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Frege, Lotze, and Boole

Jeremy Heis

1 Dummett and Sluga

In the 'analytic tradition', Hans Sluga wrote thirty years ago in his book *Gottlob Frege*, there has been a 'lack of interest in historical questions – even in the question of its own roots. Anti-historicism has been the baggage of the tradition since Frege' (Sluga, 1980, p. 2). The state of the discussion of Frege among analytic philosophers, Sluga claimed, illustrated well this indifference. Despite the numbers of pages devoted to Frege, there was still, Sluga claimed, little understanding of the sources of Frege's ideas, his intellectual debts, and the historical circumstances of his thought. Sluga singled out Michael Dummett's *Frege: Philosophy of Language* 'as a paradigm for the failure of analytic philosophers to come to grips with the actual, historical Frege'. In that book, Dummett famously wrote that the logical system that Frege put forward in his 1879 *Begriffsschrift* 'is astonishing because it has no predecessors: it appears to have been born from Frege's brain unfertilized by external influences' (Dummett, 1981a, p. xxxv) – and Dummett devoted almost none of its 700 pages to the relationship between Frege and his contemporaries.

What followed was an academic controversy remarkable for covering so many topics and for filling so many pages. In addition to his book from 1980, Sluga criticized Dummett's reading of Frege at length in five papers published between 1975 and 1987.[1] Dummett responded with four papers published between 1976 and 1982, and published in 1981 a book, *The Interpretation of Frege's Philosophy*, whose 600 pages are devoted largely to defending his interpretation against Sluga.[2] The discussion ranged over almost every significant topic in Frege's philosophy of logic and language. What is of particular interest more generally, however, is the explicitly historiographical and methodological character of the debate. The explicit topic of debate was not just, for example, 'What is the correct interpretation of Frege's context principle?' The topic was also 'What *method* should we use in giving an interpretation of that principle?' Sluga and Dummett disagreed

on how much Frege took from Kant and from nineteenth-century logicians. But even more fundamentally, they disagreed on the *value* for a historian of *philosophy* in placing a philosopher in his historical context. For this reason, perhaps the most central issue in the debate was Sluga's assertion that a philosopher writing on Frege should acknowledge the pervasive influence of the philosopher Hermann Lotze on Frege. Sluga argued that the proper interpretation of Frege's 'realism', of his conception of objectivity, and of his context principle *required* seeing these doctrines as derived from Lotze's philosophy. For Dummett, on the other hand, Lotze's writings do not afford 'any useful comparison with Frege's views': though 'a great many of Frege's leading ideas appear, in Sluga's book, as having been derived from Lotze', in fact this interpretive claim only 'results in a far-reaching misinterpretation of Frege's thought' (Dummett, 1981b, pp. 396, 525).[3]

Sluga saw Dummett's failure to acknowledge the indebtedness of Frege to Lotze as symptomatic of a more general failure of Dummett's, and indeed, of analytic philosophy itself. Analytic philosophers since Frege have been uninterested in comprehending 'concrete historical processes' or in the 'examination of actual historical discourse' (Sluga, 1980, pp. 2, 186). They have preferred instead to engage in an 'unhistorical kind of meaning analysis' that abstracts from the 'subjective, historical, and personal' features in a philosopher so as to provide a 'rational reconstruction' of the philosopher's thoughts (pp. 181, 3). Analytic philosophers, Sluga seems to be arguing, assume that they can solve the philosophical questions that currently vex them without first reflecting on the histories of these questions – that is, why we ever came to think that these questions were the important ones. Similarly, they presume to know what questions a past philosopher was trying to answer and what her words mean, without going through the hard work of placing that philosopher in her historical setting. Sluga thinks these assumptions are mistaken and that the blame lies with the philosophy of language that the analytic tradition derived from Frege.

> There are deep reasons why the writings of the analytic tradition are unhelpful at this point. From its very beginning, the tradition has been oriented towards an abstract, formal account of language and meaning, and not towards the comprehending of concrete historical processes. Frege himself considered his task that of the analysis of timeless, objective thoughts. (p. 3)[4]

For Frege, the sense of an expression is a timeless, impersonal entity that exists independently of whether any actual individual thinker ever grasps it. Thus, Sluga alleges, a Fregean historian of philosophy will approach the thoughts of a past philosopher as themselves abstract, timeless, and impersonal: she will ignore the contingent facts about the philosopher as

a person – whom she knew, what she read, what her contemporaries were saying – and the historical facts about the time in which she lived and wrote.

One might be skeptical about Sluga's diagnosis of the implicit causes of analytic philosophy's alleged anti-historicism. In any case, though, Dummett himself was quite explicit in his reasons for downplaying the significance of the investigation of various influences on Frege's thought – for preferring the method of 'rational reconstruction' and 'logical analysis' over 'historical analysis'. The first step in Dummett's defense identifies a criterion for the success of a philosophical interpretation of a philosopher like Frege:

> A good, though partial test for the perceptiveness of any exposition of the thought of a great philosopher may be had by asking how interesting the result would be simply as a piece of philosophy, for someone who had neither read that philosopher nor felt any special curiosity about the correct way of interpreting him. (1981b, p. 528)

There is surely some reason, after all, why philosophers write more on Descartes than on Gassendi; more on Kant than Jacobi; Bolzano than Drobisch. This does not mean that there are not good reasons for reading these other philosophers. But a philosopher reading historical texts differs from an historian in trying to pick out those philosophers whose work – whether or not it is historically significant in other ways – is good *as a piece of philosophy*. The second step singles out a particular reason why contemporary philosophers should see Frege's philosophy as a good piece of philosophy: 'Frege is so interesting a writer because we have got so comparatively short a way beyond the point he reached. [...] Frege's problems are therefore still our problems; his thoughts still answer to our concerns' (1991b, p. 158).[5]

These two ideas together provide an apparently compelling argument for philosophers to be indifferent to Lotze and to Frege's other contemporaries. If Frege's questions were not our own, then to see the philosophical merits in his writings would require us to imagine ourselves in the position of the typical German logician in 1879. However, we *already* know that his questions are interesting and that his answers merit philosophical reflection and close reading. This does not mean that Frege's philosophy emerged entirely free from outside influences. It is just that tracking influences does not help us to understand Frege's philosophy *as a piece of philosophy*.[6] A historian of ideas, who is concerned more with the question of historical causation, and is thus free to abstract away from the question of the philosophical value of a work, would of course be interested in questions of influence. But a philosopher, Dummett is arguing, should not be like such historians.

In this chapter, I will be presenting a contextual reading of Frege that, like Sluga's, explores Frege's relationship with Lotze. If my reading succeeds,

I will have demonstrated contra Dummett that seeing the *philosophical merits* of Frege's writings does indeed require locating them in the context of late-nineteenth=century German philosophy and logic – that contextual readings of Frege are not irrelevant or opposed to the interests of philosophical readers.

Where, then, does Dummett's argument go wrong? One possible reply would be to argue that Lotze's philosophy of logic is a real philosophical contender to Frege's, much as, say Thomas Ryckman (2005) has argued that Cassirer's and Weyl's philosophies of physics are genuine and serious philosophical alternatives to Reichenbach's.[7] In Section 3, I will argue that there is more of philosophical interest in Lotze's philosophy of logic than might first meet the eye; however, I do not think that Lotze's logical writings – standing on the far side of the logical revolution brought about by Frege and Russell – are a genuine alternative to Frege's. Another possible reply would emphasize the philosophical value of unfamiliar ideas. Perhaps reading Frege in the context of his contemporaries will uncover for us a set of questions and a vocabulary for answering these questions that, for us at least, are new. And perhaps if we try hard enough to think in those foreign terms, we will see the philosophical merit in these ways of thinking – ways of thinking that (who knows?) we might use some day. At the very least, studying thoughts not like our own can provide for us a healthy skepticism about our own questions and assumptions.[8]

Though this second reply might be compelling in other contexts, there is a significant obstacle for a historian writing on *Frege* to accept it. Dummett writes:

> Frege's formal logic has no predecessors: in the writings of nineteenth-century logicians before *Begriffsschrift,* not one hint can be found of the ideas underlying Frege's discovery of quantification theory. But Frege's formal logic is the principal factor determining the subsequent development of his philosophy, and certainly of his philosophy of language; it forms the backbone of that philosophy, which collapses if it is extracted. (1981b, p. xvii)[9]

Sluga may be correct, for instance, that both Frege and Lotze speak of the 'priority of judgments over concepts', but, Dummett contends, Frege meant this thesis to provide a key to understanding Frege's new logic and the kinds of logical analyses that can be carried out with its aid. Lotze, who was ignorant of Frege's logic, could not have understood what *Frege* meant by this phrase, and he could not even have understood the questions Frege was trying to answer. Frege's new logic introduces a break in the philosophy of logic and language, Dummett contends, and those of us whose understanding of logic and language has been shaped by Frege's discovery can gain little from reading Frege's predecessors and contemporaries. A contemporary ethicist

may profit from reading Aristotle's *Nicomachean Ethics*, but a philosopher of physics will be wasting her time studying Aristotle's *De Caelo*. Why, then, should a philosopher of logic read Lotze's *Logic*?

Dummett is surely correct that Frege's Begriffsschrift[10] was the principle factor driving his philosophy – both to understand this new tool and to explore its implications. However, Dummett concludes too quickly that this fact about Frege reduces the value of contextual readings. Here is why. One central feature of the analytic tradition's self-narrative is that the invention of the new logic did bring about a sea change in philosophy and made possible progress (maybe even definitive progress!) on philosophical issues that would otherwise be intractable.[11] But this early optimism has not been substantiated. We no longer believe that logical analysis using the tools of Frege and Russell's logic will allow for the eventual solution of every genuine philosophical problem. So, what then *is* the philosophical pay off provided by Frege's new logic? Why are we better off now than philosophers were 130 years ago?

It is precisely here – I will be arguing – where a contextualized reading of Frege can be of service. When we see what the philosophical world looked like *before* the introduction of the new logic, we can begin to evaluate the real value of the new logic for Frege and his contemporaries. We can identify the nineteenth-century philosophical problems that Frege thought himself to have solved using the new logic. It will, of course, take substantial historical work to think ourselves into the position of a philosophically sensitive logician who is unacquainted with Frege's work. But when we have gotten past the excessively bullish predictions of analytic philosophers past and have learned to see Frege's logic with fresh eyes, we can learn to appreciate in a more balanced way what makes the new logic such a powerful tool – and we can appreciate what makes Frege's philosophy of logic (entwined as it is with his new technical tool) good *as a piece of philosophy*.[12]

Though I side with Sluga in maintaining the philosophical importance of locating Frege contextually, I nevertheless do not think that there is good reason to prefer Sluga's 'historical analysis' to Dummett's 'logical analysis' and 'rational reconstruction'. The contextual reading I will present spells out the particular philosophical advances that Frege was able to make with the help of his new technical tool. Since this is my goal, cataloguing the affinities and influences of other philosophers on Frege will not be a primary goal. On the contrary, I will engage in detailed readings of particular arguments put forward by Frege and Lotze, with an eye to the ways that Frege, having taken up the questions and themes of his contemporaries, was able to make substantial progress beyond them. Indeed, in the closing section of this chapter, I will argue that Sluga's own contextual reading actually *obscures* the philosophical significance of Frege's work. (And, ironically, Sluga's contextual reading of Frege falls short precisely because he avoids

the kind of detailed logical analysis of Frege's and Lotze's arguments that he – wrongly – opposes to historical readings.)

My reading will begin in the next section by looking in particular at Frege's writings from the early 1880s that compare his Begriffsschrift with the systems put forward by George Boole and his followers. The reason for looking at these writings is threefold. First, Frege is most explicit about the philosophical significance of his new logical system when he is arguing its merits over older logical systems, like Boole's. Second, recent historical work has made it clear that these Fregean works were intended to engage an ongoing discussion within the German philosophical community about the value of Boolean logic – and, by extension, about the value of systems of logical notation.[13] These writings are therefore ideal subjects for a contextual reading of Frege. Third, we will see that Lotze himself had criticized Boolean logic in 1880. Looking at Lotze's criticisms of Boole will be an effective way, then, to see both what Frege has in common with Lotze, and – most importantly – what kind of surplus philosophical work the Begriffsschrift allows Frege to do.

2 Frege's new theory of concept formation and his criticism of Boolean logic

Almost twenty years after the publication of *Begriffsschrift*, Frege described the philosophical purpose for which the Begriffsschrift was invented:

> I became aware of the need for a conceptual notation [Begriffsschrift] when I was looking for the fundamental principles or axioms upon which the whole of mathematics rests. Only after this question is answered can it be hoped to trace successfully the springs of knowledge upon which this science thrives. (Frege, 1897, p. 235; cf. 1884, section 3)

In order to isolate these fundamental principles of mathematics, Frege needed a way of determining whether a candidate derivation of some theorem is free of gaps or in fact requires some unrecognized further concept or principle. Since even Euclid was led astray by the imprecision of ordinary language to assume certain principles without acknowledgment, it was clear to Frege that a logically improved language was needed for carrying out inferences (1882, p. 85; 1979, p. 253). Furthermore, any incomplete analysis of the fundamental concepts out of which mathematical judgments are composed will lead to an incomplete analysis of the conceptual content of a mathematical judgment; and without a complete analysis of the content of a judgment, a candidate proof of that judgment might contain a hidden gap or be insufficiently general.[14] So, for Frege, (1) the precise analysis of mathematical concepts, (2) the construction of gap-free proofs within a new symbolic language, and (3) the determination of the basic laws of mathematics are all essentially connected elements in the same project.

Frege called the symbolic language that he invented for this project a 'Begriffsschrift' because a language that allowed for the successful execution of Frege's project would be a characteristic language for mathematics or a 'lingua characterica', in Leibniz's sense.[15] The symbolic language that Leibniz described would perform three interrelated functions. First, the language would 'compound a concept out of its constituents rather than a word out of its sounds' (1880–1, p. 9). Since the primitive symbols of a *lingua characterica* would express simple concepts, and symbols for compound concepts would be composed from the symbols for their component concepts, the *content of a concept* could be directly read off from its symbolic expression. Second, once the complete analysis of concepts has been completed and expressed perspicuously in the symbolic language, all *inferring could become calculating*. The language would be, in Leibniz's terms, a 'calculus ratiocinator' – a calculus for carrying out inferences. Third, Leibniz hoped that then the *truth of every judgment* could be determined by calculating with the symbolic expression that expresses that judgment.

In a series of papers written between 1880 and 1883, Frege argued that only a symbolic language like Begriffsschrift – and not the algebraic logical languages proposed by Boole and his followers – provides the necessary tools for a characteristic language.[16] One reason is that the Boolean logicians lack an adequate representation of *generality*.

> It is true that the syllogism can be cast in the form of a computation… [E]ven if its form made it better suited to reproduce a content than it is, the lack of representation of generality corresponding to mine would make a true concept formation – one that didn't use already existing boundary lines – impossible. (1880–1, p. 35)

For instance, without Frege's *quantificational* notion of generality, a Boolean logician could not fully analyse the concept <F is a hereditary property in the *f*-series>, which in modern notation is '$(\forall x)(Fx \rightarrow (\forall y)(f(x, y) \rightarrow Fy)$'.[17] This inadequate analysis would become clear when we represent proofs of theorems containing that concept. For instance, Frege needs only logical primitives and logical laws to prove 'if z follows x in a sequence, and if y follows z, then y follows x in that sequence' – even though logicians before Frege had thought that this fact rested on intuition or a non-logical rule of inference.[18]

This difference between the expressive power of the Begriffsschrift and that of Boolean and traditional logic is a fact familiar to any undergraduate student of logic today. What is less familiar now – and is absolutely essential for understanding Frege's relationship to his contemporaries[19] – is that Frege thinks that this difference in expressive power is based in a further difference in *how concepts are formed* in the old and new logic. He illustrates the

new method of forming concepts made possible by the Begriffsschrift with the following example:

> The *x* [in '$2^x = 16$'] indicates here the place to be occupied by the sign for the individual falling under the concept. We may also regard the 16 in $x^4 = 16$ as replaceable in its turn, which we may represent, say, by $x^4 = y$. In this way we arrive at the concept of a relation, namely the relation of a number to its 4th power. And so instead of putting a judgment together out of an individual as subject and an already previously formed concept as predicate, we do the opposite and arrive at a concept by splitting up the content of a possible judgment. (1880–1, p. 17; cf. 1879, p. 16)[20]

The concepts <4th power>, <4th root of 16>, <base 2 logarithm of 16> are all formed in this example, not by compounding simple concepts by addition (or inclusion and exclusion), but by starting with a relational expression, '$2^4 = 16$', and replacing one or more singular terms by variables. None of these concepts need be explicit in first framing the judgment, which can be put together from the relational expression '$x^y = z$' and the singular terms '2', '4', and '16'. The variables in these newly formed concepts can then be bound by the sign for generality to form quantified relational expressions.

There are a number of fundamental ways in which Frege's Begriffsschrift departs from traditional and Boolean logics: it allows for the expression of relations, it represents generality with quantifiers, and it introduces a division between constants and variables. Of course, these three innovations are indissolubly linked. The distinction between *variables* and constants is significant only because variables can be bound by quantifiers. The *quantificational* notion of generality is significant only because *relational* expressions can allow for polyadic quantified sentences like 'Every number has a successor'. It is extremely important for Frege, though, that each of these three features is similarly interlinked with the new way of forming concepts made possible by his notation. A further example will make this clear. Suppose you are trying to determine whether 121 is uniquely decomposable into primes, and suppose you already know that 121 is a square and that 11 is uniquely decomposable into primes. You then arrive at the judgment:

$$S(11, 121) \rightarrow D(121)$$

with 'S(x, y)' expressing <x is a square of y> and 'D(x)' expressing <x is uniquely decomposable into primes>. By replacing '121' with a variable, we form the concept <x is decomposable if it is a square of 11>. By quantifying that variable, we get a new quantified expression

$$(\forall y)(S(11, y) \rightarrow D(y)).$$

After some further steps, we reach the judgment

$$(\forall x)(\ D(x) \rightarrow (\forall y)(\ S(x, y) \rightarrow D(y))),$$

which says that unique decomposability into primes is hereditary in the series of squares. Finally, replacing the predicates 'D(x)' and 'S(x, y)' with predicate variables, we can form the relational concept <F is a hereditary property in the *f*-series>. In this way, the uses of variables, quantifiers, and relations are all bound up with Frege's new way of forming concepts in Begriffsschrift.

Given the central importance of Frege's new way of forming concepts, it is then not surprising that Frege criticizes Boolean logic for adhering to an older, inadequate theory of concept formation. For Boole, Frege argues, all concepts are formed by taking the unions, intersections, or complements of the extensions of given concepts. (In Boole's notation: from the classes x and y, the Boolean logician forms the new classes xy, $x+y$, and $(1-x)$).[21] When compared to the extraordinary expansion of inferential and expressive power made possible by the new way of forming concepts in Begriffsschrift, the Boolean theory of concept formation appears to be an insignificant departure from the traditional view, enshrined in logic texts since Aristotle, that concepts are formed by noticing similarities or differences among particulars and *abstracting* the concept, as the common element, from these similarities or differences. Frege writes:

> My concept-script commands a somewhat wider domain than Boole's formula-language. This is a result of my having departed further from Aristotelian logic. For in Aristotle, as in Boole, the logically primitive activity is the formation of concepts by abstraction, and judgment and inferences enter in through an immediate or indirect comparison of concepts via their extensions. [...] As opposed to this, I start out with judgments and their contents, and not from concepts... I only allow the formation of concepts to proceed from judgments. (1880–1, pp. 14–15)[22]

Because of this limitation, Boolean logic is simply 'not suited for the rendering of a content' (1882–3, p. 93). There could never be a *lingua characterica* constructed from a Boolean symbolic logic.

Frege's primary criticism of the Boolean logicians that he knew, then, was that the theory of concept formation implicit in their work was too weak to allow for the rich expressive capacities required by a *lingua characterica* for arithmetic like Frege's Begriffsschrift. Frege added to this a second, corollary criticism – a criticism that, as we will see below, picks up and develops a theme familiar to German logicians in the 1870s. Frege's theory of concept formation allows for the same judgment to be decomposed in multiple ways, by replacing one or more constants with variables. (In Frege's example, the three concepts <4th power>, <4th root of 16>, <base 2 logarithm of 16> are

all formed from one relational expression, '$2^4 = 16$'). Boolean logic, lacking relational expressions and the syntactic distinction between variables and constants, does not allow for the same judgment to be decomposed into constituent concepts in more than one way. Now, since expressions for judgments can be decomposed in multiple non-trivial ways, Frege argued that inferences in Begriffsschrift can thus exploit this multiple decompositionality: they can exhibit to us structures or patterns in our premises that were not already apparent to us in first forming these sentences (1880–1, pp. 33–5; 1884, section 88). The conclusions of these inferences can then be genuinely new and surprising extensions of our knowledge.[23] Because forming concepts in the new Fregean way allows us to see new patterns and so to perform epistemically ampliative inferences, Frege describes concepts formed in Begriffsschrift as 'fruitful'.[24] Boolean logic, wedded to the old method of forming concepts, cannot express the content of these fruitful concepts and so cannot explain how deductive inference can expand our knowledge. Boolean logic leaves us wondering what the point of deductive inference is, thus reinforcing 'the impression one easily gets in logic that for all our to-ing and fro-ing we never really leave the same spot' (1880–1, p. 34).

3 Lotze's theory of concepts and his criticism of Boolean logic

Frege argues that his Begriffsschrift, unlike Boolean logic, can serve as a *lingua characterica* for mathematics and thus allows Frege to isolate the 'springs' of mathematical knowledge. He argues that the Begriffsschrift, unlike Boolean logic, avoids the expressive limitations of the traditional theory of concept formation and explains how deduction can be epistemically ampliative. In this section, I provide a historical context for these arguments by looking to the writings of the German philosopher and logician Hermann Lotze.

There are good historical reasons to look to Lotze. Lotze's *Logic* was perhaps the most widely read logic text in Germany during Frege's early career,[25] and it was a work that we know Frege read. [26] The only philosophical course that Frege took as a graduate student was from Lotze, and Frege's colleage in Jena, Bruno Bauch, claimed that Frege himself had told him that Lotze's work was of 'decisive importance' for his own.[27] And, as we will see, Lotze's *Logic* (1st ed., 1874; 2nd ed., 1880) also presents an objection against the suitability of the traditional theory of concepts for capturing the structure and formation of specifically mathematical concepts. This objection is not only broadly similar to Frege's, but Lotze, like Frege, turned it against Boolean logic in the second edition of his *Logic*.[28]

Before looking in detail at Lotze's theory of concepts and his objections to Boolean logic, it is important to recognize that Lotze's conception of logic differed fundamentally from Frege's. Frege, in his notes on Lotze's *Logic*, claimed (against Lotze's opposing view) that logic is chiefly concerned with *inference* (Frege, 1979, p. 175). Though this is a standard view today, it was

not common among nineteenth-century German logicians.[29] Lotze, by contrast, shared the view (dominant at that time) that philosophers of logic should stake out a middle position between a modest Kantian 'formal' logic and an extremely ambitious logic, like Hegel's, that combined traditional logical topics with a full metaphysics (Lotze, 1843, pp. 1–36).

Lotze's *Logic* begins with an account of how the operations of thought allow a subject to apprehend what is true and distinguish it from the mere current or stream of ideas [Vorstellungsverläufe] (section II).[30] In this current of ideas, some ideas flow together only because of accidental features of the world or because of idiosyncratic features of a particular subject's mind; other ideas flow together because the realities that give rise to them are in fact related in a non-accidental way. How are these two cases to be distinguished? Lotze insists that it is the task of thought to so distinguish them; and it is the task of logic to investigate what thought does when it distinguishes them.

On Lotze's view, before thought can separate the true ideas from the false ones, it must first act on ideas so that they *become capable of being* true or false. Thought accomplishes this by the addition to the stream of ideas of certain 'accessory thoughts' or 'accessory notions' that ground this stream in reality (section VI). An animal may connect together the idea of a tree with the idea of its leaves and associate them through memory with the idea of a tree with no leaves. But a human will add to these connections of ideas the accessory thought of a *thing* and its *property* and will see these ideas as corresponding to a particular tree that before had leaves and now has lost them. The human subject connects these ideas of the tree and its leaves with this idea of a tree with no leaves *because they are ideas of the same thing*, and these ideas are separated in the human subject *because they are ideas of the very same tree now with and now without its leaves*.

This will no doubt remind many readers of Kant's argument in the *Critique of Pure Reason* that objective validity requires the application of categories to intuitions, and that the ordering of representations in a 'synthetic unity of apperception' differs from a mere association of ideas. A contemporary reader might also wonder what this has to do with logic as it was traditionally understood – as the doctrine of concepts, judgments, and inferences. For Lotze, these elements of 'transcendental philosophy' are integral parts of a unitary investigation that also includes the subject matter of traditional 'formal' logic. Thinking allows a subject (among other things) to connect together representations so as to represent a *thing* and its *attribute*; to move from associating the ideas of stick and pain to representing the stick as the *cause* of an *effect*; and to frame a judgment as a *conclusion* that follows from a universal and particular *premise*.[31] All three of these activities are essential parts of the overall project of 'reducing coincidence to coherence' (section XI)[32] – even though Frege (and Kant) would recognize only the third as a topic for *logic*.

These ideas have significant implications for Lotze's theory of concept formation. In the chapter titled 'The Theory of the Concept', Lotze emphasizes that even the stream of ideas is itself partially a product of thought,

since only thought can transform a mere series of given impressions into ideas [Vorstellungen]. Therefore, Lotze argues that the formation of concepts is in fact the third operation in a three-step process, and he gives a lengthy discussion of the first two steps – the objectification of impressions into ideas and the 'composition, comparison, and distinction of the simple contents of ideas' – before finally turning to a discussion of concepts themselves.[33]

For Lotze, the formation of concepts is an instance of the characteristic function of thought: 'to separate the merely coincident in the manifold of ideas that are given to us, and to combine the content afresh by the accessory notion of a ground for their coherence' (section 20). Certain streams of ideas are connected in my mind: at one moment together in my consciousness is <spherical, hard, small, blue>; at another is <spherical, hard, small, red>; at another is <spherical, hard, small, green>. I form from these streams of ideas a new universal representation – <spherical, hard, small>. By treating this new universal as a unity, I am isolating a relation within my stream of ideas as belonging together in a non-accidental manner: I say that the ground for my having the series of ideas that I've had is that there is *a kind of thing* to which these series of ideas correspond.

On the traditional view, a concept is a sum of marks, formed by abstraction, which differs from non-conceptual representations chiefly in virtue of its generality. Lotze protests, however, that this picture does not capture at all *the* essential function of thought when it forms concepts: the *grounding* of the connection of the ideas. A concept is not just a universal representation: it must contain a rule or a determinate law (section 121) that explains why certain marks belong together. Lotze makes his point vivid with examples of degenerate general representations, like the 'concept' <red, juicy, edible, body> formed by abstraction from cherries and raw meat (section 31). This universal is degenerate *because* it was formed, not by looking for the rule that unites the marks that together compose a concept, but by identifying common elements in a series of particulars and *abstracting* away from the differences. As a result, the common elements or marks are simply *listed*, not compounded according to a determinate rule, and the knowledge that some particular falls under the 'concept' tells us little about it.

Lotze therefore replaces the model of the concept as the *sum* of its marks with what he calls the '*functional*' model, where the structure of the concept is expressed by some complicated interrelation of its marks:

> As a rule, the marks of a concept are not coordinated as all of equal value, but they stand to each other in the most various relative positions, offer to each other different points of attachment, and so mutually *determine* each other; ... an appropriate symbol for the structure of a concept is not the equation $S = a + b + c + d$, etc, but such an expression as $S = F\ (a, b, c$, etc.) indicating merely that, in order to give the value of S, a, b, c, etc,

> must be combined in a manner precisely definable in each particular case, but extremely variable when taken generally. (section 28)

In the functional model, the marks are *interdependent* and vary together according to a rule. Every particular that falls under a concept S = *F* (*a*, *b*, *c*, etc.) has its own specific way of exhibiting the marks of *S*; still, though, *how* a particular *S* exhibits a mark will in general be determined by how it exhibits other marks. This is particularly clear for actual mathematical functions: at each particular point on a plane curve of second degree, $a_2x^2 + a_1x + b_2y^2 + b_1y + cxy + d = 0$, its ordinate y is determined by its absissa x. Every particular that falls under <triangle> (=<figure *with* three sides *meeting at* three angles>) will not just have three sides and three angles, but the specific angles that it has will be determined by the magnitude of its three sides. The mark <angle> is then like a *dependent* variable in the concept <triangle>: the angles are *functionally* dependent on the sides. The concept <triangle> is not <three sides> + <three angles>, since the two marks 'are not coordinated as all of equal value'. However, when concepts are formed by *interrelating* component universals like interdependent variables in a function, thought is in a position to capture real, nontrivial relations among the things themselves: it represents its connected ideas as *grounded* or justified by relations that hold in reality.

Frege argued against the traditional, abstractionist model of concept formation that it fails for mathematical concepts and cannot explain how deductions can extend our knowledge. Lotze, after concluding his discussion of the different kinds of syllogistic inferences (the figures of the syllogism, along with syllogisms where the major premise is disjunctive or hypothetical in form), begins his discussion of 'mathematical inferences'with the following argument:

> The above considerations have taught us that there have to be still other logical forms of thought beyond the Aristotelian figures of the syllogism, forms that provide for the first time a fruitful application to the content of knowledge. [...] Every inference should be an acquisition of new knowledge from the premises, from which this knowledge *comes to be*, although it is not already contained in them analytically. [...] When the mind seeks a necessary law in the combination of manifold marks, it first believed it could find it in that general concept, but this concept itself came to be only through summing marks, and we can therefore not ground a conclusion through this without surreptitiously presupposing the thing we are seeking. [...] We have sought to compensate for this deficiency of the subsumptive mode of inference through the assumption of constitutive concepts; but in order to find these concepts and their logical form, we must oppose the Aristotelian figures with a series of different [inferences], which are grounded on the *content* of concepts. (1843, p. 190; cf. 1880, section 105)[34]

Lotze goes on to isolate three different types of non-Aristotelian inferences that are employed in mathematics[35] and argues that these inferences require concepts whose structures are not captured in the traditional way. These mathematical concepts exhibit most clearly the value of forming concepts, since inferences containing these concepts can extend our knowledge in highly nontrivial ways. For example:

> Analytic geometry possesses in the *equations* by which it expresses the nature of a curve just that constitutive concept of its object that we are looking for. A very small number of related elements [abscissae and ordinates, plus constants and their arithmetical combination] ... contain, implicit in themselves and derivable from them, all relations that necessarily subsist between any parts of the curve. From the law expressing the proportionality between the changes of the ordinates and the abscissae every other property of the curve can be developed. (section 117)

Both Frege and Lotze, then, want to identify forms of inference that lead to new mathematical knowledge. Both think that the traditional Aristotelian forms fail to do that: according to Frege, an Aristotelian inference only 'takes out of the box what we have already put in it' (Frege, 1884, section 88); according to Lotze, it is just 'a tautological repetition of its presuppositions' (Lotze, 1880, p. 190).[36] Both identify the cause of this futility in the fact that the concepts deployed in syllogisms need only be sums of unordered marks (1880, section 122). Both argue that epistemically ampliative inferences – like those in mathematics – require a different kind of concept, whose content has a 'functional' rather than an Aristotelian structure (what Lotze here calls 'constitutive concepts' as opposed to merely 'general concepts'). And both think that we logicians can discover these more complicated conceptual forms only by reflecting on inferences that do not fit the Aristotelian patterns.

Furthermore, Lotze, like Frege, turned this criticism of the traditional model against Boole's logic itself in 1880 (in an appendix entitled 'A note on the logical calculus', added to the second edition of his *Logic*) – the same year in which Schröder's review of *Begriffsschrift* appeared and within a year of the period in which Frege wrote 'Boole's Logical Calculus and the Begriffsschrift.'[37] The bulk of Lotze's discussion is given over to collecting together objections to Boole's procedure that were also given by others. What is interesting and novel in Lotze's criticism, though, follows directly from Lotze's non-Aristotelian, functional model of the structure and formation of concepts. Boolean operations like logical addition and multiplication on symbols for concepts can express 'merely the simultaneous presence of their elements [viz., their marks]' (1880, p. 278). Thus, the only 'concepts' that can be adequately represented by Boolean logic are the degenerate concepts like <red, juicy, edible, body>, which do not allow us to infer anything new

about the objects that fall under them. In particular, the expressive resources of Boolean logic would never be enough to represent the structure of mathematical concepts 'for the form which the result of the calculation is finally to take, is here completely and solely determined by the definitely assignable nature of the connection which this science requires to be introduced between its elements [namely, the marks in a concept]'. The algebraic combinations of simple symbols simply cannot express 'the reciprocal determination of the component parts', that is, the functional interdependence of the marks that make up a concept (p. 279).

4 Frege's advance over Lotze

Frege and Lotze identify the same set of problems in the traditional logic, and they think that the failure of traditional logic to represent inferences in mathematics rests on a commitment to a faulty understanding of how concepts are formed and so also how they are structured. Both think that Boolean logic, however much it might improve on traditional syllogistic as a tool for problem-solving, simply repeats the fundamental flaws in Aristotelian logic. And they share a general conception of how these flaws would be rectified in an improved account of concepts and inferences. These historical facts, I contend, allow us to understand Frege better. They explain, for instance, why Frege chose to argue for the superiority of the Begriffsschrift over Boolean logic by highlighting the theories of concept formation implicit in each. And they identify some of the constraints that Frege thought a symbolic logic like the Begriffsschrift had to satisfy. In this way, Sluga was correct to argue for the value of a reading of Frege that places him in his historical relationship with Lotze.

Dummett, in criticizing Sluga's concern with discovering the influences on Frege's thought, writes:

> In expounding a philosopher, one has of course to report the theses which he held in common with his contemporaries or borrowed from his predecessors; but what makes him interesting will usually be the ideas that were original with him. Emphasis on the historical background will often be useful in making the original ideas stand out prominently from the derivative ones; but the historical method, as Sluga employs it, makes it hard to see what was original about Frege at all. (Dummett, 1981b, p. 529)

The contextualized reading of Frege that I have given here avoids the mistake that Dummett attributes to Sluga. In fact, the value of such a reading *for a philosopher* is the opposite of what earlier we saw Dummett assert. Locating Frege historically puts us in a unique position to see the original philosophical insights that Frege – armed with his Begriffsschrift – could bring. In this

closing section, then, I will show that – although Frege shared with Lotze some very significant common theses – the invention of his Begriffsschrift put Frege in a position to answer questions that Lotze left open, cash out ideas that Lotze left metaphorical, and correct philosophical errors that Lotze fell into.

Lotze argued that Boolean logic does not capture the interdependence of component concepts in mathematical concepts, and he models the structure of concepts with the dependence of one variable on another in mathematical functions. However, there is no detailed positive story forthcoming with which we can flesh out the model: Lotze does not pretend to have isolated all of the types of inferences that are used in mathematics, and he does not think that there are a few basic operations by means of which *all* compound concepts can be composed. Though Frege also traffics in metaphors (in his Begriffsschrift, the elements in definitions are 'organically' connected to one another (Frege, 1884, section 88)), the quantifiers, relations, and variables in his logical symbolism allow for a precise characterization of this 'functional' interdependence.[38] Similarly, Lotze recognized that deduction can be epistemically ampliative, and, like Frege, he thought that traditional Aristotelian inferences cannot allow us to acquire new knowledge. But he had no spelled-out story of how this was really possible. Frege, on the other hand, can show in his Begriffsschrift how the same judgment can be decomposed differently from the way in which it was first formed; he can therefore provide a positive explanation of how we can learn new things from deductive inferences. According to Lotze, concepts fruitfully employed in inference cannot be formed by abstraction, and he helpfully suggests (as Frege did) that we can identify these new kinds of concepts by reflecting on forms of inference that do not fit the traditional patterns. Again, however, Frege has a concrete positive proposal: the Begriffsschrift, with its distinction between constants and variables, functional expressions and singular terms, shows us how to form new concepts from judgments by replacing constants with variables.

All of these innovations depend on the invention of the Begriffsschrift, Frege's execution of Leibniz's project of a characteristic language. It is extremely significant, then, that Lotze, though he did not declare Leibniz's project impossible, had no optimism about its successful execution, and doubted its central importance for logic. His skepticism was well motivated at the time. Leibniz was convinced that the truth of a judgment consisted in the containment of the predicate in the subject, that the axioms of a science were really definitions, and that therefore all inferring was unpacking the definitions of the subject and predicate concepts. For this reason, Leibniz thought the project of a universal characteristic would consist primarily in identifying simple concepts. Lotze argued, however, that formulating a characteristic language for a science would require two other highly non-trivial tasks besides identifying simples. For one thing, in order to determine the truth of a judgment by calculating, we need to identify the

'general laws' of the various special sciences, and discovering these axioms necessitates 'dissecting our judgments and tracing them back to simple principles' (section 198). Second, Leibniz assumed that all concepts could be formed from simple concepts by algebraic operations, and he overlooked the fact that the components in a compound concept mutually determine one another; in fact, Lotze argued, Leibniz would have needed to know all of the special *rules* by means of which the marks in a concept could interrelate to form new structured concepts. On this second point, all of the criticisms of the concept as a sum of marks come into play, and it is no surprise that Leibniz's proposals for a *lingua characterica* look a lot like Boolean logics.[39]

These criticisms of Leibniz are sound, and they are damning against any attempt to carry out Leibniz's program for mathematics using a symbolic language like Boole's. Nevertheless, Frege's Begriffsschrift – which isolates the basic laws of logic (and therefore also arithmetic) and isolates all of the ways in which concepts can be formed by interrelating component concepts – shows that a symbolic language can be devised that avoids these criticisms. Lotze, dissatisfied with the Boolean and Leibnizian proposals, thought that symbolic logic was a waste of time and destined simply to fall prey to the philosophical and expressive shortcomings in the traditional logic.[40] Again, however, Frege's Begriffsschrift shows Lotze to be mistaken: the flaws that Lotze identified in traditional logic in fact could be solved *only with* a new symbolic language.

Sluga obscures the decisive advance that Frege made over Lotze by over-emphasizing the affinity between Lotze's conception of concepts as functions and Frege's idea that concepts are functions.[41] For Lotze, a compound concept is (not itself a function, but) the value of applying a function to a collection of marks. For Frege a concept is a function, and the value of the function when applied to an argument is a sentence. This difference cannot be ignored, since it allowed Lotze to hold onto the subject/predicate analysis of sentences, while Frege rejected it. For example, Frege argued that the same sentence could be viewed as the result of applying different functions to different arguments – 'Cato killed Cato' is the result of applying the function 'killing Cato' to 'Cato' and also the result of applying 'being killed by Cato' to 'Cato' (Frege, 1879, section 9). It was thus because Frege hit on the idea of the function/argument analysis of sentences (as opposed to the traditional subject/predicate analysis) that he was able to discover the possibility of multiple decompositionality. This, in turn, made it possible for Frege (unlike Lotze) to give concrete and philosophically satisfying accounts of how concepts can be formed in new ways and how deductions can expand our knowledge. When Sluga, however, views Frege's idea that concepts are functions as derived from Lotze's different idea, he papers over what was new and important in Frege's logic and philosophy.

The fundamental thesis of Frege's philosophy of mathematics is that 'arithmetic is a branch of logic and need take no ground of proof from either

experience or intuition' (1893, p. 1). Earlier, Lotze called mathematics 'an independently progressive branch of universal logic' (1880, section 18). It is therefore tempting to conclude, as Sluga does, that 'among the many things that Frege owes to Lotze, the most important is perhaps the idea of logicism' (1980, p. 57). Sluga elaborates:

> Whatever the details of Lotze's position, it is clear that in some sense he subscribed to the claim that arithmetical propositions are grounded in general logical laws alone ... Though Lotze claimed that arithmetic was really part of logic he never tried to show that conclusion could be established in detail nor did he list the additional logical principles which he considered necessary for that task. It was Frege who set out the necessary details. (1984, pp. 343–4)[42]

However, as Dummett correctly argued (1981b, pp. 525–6), there are very fundamental differences between Frege's logicism and Lotze's philosophy of mathematics. Lotze claims to be 'in entire agreement with Kant' that the truths of arithmetic and geometry are *synthetic* (section 353). Mathematical judgments and inferences rest on a pure form of *intuition*: geometry on a pure intuition of space (section 354ff; section 152); arithmetic on a pure intuition of quantity and an intuition of our own mental 'operations' (section 353, section 361). 'No mere logical analysis', he writes, could inform us of the truth of arithmetical equations (section 361), whose self-evident truth is analogous to that enjoyed by the 'simplest principles of mechanics' (section 364).

In fact, the conflict between Frege and Lotze here is even more fundamental than Dummett or Sluga realized. Frege described his project in *Foundations of Arithmetic* as showing that arithmetical truths are 'analytic' – they depend only on 'general logical laws and definitions' – and he says that his thesis would be refuted if these laws and definitions were 'not of a general logical nature, but belong to the sphere of some special science' (Frege, 1884, section 3). But this is precisely what Lotze asserts. Above, I noted that Lotze isolates three kinds of non-Aristotelian inferences used in mathematics. These forms of inference, Lotze acknowledges, are 'confined to the region of mathematics, and primarily to the relations of pure quantities' (section 111).[43] Because of the unique subject matter of mathematics – the nature of space for geometry, the nature of pure quantity for arithmetic – these forms of inference are applicable in these sciences but not elsewhere.[44]

Lotze recognizes that the limited applicability of the three mathematical forms of inference presents a prima facie objection to including a discussion of them in a treatise on logic. But he replies:

> The fact that the use of [these forms of inference] is confined to mathematics, cannot hinder us from giving [them] a place in the systematic series of forms of thought. For in the first place we must not forget that

> calculation in any case belongs to the logical activities, and that it is only their practical separation in education which has concealed the full claim of mathematics to a home in the universal realm of logic. (section 112)[45]

These two sentences, however, seem to be in open tension with one another. The second quoted sentence might motivate some readers to side with Sluga and Gottfried Gabriel to argue that 'Lotze and Frege both subscribed to the reducibility of arithmetic to logic' (Sluga, 1980, p. 73), while the first sentence might lead others to side with Dummett and deny this. But the tension here is only apparent, and Lotze seems confused only if we fail to appreciate just how different our post-Fregean conception of logic is from Lotze's. As we saw above, logic for Lotze is the study of the operations of thought that allow for the reduction of coexistence to coherence. On this view, logic begins earlier than it did for Frege, with a discussion of how thinking transforms merely subjective states into objective ideas that can be true or false. And it also extends more broadly than it did for Frege, covering all of those ways in which thinking introduces an order into our ideas that can then model the reciprocal interactions of the things that exist independently of our ideas. In following out these later developments of thinking, Lotze investigated mathematical concepts and inferences, arguing that mathematics, with its functional concepts and non-Aristotelian inferences, uniquely fulfilled the task of thought. In this sense, the mathematical inferences belong necessarily in logic. But it simply does not follow that these inferences are of universal scope, and it does not follow that Lotze agreed with Frege that arithmetical truths are non-intuitive.

As Sluga and Gabriel point out, Lotze makes a similarly Fregean-sounding claim about mathematics earlier in the book:

> All ideas which are to be connected by thought must necessarily be accessible to [...] quantitative determinations ... I exclude [from our present investigation] the investigation of the consequences which may be drawn from these quantitative determinations as such: they have long ago developed into the vast structure of mathematics, the complexity of which forbids any attempt to re-insert it into universal logic. It is necessary, however, to point out expressly that all calculation is a kind of thought, that the fundamental concepts and principles of mathematics have their systematic place in logic. (section 18)

These sentences appear in the long discussion of the formation of concepts with which the book begins. As I briefly mentioned above, the formation of concepts for Lotze is the culmination of a three-step process that begins with merely subjective impressions. The first operation of thought is the 'objectification' (section 3) of the subjective impression, whereby I distinguish my *act* of sensing from its content, which is that which is sensed (section 2).

With these objective contents in hand ('the red', 'the toothache'), a thinker can now interpret the relations that impressions have to one another as in fact 'aspects of the *content* of the impressions' themselves (section 9). This is the second operation of thought, 'the composition, comparison, and distinction of the simple contents of ideas', wherein thought distinguishes one content from the content of other ideas, and 'estimates by quantitative comparison its differences and resemblances' (section 19). Lotze's point in the quoted passage, then, is that the contents of sensation, for instance the brightness or saturation of this or that red, can *always* be compared to one another *quantitatively*. And so both the concept <quantity> and the various principles of quantity 'have their place' in logic.

Readers familiar with the history of German philosophy will recognize this argument at once. Kant had argued (in the chapter entitled 'The Anticipations of Perception' in the first *Critique*, 1781/1787) that all sensible qualities come in a degree, and so the matter of any empirical intuition can be compared with that of another intuition quantitatively. This argument is part of Kant's explanation for the applicability of the concept <quantity> and for the necessary applicability of mathematics in experience – rest, he thinks, on the conditions of the possibility of (objective) experience. Though it seems odd to us to find this in a logic text, Lotze is not confused when he gives this argument from 'transcendental logic' in his book, since his conception of logic is so much broader than Frege's.[46] But we only misunderstand him when we try to see this passage as an attempt to answer the question – Is arithmetic analytic? – for which Frege's logicism is an answer. It was only because Frege had his Begriffsschrift that he could set about determining whether arithmetic is analytic. On the other hand, because Lotze did not pose this question, he failed to see the philosophical payoff that a new logical language could provide.[47]

Notes

1. Sluga (1975), (1976), (1977), (1984), (1987).
2. Dummett (1981b), and Chapters 4–7 reprinted in Dummett (1991b).
3. Sluga makes these kinds of historical claims at, for instance, pp. 55, 60, and 181 of Sluga (1980).
4. Sluga contrasts this faulty Fregean philosophy of language (and historiography) with a less blinkered 'Wittgensteinian' one on p. 186.
5. Richard Rorty (1984, p. 57) advocated a stronger historiographical position, according to which the standards we use in determining how interesting a piece of philosophy is as a piece of philosophy are *always* our standards: the ones we use in evaluating the philosophy written by other philosophers writing today.
6. 'Sluga is so keen to discover sources for Frege's ideas ... that he fails to convey what was great about Frege' (Dummett, 1981b, p. 529).
7. Jarmo Pulkinnen (2005) argues against Dummett that historians of philosophy should give only causal explanations for why certain ideas emerged – explanations that abstract from the philosophical merits of those ideas. This seems to me too radical.

8. 'Historical reconstructions remind us of all those quaint little controversies the big-name philosophers worried about, the ones which distracted them from the 'real' and 'enduring' problems which we moderns have managed to get in clearer focus. By so reminding us, they induce a healthy skepticism about whether we are at all clear and whether our problems are all that real.' (Rorty, 1984, p. 71). See also Hylton (1990, p. 6).
9. One need not assent to the exaggeration in Dummett's first sentence to see his point.
10. I use the following convention: *Begriffsschrift* is the book written by Frege in 1879; Begriffsschrift is the logical system propounded in that book.
11. See, for instance, Russell (1914/1926, pp. 243–4).
12. I would be gratified if the following discussion of Frege and Lotze were read in the spirit of Peter Hylton's book on Russell. Noting the failure of modern logic to translate philosophy into a progressive science (p. 391), he argues that it is necessary for historians of analytic philosophy to identify the philosophical issues that occupied Russell's British contemporaries and to trace out how Russell uses the new logic to make progress on these issues. He is not interested in 'influences' or historical causation *per se*, but on the way that Russell's philosophy (along with his new logic) could or could not make up for the *philosophical* deficiencies of his contemporaries.
13. Pulkkinen (2005, chapter 4) has pointed out that Frege's writings on Boole were part of a larger discussion of the philosophical significance of Boolean logic that was carried on in Germany between 1877 and 1882. Although Boole's *Laws of Thought* was published in 1854, it received virtually no attention in Germany until Alois Riehl, Ernst Schröder, Wilhelm Wundt, Hermann Lotze, Hermann Ulrici, Friedrich Lange, Louis Liard, Leonard Rabus, and, of course, Frege all wrote works within that five-year period defending or praising Boolean logic. See also Peckhaus (1988).
14. Frege, of course, thinks that the converse point holds as well: Without a complete analysis of inferences into their simplest components, we would be unable to have complete analyses of the concepts employed in those inferences. A good example of this point is Frege's analysis of theorems about sequences – if we did not see that these inferences do not require intuition but rest only on logical laws, we would not be in a position to see that the concept <*x* is a hereditary property> is in fact a compound concept analysable into logical primitives.
15. Frege most likely learned of Leibniz's idea, and the terms '*lingua characterica*' and '*Begriffsschrift*' from Friedrich Trendelenburg's 1867 essay 'On Leibniz's Project of a Universal Characteristic', which Frege cites at 1879, pp. v–vi. Frege describes what he takes a Leibnizian *lingua characterica* to be and argues that his Begriffsschrift is an instance of such a language at Frege (1880–1, pp. 9–10). (1882–3, pp. 90–1), (1897, p. 235). Leibniz had conceived of a *lingua characterica* as a *universal* characteristic. Though Frege thought that the Begriffsschrift could be employed outside of arithmetic (1882, p. 89) and he speculated that it could be extended to other areas (1879, p. vi), he only claimed that he had succeeded in formulating a characteristic language for arithmetic. (In this paper, I will not be considering the suitability of the Begriffsschrift for acting as a *lingua characterica* for other sciences.)
16. These writings were prompted by Schröder (1880), which compared *Begriffsschrift* unfavorably with the works of Boolean logicians. On the Frege-Schröder controversy, see Peckhaus (1997, pp. 287–96; 2005) and Sluga (1987). It is important

to remember that Frege's criticisms of Boolean logic were aimed against the Boolean writings that he knew: Schröder's *Operationskreis* (1877), as well as the work of Boole and Jevons. In particular, Boole's system was modified and greatly expanded by Peirce, who (exploiting ideas from De Morgan) arrived independently in 1883 at a system expressively equivalent to the first order fragment of Begriffsschrift. But in the Boolean works that Frege knew, there were no equivalents to Frege's use of variables, quantifiers, and relations.

17. Throughout the paper, I refer to concepts in brackets, and linguistic expressions in single quotes.
18. See 1879, section 28, theorem 98. Frege emphasizes that this theorem (and ones like it) do not require intuition at 1879, section 23 and 1880–1, p. 32. The non-logical rule of inference is, of course, mathematical induction, which Frege actually derives from the laws of Begriffsschrift (1879, section 27; 1880–1, p. 31).
19. The theory of concept formation, as we will see below in the case of Lotze, was one of the most active areas of debate among nineteenth-century German logicians. See Heis (2012) (section 4).
20. Frege made the point that his logic, unlike that of Leibniz, Aristotle, or the Booleans, forms new concepts from completed judgments, and not vice-versa, from early in his career till very late: see 1882, p. 94; 'Notes for Ludwig Darmstaedter' (1919) in Frege (1979, p. 253).
21. Frege does not cite any passages when he attributes this theory of concept formation to Boole. An apt citation would have been Boole (1854, pp. 42–7), where Boole describes the 'acts of conception' whereby any simple or compound conception is formed. There, he gives two primitive operations of the mind: selecting from a given class x those individuals that also belong to a class y; and 'form[ing] the conception of that collection of things which two classes taken together compose.' (On p. 48, Boole adds the operation of taking the complement of a class.) Boole adds some brief comments about the faculties of the mind at work in these acts: attention, imagination, comparison, and abstraction (1854, p. 43; 1847, p. 16).
22. Frege will sometimes characterize the traditional view as one according to which concepts are formed by abstraction, and sometimes as one according to which concepts are formed by Boolean combinations of simpler concepts. Frege is on good ground in moving back and forth between these descriptions, since the two ideas were indissolubly linked in the tradition. See Heis (2012) (section 4).
23. Readers interested in a detailed explanation of how Frege's argument works here may consult Dummett (1991a, pp. 36–42).
24. I think that when Frege calls a concept 'fruitful' he means that inferences involving that concept can extend our knowledge. Jamie Tappenden (1995) thinks that Frege uses the word 'fruitful' to pick out those concepts that are *mathematically* significant in other, more interesting ways. As I explain below, however, other German *logicians* in the 1870s were using the word 'fruitful' to pick out those ways of forming concepts that allow for inferences that extend our knowledge. Moreover, Frege's account of fruitful concept formation by decomposition does successfully explain how inferences can extend our knowledge. It seems better, then, to read Frege's use of the word 'fruitful' to describe concepts in the way that other *logicians* were using that word.
25. On Lotze's place among late-nineteenth-century German logicians, see Gabriel (1989a).

26. A short document from Frege's *Nachlass* is in fact a philosophical commentary on the introduction to Lotze's *Logic*. See Dummett (1991b, pp. 65–78).
27. See (Schottler 2006, p. 45). The most extensive discussion of the relationship between Frege and Lotze is Gabriel (1989a) and (1989b). See also Carl (1994, pp. 47ff; 2005); Dummett (1981b, 1991b, pp. 65–125); Gabriel (2002); Milkov (2007); Peckhaus (2000); Schmit (1990); Sluga (1976), (1977), (1980), and (1984). Most of the discussion of the relationship between Frege and Lotze has understandably focused on Frege's so-called 'platonism' and Lotze's theory of objectivity and validity. Less attention has been given to the issues I discuss in this paper.
28. Gabriel has pointed out that Frege could take Lotze as an ally or even as a source for his attack on Boolean logic (1989b, pp. xxv–vi). In this paper, I greatly expand on these brief remarks from Gabriel – and I argue in the closing section of the paper that Gabriel takes these affinities too far.
29. On the rival conceptions of logic in nineteenth-century Germany after Hegel, see Heis (2012), section 3; Peckhaus (1997, pp. 130–63); Vilkko (2002, chapters 3–4). Frege does not mention it, but his view that logic is primarily concerned with inference was a common view among *British* logicians, like Whately, Mill, and Boole.
30. References to Lotze's *Logic* will be to section numbers, which are common to the German original and the English translation. The translations will be from the 1888 English translation, edited by Bosanquet, though with some modifications of my own here and there.
31. These are the three examples Lotze gives of 'accessory notions' in Section 6.
32. Lotze's contemporary, Friedrich Ueberweg (1857, section 28), also noted that Lotze's notion of logic was close to Kant's *transcendental* logic.
33. I will return to the first two steps below, p. 33.
34. Christoph Sigwart (1878, section 75.2) argued that forming concepts by summing marks is an 'unfruitful' method of concept formation, and he therefore rejects the attempts (like Leibniz's) to represent compound concepts as algebraic combinations of simples. Similarly, Schröder (1890, pp. 101, 566–8) defends Boolean logic against the view – which he attributes to Lotze but suggests is extremely widespread – that a symbolic logic that treats of algebraic relations among concept extensions is 'unfruitful.'
35. The three forms of inference are inference by substitution, inference by proportion, and inference from constitutive equations. See also Peckhaus (1997, pp. 159–163).
36. Lotze explains in more detail why reasoning in the Aristotelian way cannot produce new knowledge also at 1880, section 98.
37. In this note, Lotze discusses the writings of Boole, Jevons, and Schröder. (Although Frege's *Begriffsschrift* appeared in 1879, Lotze does not mention it, and there is no evidence that Lotze ever knew Frege's work.) As Pulkinnen (2005, p. 123) points out, Lotze's criticism of Boole was the most comprehensive criticism of Boolean logic written in Germany at the time. (Pulkkinen does not mention, though, that Lotze's criticism draws on his critique of the traditional theory of concepts, and he does not point out the affinities between Frege's and Lotze's discussions.) On Lotze's note, see also Peckhaus (1997, pp. 159–163).
38. Frege does not illustrate what he means by the 'organic interconnection' of elements in his definitions. One apt illustration might be his notion of quantifier dependence – where '$(\forall x)(\exists y)R(x, y)$' expresses a different relation among the variables than '$(\exists y)(\forall x)R(x, y)$'. The Begriffsschrift captures these 'interconnections' because it includes relations and polyadic quantifiers – elements that Frege thinks depend on his new way of forming concepts.

39. Trendelenburg also criticizes Leibniz's project for apparently requiring the faulty view that concepts are algebraic combinations of marks (1867, p. 24). Gabriel nicely points out that Frege's criticism of Boole can also be seen as a defense against Trendelenburg's criticisms (1989a, pp. xxiv–v).
40. Lotze's hope was that German philosophers would seek 'not merely to *calculate* the course of the world, but to *understand* it' (section 365).
41. Sluga (1980, pp. 53, 56–7). The connection between Lotze's and Frege's theories of concepts and functions was made earlier by Thiel (1968, p. 155), and ultimately by Bauch (1918, pp. 47–8). Other writers have questioned the connection: Gabriel (1989a, pp.xxv–vi); Kreiser (2001, p. 150).
42. Gottfried Gabriel has argued for a similar conclusion (1989a, p.xxi).
43. Lotze makes this point in detail for each of the three 'mathematical' forms of inference in sections 111, 115, and 118.
44. Similarly, the primary technical result of Frege's *Begriffsschrift* is the purely logical analysis of mathematical induction. Lotze, on the other hand, though he discusses mathematical induction in section 210, never feels the need to reduce it to forms of deduction that are universally applicable; indeed, he never even suggests that it is reducible.
45. Both Gabriel and Sluga quote this sentence (but not the first!) in support of their interpretation.
46. Clinton Tolley has pointed out to me that Frege does sometimes use the word 'logic' more broadly – in his unpublished works called 'Logic' and in his late 'Logical Investigations'. But I have in mind the narrower notion of logic that Frege uses (say, in 1884, Section 3) when he is arguing for logicism.
47. Many thanks to Penelope Maddy, Richard Mendelsohn, Thomas Ricketts, Gila Sher, Clinton Tolley, Mark Wilson, and the participants of the Southern California History and Philosophy of Logic and Mathematics Group. I owe special thanks to Erich H. Reck, as both an editor and philosophical interlocutor, for helping me to improve this paper significantly.

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