinite space that has boundaries. And so a person will be duped who believes that he has an imageable concept of an infinite space. He will imagine [bilden] what is very large, not what is infinite" (409). The only way to form an intuition of an infinite figure would be to repeat infinitely many times the construction procedure for drawing a finite figure. But this is "an ability no mathematician possesses": "he who would want to compose the infinite from finite straight lines must take them infinitely many times. Since now the collection of them that he has taken is always finite, the infinite will never come to be" (411). The actual infinite is never found in reality, only the potential infinite (418). When Euclid says in his axiom that the two lines will intersect "if produced into infinity," he only means that if the lines are extended far enough, they will intersect. This intersection point may be as far away as Sirius, but it will always be at a finite distance (407, 12).

VII. Kant's "Über Kästners Abhandlungen"

Kästner inferred from the impossibility of an actually infinite spatial figure not only that Schultz's proof was impossible, but also that there can be no pure intuition of space at all. Kant was intent on responding to both Kästner's and Eberhard's criticisms of his theory of space and geometry, but – having already responded to vol.1 of the *Magazin* in his book *On a Discovery* and always concerned about his limited time – deputized Schultz to write a reply for him.[89] Kant did, however, take the time to personally write a short essay ("Über Kästners Abhandlungen," Ak 410-23) replying to Kästner's essays in particular, and asked Schultz to include these pages in his review. Kant's goals in writing this short essay were threefold. First, he thought that an essay from Kästner (whom Kant admired)

[89] Schultz's review is reprinted in Ak 20: 385-423.

merited more attention than the essays from Eberhard (whom Kant despised). Second, Kant wanted to clarify his conception of the infinity of space in response to Eberhard's objections. And, third, Kant wanted to distance himself from Schultz's treatment of parallels. Though Kant had referred to Schultz's *Entdeckte* as "your ingenious theory of parallel lines" in 1784, it is clear by 1790 that Kant – either from studying the work himself or from reading Kästner's criticism of it – had grown skeptical. In two letters to Schultz, he urges him not to mention the theory of parallels at all in his review (Ak 11:184, 200-1) – a suggestion that Schultz did not follow[90] – and when sending Schultz his essay on Kästner he writes:

> I believe that the enclosed [essay] might (if I may flatter myself) offer some new material for bringing your theory into agreement with what the Critique says in the section on the antinomy of the infinite in space. (Ak 11:184)

Kant is referring here, of course, to the second part of the first antinomy, whose thesis is that the world is enclosed in boundaries in space, and whose antithesis is that the world has no bounds in space, but is infinite with regard to space.

The conflict between Kant's resolution of the antinomy and Schultz's theory is made vivid by looking at the two ways Schultz defends his view that infinite spatial magnitudes (such as his angle surfaces) can be constructed. In each of these defenses, Schultz begins with a genuinely Kantian idea but applies it in a heterodox way. First, Schultz argues, the fact that there are no *empirical* intuitions of infinite magnitudes does not show that there cannot be an infinitary *pure* intuition. In reply to objections from G.A. Tittel (an ally of Feder), Schultz argues that Kant's critics misread the pure intuition

[90] In his review, Schultz inserts into Kant's "Über Kästners Abhandlungen" a page defending his theory of parallels against Kästner's criticism, and claiming that the only possible proof of the axiom of parallels required constructing infinite portions of the plane bound by angles (Ak 20:221-2). (Note that this passage is not included in Allison's partial translation in *The Kant-Eberhard Controversy*.)

of space as if it were like an image or visual experience (*Prüfung* vol.1, 107), and then infer that it could not be infinite. (Indeed, Eberhard, in confusing intuitions with images, does precisely this.) Infinite three-dimensional space, as the form of sensibility, is not like an empirical intuition. Schultz continues:

> what concerns a proper geometrical construction consists not in its sensible picturing [Abbildung] but rather in pure intuition; so in my opinion the infinite can be constructed in geometry just as completely as the finite … Thus I have e.g. an entirely intuitive concept of a straight line AB produced on the side B without end, and I can draw it in thought without end just as well as I can draw a finite line. (*Darstellung*, 44-5)

This last move, however, goes far beyond the first. To say that there is an intuition of the infinite space within which empirical intuitions and constructed geometrical figures take place is not yet to say that there can be infinite *constructed* figures within that space.

Second, Schultz argues that intuitions of infinite figures are just as admissible as those of finite figures because (as Kant argued in the Metaphysical Exposition of the Concept of Space, A24-5/B39) the representation of any finite, bounded figure presupposes the representation of an infinite space from which it is limited off. Eberhard, thinking of all pure intuitions on the model of empirical images, argues that an intuition of space as a whole would have to be an "infinite aggregate" (*PM* 2, 86), which could never form a whole. Schultz replies:

> Mr. Eberhard regards it as an entirely false idea that particular spaces are parts of one and the same, single space, *because no infinite aggregate can be a totality,* but this rests entirely on the completely erroneous notion that we can only arrive at the concept of the whole of a single, infinite space by first composing it as an aggregate out of infinitely many parts. He thus neglects the unique feature, which is found in no other composite besides space and time. This is the fact that space is given to us in pure intuition as a magnitude, in which the *possibility* of representing parts *already presupposes the representation of an infinite whole.* (Ak 20: 402)

So far, so good. But Schultz continues:

> Moreover the finite would not be given to us, if the infinite were not given to us, i.e. the construction of the finite presupposes the construction of the infinite, so that the former without the latter would be entirely impossible. Since every determinate magnitude can be increased without end, I cannot thus think a finite magnitude in any other way than by representing it as a limited part of the unlimited or infinite. So space, according to the original intuitive representation that we have of it, is only a single space extending without end from all possible sides. This is our first essential geometrical construction, or that pure intuition in which space, as the object of geometry, is originally presented. … For what does the second postulate of Euclid say: to extend a straight line without end, which is in truth nothing other than: I should represent it as one part of an infinite line, and thus produce it in thought without end. (*Darstellung*, 46, 8)

However, to say that finite constructed figures presuppose infinite space is not to say that finite figures presuppose infinite constructed figures. And this additional move would not follow unless single infinite space were itself constructed, as Schultz here alleges.

In each of these arguments, Schultz defends his infinite angle surfaces by arguing that the pure intuition of an infinitely long line is just as admissible on Kantian principles as the pure intuition of space as a whole. Unfortunately, Kant clearly argued the contrary. Kant asserts that "I cannot represent to myself any line … without drawing it" and that drawing is a "successive synthesis" in which the representation of the parts of the line precede the representation of the whole (A162-3/B203-4; cf. B137-8, B154). Further, the "true transcendental concept of infinitude" according to Kant is that "the successive synthesis of units required for the enumeration of a quantum can never be completed" (A432/B460). An intuition of an infinitely long line is therefore impossible, and Kant prefers to describe the drawing of a line as a *progressus in indefinitum* not a *progressus in infinitum* (A510-1/B538-9).

Kant employs a similar line of reasoning in resolving the first antinomy, concerning the magnitude of the world whole. An intuition of a world whole would require an intuition of the space occupied by that world whole, and "the synthesis of the

manifold parts of space, by means of which we apprehend space, is successive" (A412/B439), taking place in time and proceeding from part to whole. The world whole cannot be infinite, then, given the transcendental concept of infinitude. Indeed, there cannot be an intuition of the world whole at all (A518/B546). Thus, even though the synthesis of the world is an empirical synthesis of empirical objects in space and the synthesis of a line is a pure synthesis of parts of space, both syntheses are successive. And this is sufficient to show – contrary to Schultz's first argument – that an intuition of an infinite spatial line or figure is no more possible than an intuition of the world whole. Of course, this does not mean that the world whole is finite, either; measurement (the determination of the magnitude of a quantum) requires the synthesis of the parts (A412/B439, A426-8/B454-6), and so the world whole (the synthesis of which is never complete) has no determinate magnitude at all. As with a straight line, the synthesis of the world whole proceeds *in indefinitum* not *in infinitum* (A518/B546). Since the representation of the world whole would have to proceed from part to whole, we cannot say that the world whole is a *totality* at all (A500-1/B528-9). Schultz was right in his second argument to protest that space as an infinite given whole can be a totality without its representation being composed out of the representation of its parts, but he was surely wrong to think that a figure in space (whose synthesis, like that of the world whole, is successive) could be an infinite totality. Schultz motivated his proof of Euclid's axiom by claiming that his replacement axioms – that every angle determines an angle surface, that larger angles have larger surfaces and equal angles equal surfaces – satisfied the Kantian conception of axioms as indemonstrable theoretical propositions that can be exhibited in intuition. Kant is claiming that in fact Schultz's axioms are not genuine axioms after all.

Kant therefore entirely agrees with Kästner's criticism of Schultz's treatment of parallels, as he makes clear in "Über Kästners Abhandlungen." The critical theory of space is "in perfect agreement" with Kästner's contention that the actual infinite is never found in reality, only the potential infinite (Ak 20: 421), and in Kästner's essays "it is very correctly pointed out" that "infinite space would never come to be by means of composing it from finite spaces" (Ak 20: 417). Kästner had inferred, however, from the impossibility of an infinite constructed figure (like an infinitely long line or infinitary planar surface) to the impossibility of an intuition of infinite space as a whole. Schultz reasoned in the opposite direction, from the permissibility of an intuition of infinite space as a whole to the permissibility of infinite constructed figures (like his angle surfaces). Kant believed that both philosophers were wrong. He makes room for this middle position through his famous distinction between metaphysical and geometrical space, which is articulated in a passage from Kant's essay that has been widely discussed by recent commentators.[91]

> Metaphysics must show how one has the representation of space, geometry, however, teaches how one describes a space, i.e., can present it a priori in a representation ... In the former, space is considered as it is given, before all determinations, in accordance with a certain concept of an object. In the latter it is considered as it is generated [gemacht]. In the former it is original and only one (single) space. In the latter it is derived and there are (many) spaces.

Space as it is considered in metaphysics is *single* and *original*; spaces as considered by geometry are *many* and *generated* or constructed. Schultz's infinitely long lines and infinitely large area surfaces are then geometrical spaces, since they are many and constructed. For this reason, he cannot argue for their admissibility by appealing to the

[91] Ak 20:419-20; in *Kant-Eberhard*, 175-6. See Carson, "Kant on Intuition in Geometry"; Friedman, "Geometry, Construction, and Intuition in Kant and His Successors," 188ff.; Friedman, "Kant on Geometry and Spatial Intuition."

admissibility of space as a whole without confusing Kant's distinction. And he certainly cannot say of singular space as a whole – which is metaphysical space, original and not generated – that it is "our first essential geometrical construction."

Kästner had argued that an intuition of infinite space is impossible on the grounds that it would require taking the construction of a finite space and iterating it infinitely many times. Kant agrees that there can be no actually infinite geometrical space, but he argues that there can be an intuition of infinite metaphysical space, whose infinity does *not* consist in an infinitely iterated construction procedure:

> Now one can only view as infinite a magnitude in comparison to which each specified homogeneous magnitude is equal to only a part. Thus, the geometrician, as well as the metaphysician, represents the original space as infinite and, indeed, given as such. For the representation of space (together with that of time) has a peculiarity found in no other concept, viz., that all spaces are only possible and thinkable as parts of one single space, so that the representation of parts already presupposes that of the whole.

Space as it is considered in metaphysics is *infinite* and the representations of its parts *presuppose* the representation of the whole; a space as considered by geometry is *finite* and the representations of its parts *precede* the representation of the whole. Schultz's area surfaces, being geometrical surfaces, cannot be infinite as he claims. He was right to argue against Eberhard that the pure intuition of (metaphysical) space can be a totality without being aggregated out of finite spaces, but he was wrong to think that this lends any legitimacy to his infinite geometrical spaces.

A key aspect of the dispute between Eberhard, Kästner, and Schultz concerned the infinite extensibility of a straight line. To Eberhard, the impossibility of an intuition of an infinite straight line shows the impossibility of any intuition of infinite space; to Kästner, an infinite straight line cannot come to be by iterating the construction of a finite line; and

to Schultz, the admissibility of extending a line rests on the fact that every finite line is already a part of an infinite (but still constructed) line. Kant believes that all three have gone wrong:

> Now, if the geometer says that a straight line, no matter how far it has been extended, can still be extended further, this does not mean the same as what is said in arithmetic concerning numbers, viz., that they can be continuously and endlessly increased through the addition of other units or numbers. In that case the numbers to be added and the magnitudes generated through this addition are possible for themselves, without having to belong, together with the previous ones, as parts of a magnitude. To say, however, that a straight line can be continued infinitely means that the space in which I describe the line is greater than any line which I might describe in it. Thus, the geometrician expressly grounds the possibility of his task of infinitely increasing a space (of which there are many) on the original representation of a single infinite space, as a singular representation, in which alone the possibility of all spaces, proceeding to infinity, is given.

Against Kästner and Eberhard, Kant argues that the extendibility of any line depends on a prior intuition of infinite space. But he argues against Schultz (and with Kästner) that this infinite space is not itself drawn or constructed (and so not a geometrical space).

Kant's clear conclusion is that the new axioms that Schultz proposes to prove Euclid's are not genuine axioms at all.[92] Of course, Kant would not have accepted a purely discursive proof (such as Wolff's demonstration or Kästner's hoped for proof from a new definition of <straight line>), and he would not have been content to argue, as Eberhard did, that at least God could prove it. An empirical demonstration would also have been out of the question for Kant, since geometrical propositions "cannot be proven through looking, measuring, and weighing, but rather a priori" (Ak 20: 417). If I am correct that he would not have accepted Euclid's axiom, either, then Kant is left in the uncomfortable position of requiring a proof, but having to reject on philosophical or

[92] I strongly disagree, then, with Webb's contention ("Hintikka on Construction," 231) that Kant accepted Schultz's proof.

mathematical grounds all of the candidate replacement axioms on offer. Or at least, this is how Kant's student Kiesewetter saw things in his 1799 *Anfangsgründe*. Kiesewetter, as we saw, rejected Euclids axiom and demanded for it a proof, though "one has sought in vain up till now for an acceptable proof of it" (§68). In particular, Kiesewetter rejected Schultz's approach (without mentioning him by name) on strictly Kantian grounds:

> It will not be possible to provide a direct proof that the angles [of a transversal with two parallels] lying on one side are equal to two right angles from the mark of the parallelness of two lines, since the infinite extending of lines, without them intersecting, does not admit of construction (intuitive exhibition) [keine Construction (anschauliche Darstellung) zulässt].

I have been arguing that Kiesewetter has Kant right here: Euclid's axiom required a proof, but there really was no proof on offer that could meet Kantian strictures. Though this was surely an uncomfortable place to be in, it was in hindsight the only acceptable position to take. For, as we now know, there simply were no proofs to be had.[93]

[93] [[acknowledgements]]

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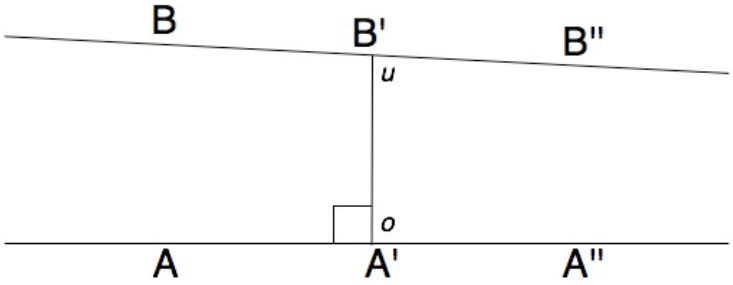
Figures



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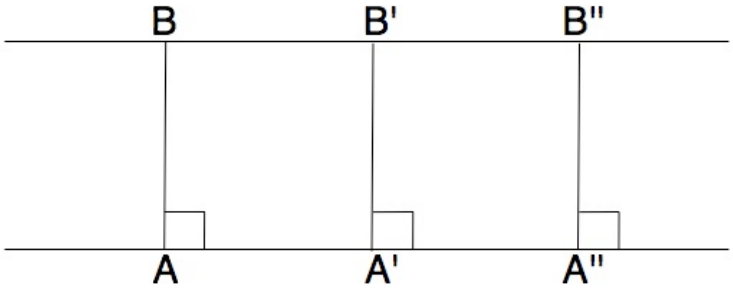

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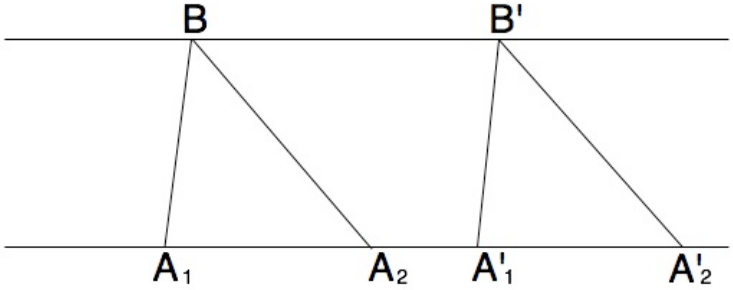

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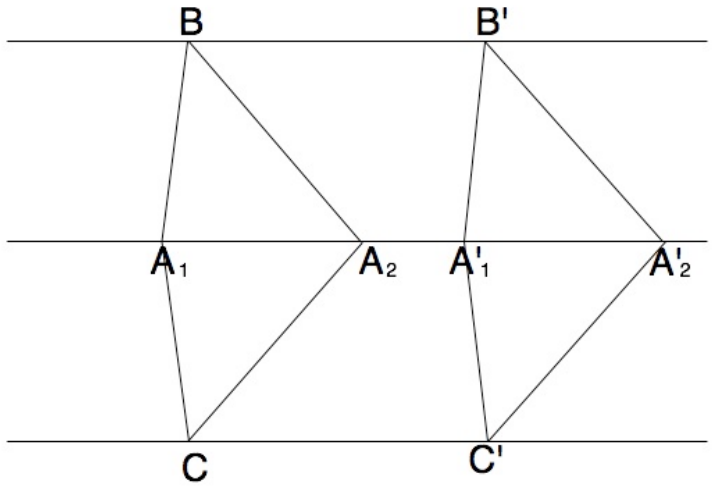

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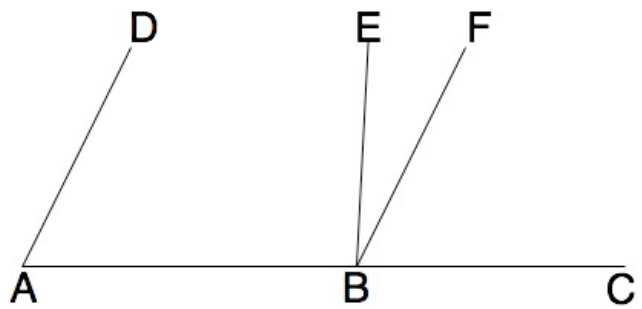

Figure 5.