Canadian Journal of Philosophy, 2014
Vol. 44, Nos. 5–6, 605–630, http://dx.doi.org/10.1080/00455091.2014.971689

Kant (vs. Leibniz, Wolff and Lambert) on real definitions in geometry

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(*Received 23 September 2014; accepted 29 September 2014*)

This paper gives a contextualized reading of Kant's theory of real definitions in geometry. Though Leibniz, Wolff, Lambert and Kant all believe that definitions in geometry must be 'real', they disagree about what a real definition is. These disagreements are made vivid by looking at two of Euclid's definitions. I argue that Kant accepted Euclid's definition of circle and rejected his definition of parallel lines because his conception of mathematics placed uniquely stringent requirements on real definitions in geometry. Leibniz, Wolff and Lambert thus accept definitions that Kant rejects because they assign weaker roles to real definitions.

Keywords: Kant; Lambert; Leibniz; Wolff; Euclid; geometry; definition

Two trends have characterized recent work on Kant's philosophy of mathematics. On the one hand, Kant's readers have been providing richer and richer contextual interpretations of his philosophy of mathematics. Of course, interpreters have long emphasized that Kant, in taking mathematical judgements to be synthetic, departed from the Leibnizian and Wolffian views that all mathematical judgements are derivable syllogistically from definitions alone. But starting with works by Friedman (1992) and Shabel (2003), there has been greater attention to how Kant's conception of mathematics was modelled on early modern proof methods, which derived ultimately from Euclid's *Elements.* For example, building on this work, Sutherland (2005) has explored Kant's conception of equality, similarity and congruence in the context of Leibniz's and Wolff's mathematical works, and has recently (Sutherland 2010) examined Kant's reception of Leibniz's geometrical 'analysis of situation' and Wolff's use of similarity in recasting Euclid's geometrical proofs.

On the other hand, Kant interpreters have been expanding their investigations beyond the widely discussed questions of the syntheticity of geometrical axioms and proofs to look at a wider range of topics within Kant's philosophy of mathematics. Recent investigations have taken on Kant's theory of mathematical postulates (Laywine 1998, 2010), Kant's conception of the mathematical method

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(Carson 1999, 2006) and Kant's theory of geometrical concepts and definitions (Dunlop 2012). This paper builds on these trends to give a contextualized reading of Kant's theory of real definitions in geometry. In particular, I look at two specific cases, the definitions of *circles* and *parallel lines*, to illustrate how Kant's theory of geometrical definition differed self-consciously in sometimes subtle (but philosophically significant) ways from the theories put forward by Leibniz, Wolff and Lambert. These cases show that Kant's conception of the mathematical method (as resting on the construction of concepts in pure intuition) motivated him to give an extremely demanding theory of geometrical definitions – a theory that ruled out both Leibniz's and Lambert's preferred definitions of *parallel lines*.

Beyond its significance for our understanding of Kant's philosophy of mathematics, the present investigation is historically significant for four additional reasons. First, a striking feature of eighteenth-century German work in the philosophy of mathematics is its explicit engagement with Euclid. Indeed, Wolff, Lambert and Kant each thought of themselves as giving a philosophy of mathematics that was most faithful to the Euclidean model, despite the fact that they disagreed with one another in fundamental ways.[1] (For this reason, it is not enough to say that Kant's philosophy of mathematics is modelled on Euclid's geometry, since this simply raises the question *Whose Euclid?*) An important front in the battle between Euclid's defenders and critics concerned Euclid's definitions. Leibniz, for instance, devoted his longest geometrical work ('In Euclidis πρῶτα') largely to a line-by-line criticism of Euclid's definitions (Leibniz 1858), and one of Salomon Maimon's chief criticisms of Kant centred around what Maimon saw to be Kant's unreflective acceptance of Euclid's definition of *circle* (Freudenthal 2006). Looking at the specific stances Leibniz, Wolff, Lambert and Kant took on two of Euclid's definitions will allow us to see this battle close up.

Second, some commentators (Laywine 2001, 2010; Webb 2006, 219; Hintikka 1969, 43–44) have noted the affinities between Lambert's philosophy of geometry and Kant's own. Looking at the theory of parallel lines, where Kant and Lambert disagree, will put us in a better position to see where the two thinkers (though united in their opposition to Leibniz and Wolff) nevertheless diverge.

Third, a detailed look at Leibniz, Wolff and Lambert's theories of real definitions in geometry will show that these philosophers (like Kant) agree that constructions do in fact have a role in demonstrating the real possibility of defined concepts in geometry. That is, the debate is not whether constructions play such a role, but *how* constructions demonstrate the possibility of defined concepts. This may seem surprising, since one might have thought that it was Kant's unique contribution to the philosophy of geometry to insist on the role of constructions (against Leibniz and Wolff's purely conceptual or 'discursive' conception of mathematics). In fact, I will show, the situation was more complex and interesting: while Kant alone insisted on the role of constructions in *inference*, each philosopher claimed a role for construction in mathematical *definitions* – though each philosopher conceived of this role differently in order to fit their differing conception of mathematical proofs.

Fourth, the debate over the proper real definition of *parallel lines* was central in the most significant eighteenth-century debate in the philosophy of geometry: the status of the theory of parallel lines. Though both the Leibniz and the Kant Nachlass contain writings devoted to the theory of parallels, virtually no scholarly work has been done on these writings.[2] This essay will help rectify this situation. Moreover, the debate over the theory of parallels – and over the status of Euclid's axiom of parallels in particular – has a significance that far transcends the comparatively narrower topic of eighteenth-century German philosophical debates. The axiom of parallels is the fifth postulate in his *Elements*:

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles. (Euclid 1925, 202)

From the ancient world through the eighteenth century, many geometers believed that this axiom lacked the self-evidence of the others and thus required a proof. After centuries of debate and failed proofs, it was shown in the late nineteenth century that it cannot be proven from Euclid's other axioms, and most philosophers came to believe that the axiom is either an empirical claim, a conventional stipulation, or an implicit definition. Any of these options are contrary to Kant's view that the axioms of geometry are synthetic a priori truths. For this reason, many philosophers – following logical empiricists such as Reichenbach – concluded from these developments in the theory of parallels not only that Kant's philosophy of mathematics was refuted, but also that the very ideas of pure intuition and synthetic a priori truths were doomed. Given this wider historical context, it is surely significant to understand the eighteenth-century debates over the theory of parallels, especially those that Kant himself participated in.

The history of philosophical debates over the theory of parallels is a vivid illustration of the way that seemingly recondite, and highly specialized debates over mathematical methodology can very quickly open up onto much wider philosophical themes. In particular, Kant's claim that (contrary to Leibniz and Wolff) our cognitive faculty splits into two stems, sensibility and understanding, with the representations of the imagination a result of their joint interaction, is essentially connected to his unique view that mathematical knowledge is rational knowledge from the construction of concepts in pure intuition. A compelling case for this conception of mathematics would *ispo facto* support Kant's conception of our cognitive faculties; conversely, a prior commitment to a certain view of our cognitive faculties would compel a geometer to approach certain concrete mathematical questions in specific ways. A close look at Leibniz, Wolff, Lambert and Kant's take on the very concrete questions of how to define *circle* and *parallel lines* provides a clear illustration of what it looks like when the rubber meets the road.

This paper is organized into six sections. In Section 1, I explain how Kant's theory of real definition in mathematics flows out of his conception of mathematical knowledge as rational knowledge from the construction of concepts. In Section 2, I contrast Kant's and Wolff's take on Euclid's definition of *circle*, turning in the next section to Kant's criticisms of both Euclid's definition of *parallel lines* and the alternative proposed by Leibniz and Wolff. In Section 4, I explain why Kant took to be merely nominal the definition that Leibniz took to be real, and in Section 5 I show why Lambert accepted (whereas Kant rejected) Euclid's definition of *parallel lines*. Section 6, which is an appendix to the main argument of the paper, considers just how strong of a theory of real definitions Kant is committed to.

1. Kant on real definitions in geometry[3]

Kant was committed to a particularly strong thesis about mathematical concepts and definitions. He believed that possessing a concept, having its definition, and being able to construct instances of it were all coeval abilities. On Kant's view, one cannot possess a mathematical concept without knowing its definition, and one cannot know the definition without knowing how to give oneself objects that fall under the concept. Kant's way of putting the first point is to insist that mathematical concepts are 'made', not 'given':

> In mathematics we do not have any concept prior to the definition, as that through which the concept is first given ... Mathematical definitions can never err. For since the concept is first given through the definition, it contains just that which the definition would think through it. (A731/B759)

His way of putting the second point is to insist that a mathematical definition always contains the construction of the concept. As he put it in a letter to Reinhold:

> the definition, as always in geometry, is *at the same time* the construction of the concept. (19 May 1789, Letter to Reinhold, Ak 11:42)

For Kant, the 'construction of a concept' is 'exhibition of a concept through the (spontaneous) production of a corresponding intuition' (Kant 2002, Ak 8:192). So to construct the concept <circle>[4] is to produce a priori an intuition of a circle. Since <circle> is a geometrical concept, for Kant it follows that one cannot think of a circle without knowing its definition, and one cannot know the definition without being able to construct circles in intuition. Kant in fact says precisely this: 'we cannot think a circle without describing it in thought' (B154), that is, without constructing a particular circle by rotating a line segment around a fixed point in a plane.[5]

Since having the definition of a mathematical concept is always at the same time having the ability to produce intuitions of instances of it, Kant says that all mathematical definitions are 'real', indeed 'genetic' definitions. According to Kant, 'real definitions present the possibility of the object from inner marks'

(*Jäsche Logic* [*JL*], §106).[6] Since mathematical definitions 'exhibit the object in accordance with the concept *in intuition*' (A241-2), they present the *possibility* of an object falling under the defined concept by enabling the mathematician to construct an *actual* instance of the concept. Kant therefore calls them 'genetic' definitions because they 'exhibit the object of the concept *a priori* and *in concreto*' (*JL*, §106).[7] Because anyone who grasps a mathematical concept knows its definition, and because the definition allows one to construct instances of it, it follows that one cannot possess a mathematical concept and still doubt whether or not it has any instances.[8]

Not every concept is made, and not every concept has a real definition. In general, one can possess a concept without knowing what its definition is and whether or not it has instances. Why does Kant endorse this strong thesis for mathematical concepts in particular? The short answer is that these requirements make mathematical knowledge – 'rational cognition from the construction of concepts' – possible. Consider first the thesis that mathematical concepts are 'made', not 'given'. For Kant, that mathematical knowledge is 'rational knowledge from the construction of concepts' means that mathematical proofs proceed by producing a particular object, noting its properties, and then drawing general and a priori conclusions from it (A713/B741). This procedure introduces the danger of illicit generalization. Suppose I want to prove some property of all triangles, say that in any triangle the angle opposite the largest side is the largest angle (Euclid 1925, I.18). To prove this, I draw a particular triangle (let us suppose, an isosceles triangle whose base is the largest side), show that it has the desired property, and generalize from this that all triangles have this property. But how can I take care that in my reasoning I have not illicitly made use of a feature of the triangle (say, its being isosceles) that does not hold of all triangles? What is required is a way of keeping track of precisely those properties that are true of all and only triangles – that is, the definition of *triangle* (A716/B744). Without the definition of *triangle*, I would not be able to reliably make inferences about all triangles from a particular drawn triangle. A mathematical concept without a definition could not then be used for acquiring 'rational cognition from the construction of concepts', which is just to say that it would not be a mathematical concept after all.

Similar reasons explain why Kant thinks that a mathematical concept must contain its own construction. Mathematical concepts are just those concepts that can be deployed in drawing a priori conclusions from the construction of concepts. A concept that did not contain its own construction could not then be used in drawing inferences mathematically: one could only reason about it discursively, as Kant imagines a philosopher would have to do with a properly mathematical concept like <triangle>. What is more, the construction procedure must be *immediately* contained in the definition. For, if it were not, it would have to be *proved* that it is possible to construct instances of the concept. Because Kant believed that mathematical proofs are constructive (and not merely discursive), the proof that a concept can be constructed would itself already require that the concept be constructed. So there could be no (mediate) proof of the reality of a definition. The concept must then have a real definition, and the possibility of constructing instances of it must be (as Kant puts it) a 'practical corollary' (i.e. an immediate consequence: *JL*, §39)[9] of the definition. (This argument that the construction must be *immediately* contained in the definition thus depends essentially on proofs' requiring constructions. A philosopher who conceives of mathematical proofs as non-constructive and composed entirely in words would therefore be free to deny the immediacy condition on geometrical definitions – as we'll see Lambert in fact does.)

One last feature of Kant's theory of mathematical concepts will be especially significant in contrasting his view with Leibniz's and Lambert's. A mathematical proof can be general (despite the fact that it deploys a particular instance) because the particular instance has been constructed according to a procedure that is contained in a definition, and this definition (as I argued above) allows the geometer to keep track of which features of the drawn figure are properly generalizable. This requires that the construction procedure be itself fully general: as Friedman puts it, it must 'yield, with the appropriate inputs, *any and all* instances of these concepts' (2010, 589). Kant calls the 'general procedure' for providing for a concept an individual intuition corresponding to it the 'schema' of the concept. The generality of these schemata secure general proofs:

> In fact it is not images of objects but schemata that ground our pure sensible concepts. No image of a triangle would ever be adequate to the concept of it. For it would not attain the generality of the concept, which makes this valid for all triangles, right or acute, etc., but would always be limited to one part of this sphere. (A140-1/B180)

If a geometer constructed a triangle using a procedure that yields only a subclass of the triangles (say the acute-angled triangles), then she could not reliably infer from her constructed individual to all triangles but only to all the members of the subclass of triangles constructible using her procedure. (In such a case, even though the constructed figure is a triangle, the concept constructed was in fact <acute-angled triangle>, not <triangle>.) Moreover, the procedure contained in the concept not only must be completely general, but the mathematician herself must know it to be so in order for her to be justified in generalizing from the constructed individual.

2. Wolff and Kant on the real definition of *circle*

For Kant, mathematics is rational knowledge from construction of concepts. Construction is an activity: the '(spontaneous) production of a corresponding intuition' (Ak 8:192). Mathematics therefore requires the possibility of certain spontaneous acts. The possibility of such an act is guaranteed by a 'postulate': 'a practical, immediately certain proposition or a fundamental proposition which determines a possible action of which it is presupposed that the manner of executing it is immediately certain' (*JL*, §38).[10] A favourite example of Kant's is

Euclid's third postulate: 'To describe a circle with a given line from a given point on a plane' (A234/B287). (An axiom is then a *theoretical*, immediately certain proposition that can be exhibited in intuition (*JL*, §35). A favourite example of Kant's is 'With two straight lines no space can be enclosed.'[11])

Kant has an elegant and satisfying explanation for why postulates are immediately certain. To think the postulate, one must of course possess the concepts contained in it; but the procedure described by the postulate is *itself* the means by which the concepts in question are first generated.

> Now in mathematics a postulate is the practical proposition that contains nothing except the synthesis through which we first give ourselves an object and generate its concept, e.g. to describe a circle with a given line from a given point on a plane; and a proposition of this sort cannot be proved, since the procedure that it demands is precisely that through which we first generate the concept of such a figure. (A234/B287)

To see Kant's point, it will be helpful to lay out Kant's way of expressing Euclid's third postulate and Euclid's definition of circle.

> Euclid's Third Postulate: To describe a circle with a given line from a given point on a plane.
>
> Circle: A ... line [in a plane] every point of which is the same distance from a single one. (A732/B760)

On Kant's view of mathematical concepts, we cannot possess the concept <circle> without having its definition. But its definition, being genetic, enables me to describe circles in pure intuition *a priori* and *in concreto*. So it is impossible that I should have the concept <circle> and not know that circles can be described with a given line from a given point. The (basic[12]) genetic definitions of mathematics are then virtually interchangeable with postulates.

> The possibility of a circle is ... *given* in the definition of the circle, since the circle is actually constructed by means of the definition, that is, it is exhibited in intuition [...] For I may always draw a circle free hand on the board and put a point in it, and I can demonstrate all the properties of a circle just as well on it, presupposing the (so-called nominal) definition, which is in fact a real definition, even if this circle is not at all like one drawn by rotating a straight line attached to a point. I assume that the points of the circumference are equidistant from the centre point. The proposition 'to inscribe a circle' is *a practical corollary of the definition* (or so-called postulate), which could not be demanded at all if the possibility – yes, the very sort of possibility of the figure – were not already given in the definition. (Letter to Herz, 26 May 1789; Ak 11:53, emphasis added)

In this letter to Hertz, Kant takes as a paradigm real definition Euclid's definition of *circle*, which was Kant's preferred example also in the discussion of mathematical definitions in the *Critique of Pure Reason*. The way that Kant describes this definition to Hertz – as 'the (so-called nominal) definition, which is in fact a real definition' – makes clear that Kant recognizes that his view of Euclid's definition is controversial, and departs from other views that were well known to his audience. Indeed, the parenthetical aside is clearly an allusion to

Christian Wolff, whose mathematical works Kant knew well, having taught them for years to his mathematics students in Königsberg. According to Wolff:

> If a circle is defined through a plane figure returning to itself, the single points of whose perimeter are equally distant from a certain intermediate point; the definition is nominal: for it is not apparent from the definition, whether a plane figure of this kind is possible, consequently whether some notion answers to the definitum, or whether it is actually a sound without meaning [mens]. For truly if the circle is defined through a figure, described by the motion of a straight line around a fixed point in a plane, then from the definition it is patent [patet], that a figure of this kind is possible: this definition is real. (1740, §191)

Wolff here rejects Euclid's definition in favour of one that explicitly describes the procedure for constructing circles:

> A circle is that figure that is described by moving a straight line around a fixed point in a plane.

That all of the points in the circle are equidistant from the centre (which is Euclid's definition) is on Wolff's view not part of the definition, but a consequence of it. In fact, Wolff lists it as an axiom.[13]

This disagreement over Euclid's definition is initially surprising, since Kant and Wolff seem to define real definitions in very similar ways. For Kant, 'real definitions present the possibility of the object from inner marks' (*JL*, §106); for Wolff, they are 'definition[s] through which it is clear that the thing defined is possible' (Wolff 1740, §191, cf. 1741, Introduction §17–§18, 1710, Introduction, §4). But the disagreement is not primarily a difference in the conception of real definition, but a difference in the conception of what it is to grasp a concept. One of the most significant and innovative features of Kant's critical philosophy of mind is the thesis that concepts 'rest on functions', or constitutively include abilities to do certain things (A68/B93). This innovation explains the divergence between Kant and Wolff over Euclid's definition of *circle*. Because Kant identifies the distance between two points as the straight line between them (Reflexion 9, Ak 14: 36) and because one cannot think a line without drawing it in thought (B154), to think of the points equidistant from a given point is *ipso facto* to think of a straight line moved around the given point. The upshot is that 'we cannot think a circle' – that is, understand its Euclidean definition – 'without describing it from a given point' (B154), and Wolff's 'real definition' coincides with Euclid's 'so-called nominal definition'.

3. Kant versus Euclid's and Leibniz's definitions of *parallel lines*

As I mentioned in the opening paragraphs of this paper, the most contentious feature of Euclid's geometry was his notorious axiom of parallels. It was virtually the consensus view in eighteenth-century Germany – a consensus shared by Leibniz, Wolff, Lambert and Kant – that Euclid's axiom is not in fact a genuine axiom. Although no one doubted its truth, everyone agreed that it was not self-evident and so not an *axiomatic* truth.[14] The debate over Euclid's axiom in eighteenth-century Germany is rich and interesting, though still largely untold. But it is less well known that there was in the early modern period a different debate, dovetailing with the first, over Euclid's *definition* of *parallel lines*. Euclid defined parallel lines as

> straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction. (Euclid 1925, 190)

Leibniz, Wolff, Lambert and Kant all participated in the debate. As we will see, Leibniz, Wolff and Kant all rejected the definition, whereas Lambert (along with Saccheri 1920, 7, 88–95, 237–41) defended it.

Kant rejects Euclid's definition in Reflexion 6 (Ak 14:31), which dates from the critical period and is the opening reflection in a series of notes (Reflexions 6–11, Ak 14:31–52) on the theory of parallels.[15] Kant begins by restating his now familiar requirement on real definitions in mathematics:

> I think that from a definition that does not at the same time contain in itself the construction of the concept, nothing can be inferred (which would be a synthetic predicate).

Mathematical definitions make immediately patent the procedure for constructing instances of the concept. Any candidate mathematical definition for which the construction procedure is not a 'practical corollary' of the definition, is thus a faulty kind of definition. From this, Kant then draws the surprising conclusion that 'Euclid's definition of parallel lines is of this kind'. Kant's point is that grasping Euclid's definition does not tell you how to construct two non-intersecting straight lines. Euclid's definition of a circle is such that one cannot think it without rotating a line segment around a fixed point in the plane, but Euclid's definition of *parallel lines* provides no guidance whatsoever. A subject could fully grasp the marks <straight line>, <coplanar> and <intersecting>, know full well how to construct straight lines, planes and intersecting lines, understand fully how these marks are put together to form the concept <parallel linesz>, but still not know how to construct them. The definition is thus not genetic. Indeed, since it provides no method for constructing parallels, the definition does not present the possibility of parallels and so is not even a real definition.

Kant's criticism of Euclid here is not unprecedented. He would have known a similar criticism that Leibniz had levelled against Euclid in the *New Essays*:

> [T]he real definition displays the possibility of the definiendum and the nominal does not. For instance, the definition of two parallel straight lines as 'lines in the same plane which do not meet even if extended to infinity' is only nominal, for one could at first question whether that is possible.

Leibniz proposes instead an alternative definition, which he claimed is genuinely real:

> But once we understand that we can draw a straight line in a plane, parallel to a given straight line, by ensuring that the point of the stylus drawing the parallel line remains at the same distance from the given line, we can see at once that the thing is

> possible, and why the lines have the property of never meeting, which is their nominal definition. (Leibniz 1996, III.iii.§18, cf. 1858, 201)

Leibniz is suggesting here an alternative definition to Euclid's (which he takes to be merely nominal):

> Parallels are 'straight lines in a plane which have everywhere the same distance from one another'.[16]

On Leibniz's view, this definition is real (indeed, genetic) because it tells us how to construct the parallel lines: take a perpendicular of fixed length on a given line, slide the perpendicular along the line, and then the line described by the end point of the moving perpendicular is everywhere equidistant from the given line. Thus, this definition 'carries with it its own possibility' (Leibniz 1858, 202; Definition XXXIV, §6). (Wolff follows Leibniz in defining *parallel lines* in terms of equidistance, and he recasts the theory of parallels using this new definition in his widely read and influential mathematical works.)

Kant, however, thinks that Leibniz's (and Wolff's) definition is no more a real definition than Euclid's. After a series of attempts to reconstruct the theory of parallels using this new definition (Reflexions 7–10), Kant ultimately concludes:

> For now the geometric proof [of a proposition in the theory of parallels] rests on [...] a concept of determinate distances and of parallel lines as lines, whose distance is determinate, [a concept] which cannot be constructed, and is therefore not capable of mathematical proof. (Reflexion 10, Ak 45-8)

Leibniz's definition runs afoul at the same point as Euclid's: in constructing it. Granted, it is immediately evident that it is possible to slide a perpendicular line of fixed length along a given straight line and thus trace out a new line everywhere equidistant from the first. But what guarantees that the line traced out is *itself straight*, as Leibniz's definition of *parallel lines* requires? Indeed, without an extra assumption that guarantees that a line everywhere equidistant from a straight line is itself straight, we cannot take ourselves to have constructed parallel lines in Leibniz's sense. As Kant's student Schultz put the point in his 1784 work on parallels, Wolff does not rule out that the concept of everywhere equidistant straight lines is a 'Hirngespinst' – a figment of the brain (7–8, cf. Kant 1998, A157/B196). (As a matter of fact, the assumption that the everywhere equidistant line is straight is just equivalent to the parallel axiom. So the constructibility of Leibniz's definition falls back onto the unsolved problem of finding a proof of the axiom of parallels.) Moreover, even if Leibniz could come up with a proof that shows that the line traced out is in fact straight, this would not make the definition a *real* definition for Kant: a construction procedure which must be *proved* to produce instances of the defined concept does not *itself* show the possibility of the concept. One could easily grasp Leibniz's definition and still doubt whether the line traced out is straight. In Section 1 of this paper, we saw Kant argue that only real mathematical definitions – definitions which themselves show their own possibility by containing their own constructions – can be employed in mathematical proofs, which proceed by constructing concepts. Kant in this note draws the obvious conclusion: Leibniz's definition of *parallel lines* cannot be employed in genuine geometrical proofs.

4. Kant versus Leibniz on real definitions in geometry

It will surely be surprising to some – given the 'dogmatism' to which Kant was allegedly prone – to see him rejecting both Euclid's definition of *parallel lines* (which, we will see, Lambert defended) and the alternative definition proposed by Leibniz and Wolff.[17] In the remainder of this paper, I will argue that it is precisely *because* Kant believed (while Leibniz and Lambert did not) that mathematics is knowledge from the construction of concepts that his requirements on the definition of parallel lines are more restrictive than his contemporaries' requirements.

Let us begin with Leibniz. Kant's notion of real definition closely follows Leibniz. Like Kant, he defines a real definition as a definition that 'shows that the thing being defined is possible', and thus also gives the essence, or inner possibility, of a thing (Leibniz 1996, III.iii.18).[18] A special kind of real definition is a causal (or genetic) definition, which 'contains the possible generation of a thing'.[19] Again like Kant, Leibniz claims that the mathematician's ability to form real definitions by putting together geometrical concepts is highly constrained. He emphasizes that 'it isn't within our discretion to put our ideas together as we see fit' (Leibniz 1996, III.iii.15), and calls 'paradoxical properties' those combinations of ideas whose possibility can be doubted (Leibniz 1970, 230). Leibniz gives many examples of merely nominal mathematical definitions: In *Discourse on Metaphysics* (§24), he gives a nominal definition of *endless helix*, and in *New Essays,* two merely nominal definitions of *parabola* (III.x.19). Eerily similar to Kant's Reflexions 5–6, Leibniz (1970, 230) contrasts Euclid's definition of *circle*, which he claims to be properly genetic, from another definition (derived from Euclid III.20) that is merely nominal, even though it provably holds of all and only circles.[20]

Despite their commonalities, the notion of a real definition plays a much different role in Kant's philosophy of geometry than it does in Leibniz's. Kant's restrictions on real definitions in mathematics are motivated by his theory of construction: mathematical concepts are those concepts for which a fully general a priori and certain proof can be given using a representative individual. Leibniz, however, famously believes that the process of reasoning about mathematical truths does not differ in kind from the reasoning about any other necessary truth. He forbids 'admitting into geometry what images tell us', he proposes a new kind of geometry 'even without figures', and believes that all geometrical truths can be reduced to identities through conceptual analysis using the principle of contradiction.[21] Now, reasoning from definitions cannot be done safely 'unless we know first that they are real definitions, that is, they include no contradictions' (Leibniz 1989, 25). On many occasions, Leibniz motivates the necessity of real definitions by criticizing Descartes's argument for the existence of God: we cannot safely infer from the definition of God unless we know that God is possible (Leibniz 1989, 24, 1970, 231). For Leibniz, then, the purpose for introducing mathematical definitions that 'explain the method of production is *merely* to demonstrate the possibility of a thing' (1970, 231, emphasis added). This weaker role for construction in mathematical proof brings with it a weaker requirement on real definitions. A definition that contains a method for constructing some, though not necessarily all, instances of a concept would fulfil Leibniz's requirement. Even one instance of a concept is sufficient to show that the concept is not contradictory. For Kant, though, fully general constructive proofs require that the constructive procedure immediately contained in the concept must be general enough to produce any and all instances of the concept.[22]

Leibniz's real definitions play a different role from Kant's in another respect. Leibniz (1970, 231–232, 1996, IV.vii.1) believed that all necessary truths, including the axioms of geometry, could be reduced to identities once fully adequate definitions have been found. Truths that on Kant's view are expressed by axioms and postulates have to be somehow built into appropriately formulated definitions. For a Kantian, this is putting weight onto the definitions that they cannot possibly bear. Leibniz's and Wolff's own work in the theory of parallels shows this clearly because they more or less smuggled the problem with Euclid's axiom into a definition of parallel lines whose reality is no more evident than the truth of the axiom.

5. Kant versus Lambert on real definitions

In fact, this was precisely Lambert's criticism of Wolff's proofs in the theory of parallels. Lambert begins his justly famous *Theory of Parallel Lines* by criticizing Wolff's definition of *parallel lines*. From his point of view, Wolff 'conceded too much to the definitions; and because he wanted to arrange them suitably for the subject matter [. . .] the difficulty is only taken from the axiom and brought into the definition' (Lambert 1895b, §5, §8).[23] Reflection on what goes wrong in Wolff's treatment of parallels (and on what goes right in Euclid's) will motivate a better conception of science, and therefore also a better grounded geometrical theory of parallels.

On Lambert's view, the fallacies in Wolff's treatment of parallels are a symptom of a deeper confusion in Wolff's (and Leibniz's) theory of science.[24] On pain of circularity, not every scientific concept can be defined; some must be simple. For this reason, propositions that contain only undefinable concepts standing in undefinable relations cannot possibly be inferred from definitions. The arbitrary combination of simple concepts into compounds thus needs to be licensed or constrained by certain basic propositions.[25] Lambert (1771, §13; cf. 13 November 1765, Letter to Kant, Ak 10:52) claimed that propositions containing only simples are thus either postulates (which affirm that certain simple concepts stand in certain simple relations) or axioms (which deny other simple relations among simple concepts). The possibility of every compound concept, like Euclid's definition of *parallel lines*, needs to be given a proof that bottoms out in postulates and axioms. On Lambert's view, Leibniz and Wolff therefore inverted the order of dependency among definitions, axioms and postulates. Axioms and postulates are not dependent on definitions as their corollaries, as Leibniz and Wolff claimed, but are preconditions of the possibility of definitions. In any properly produced science, the axioms and postulates should be presented before the definitions, not after (Lambert 1771, §23).

No doubt there was much about this view that accorded with Kant's own. In particular, Lambert's axioms and postulates, which do not follow from definitions and – being composed entirely of simples – cannot be containment truths, would have been for Kant clear examples of *synthetic* a priori truths (3 February 1766, Letter to Kant, Ak 10:65). Nevertheless, Kant could not give his full consent to Lambert's view of science: as we have seen, Kant continued to hold with Wolff (and against Lambert) that *postulates* are practical corollaries of definitions.[26] This disagreement over postulates spills over into a further divergence in their theory of real definitions.

> [W]hat one derives as axioms [Grundsätze] from the definitions according to Wolff are according to Euclid propositions that the definition already presupposes, and from which the definition is formed and proved. In this way the things that appear arbitrary and hypothetical fall away from the definitions, and one is completely assured of the possibility of everything they contain. [. . .] Definitions produced in this way define the thing itself, and insofar as one brings them forth from basic concepts [Grundbegriffe], one can call them real definitions, which are a priori in the strictest sense. (Lambert 1771, §§23–§§24)

Real definitions are definitions composed of simple concepts whose possibility has been proved from the axioms and postulates. Notice that it is no part of Lambert's view that the possibility of the concept be *immediate from the concept itself*. In fact for Lambert, the proper method (which, again, he thinks is Euclid's) is to first introduce purely nominal definitions and *only later* prove their possibility. After this proof, the 'hypothetical' concept becomes a 'derived concept'.[27]

In fact, for Lambert Euclid's definition of *parallel lines* is a paradigm real definition, since it is first introduced as a nominal definition, and then later becomes a real definition only after Euclid proves its possibility in I.31. Lambert points out that while Wolff assumed without proof that there are everywhere equidistant straight lines, Euclid does not assume without argument that there are non-intersecting lines.[28] He proves in Euclid I.27, without using the parallel axiom, that there are parallel lines (namely, straight lines that make equal alternate angles with some third straight line), and in Euclid I.31, he shows (again without appeal to the parallel axiom) how to construct them. (Euclid's axiom, then, simply shows that this is the *unique* way of constructing parallels: that there are no parallels to a given straight line other than those that make equal alternate angles with a third straight line.) On Kant's view, however, every genuine mathematical concept must itself exhibit its own possibility, and so every mathematical definition must be real from the get-go. Kant would then be unimpressed with the fact that Euclid later proves that there are parallel lines, since a legitimate definition of *parallel lines* would be such that 'the definition, as always in geometry, is *at the same time* the construction of the concept' (19 May 1789, Letter to Reinhold, Ak 11:42). Lambert, on the other hand, is untroubled to admit that Euclid's definition of parallel lines 'does not indeed indicate their possibility' (1895a, #7).

The case of parallel lines illustrates another fundamental difference between Lambert and Kant, Leibniz and Wolff. As we have seen, these other philosophers all insist that every mathematical definition is genetic. Lambert, however, does not. He argues, against Wolff, that one needs to distinguish between the mode of genesis of a concept, and the mode of a genesis of a thing. The mode of genesis of a concept is a proof of the possibility of a concept from axioms and postulates (Lambert 1915, §27, cf. 1771, §24). In this sense, the proof in Euclid I.27 and I.31 gives the mode of genesis of the concept <parallel lines>: it proves, step by step syllogistically from the axioms and postulates, that there are parallel lines. The mode of genesis of the thing, on the other hand, gives the procedure for making an instance. An example of a mode of genesis of a mathematical object is the procedure for producing a conic section (first, produce a cone by rotating a right triangle about one side, then cut the cone with a plane), and an example of a mode of genesis of a natural scientific object is the procedure for producing a salt crystal (first dissolve salt in water, then do such and such to the water . . .).[29]

Lambert claims that a real definition of a mathematical object requires only the first kind of mode of genesis. Indeed, Lambert claims that a definition that gives the mode of genesis of the thing is appropriate only for a posteriori concepts – concepts whose possibility is shown not by a proof from first principles, but from some other source, such as experience.[30] For example, using the genetic definition of salt crystals, one carries out the procedure described in the definition, observes that something is produced by the procedure, and thereby comes to know a posteriori that the concept <salt crystal> is possible. Lambert is therefore undisturbed by the fact that Euclid's definition of *parallel lines* in no way expresses their mode of genesis.

There is a further difference between Kant's and Lambert's conceptions of geometry that helps explain why Kant rejected the Euclidean definition of *parallel lines* that Lambert accepted. Although every science must on Lambert's view prove the possibility of its compound concepts, this is easier in geometry because the constructed figures themselves demonstrate the possibility of compound concepts.

> It was easy for Euclid to give definitions and to determine the use of words. He could lay the lines, angles and figures before the eyes, and thereby combine words, concepts, and things [Sache] immediately with one another. The word was only the name of the thing, and because one could see it before the eyes, one could not doubt the possibility of the concept. Furthermore, it was the case that Euclid had unstinted freedom to let everything that did not belong to the concept or was not

> present in it to fall away from the figure, which is really only a special or singular case of a general proposition, but thereby serves as an example. The figure accordingly represents the concept completely and purely. On the other hand, because the figure does not provide the general possibility of the concept, Euclid had taken care to exposit this [possibility] exactly, and here he uses his postulates, which represent general, unconditioned and simple possibilities that are thinkable for themselves, or things that can be done [Thulichkeiten], and this [possibility] he put forward in the form of problems. (1771, §12)

The possibility of a compound concept is made manifest in the constructed figures licensed by geometrical problems. Lambert's examples include Euclid I.1 (which demonstrates the possibility for <equilateral triangle>), I.22 (for <triangle>) and I.31 (for <parallel lines>).[31] These figures are not merely examples, though, because they are produced by general procedures licensed by postulates. The figures then stand in for the concept completely and purely.

On its surface, this passage is strikingly similar to Kantian passages like A713/B741. Indeed, some commentators have alleged that Lambert's comments on the role of individual constructed figures in geometry were an impetus for Kant's own views on the role of intuitions of individuals in mathematical proofs.[32] However, the mere fact that Lambert – but not Kant! – believed that the construction licensed by Euclid I.31 demonstrated the reality of <parallel lines> should give us pause. On Kant's view, the construction procedure immediately contained in a mathematical concept has to be general in the sense that we can know, with certainty, that each and every instance of that concept can be constructed in this way. Unless we can be certain of the truth of Euclid's axiom, I.31 will not show that Euclid's definition is general *in this sense*. Lambert, however, clearly thinks that Euclid's definition has been proven, and the construction from I.31 is fully general *in the sense that matters to him*, even though Euclid's axiom requires, but does not yet have, a demonstration.

Let me explain these differing notions of generality. Lambert was concerned to justify certain arbitrary combination of concepts in the face of worries that the combined concepts might together contain a hidden contradiction. He imagines a sceptic who denies the possibility of such a combination. There are two senses for Lambert in which Euclidean problems are general – two senses that suffice to refute such a sceptic but which nonetheless fall short of the kind of generality that matters to Kant. For one, Euclidean constructions can be carried out by *any person* in any time or place. In an extended and rich discussion in his 'Essay on the Criterion of Truth', he imagines Euclid responding to a sophist as a metaphysician might respond to a solipsist: 'there is no better way to refute someone who believes something impossible than if one shows him how he can himself bring it about' (§79).[33] Euclid's definition of *parallel lines* is general in this sense, since any geometer can carry out I.31 for herself any time she pleases (Lambert 1918, §20–§21). Second, Lambert highlights that Euclidean problems specify conditions that *always suffice* to provide an instance of the concept. In this sense, Euclid I.22 shows that a triangle can be formed from any three straight lines provided any two are greater than the third (Lambert 1895a, #6, 1915, §79).

Euclid's definition of *parallel lines* fulfils this requirement as well, since a geometer can carry out the construction in I.31 given *any* line segment and *any* point not on that line. Still though, the fact that *any* geometer at *any* time can construct from *any line and point* a new line that does not intersect it in no way shows what Kant wants: that the construction be certainly sufficient to produce *any and all* parallels.[34]

Why in the end do Lambert and Kant demand very different things of Euclid's definition? I have emphasized that Kant's very stringent conditions on mathematical concepts follow from his characteristic view that a mathematical proof works by drawing a fully general, certain and a priori conclusion from a single intuition of an individual. And though Lambert acknowledges the role of individual drawn figures in proving the possibility of compound concepts, his view of mathematical proof is thoroughly Wolffian: the proof of a theorem from axioms and postulates 'depends on the principle of contradiction and in general on the doctrine of syllogisms' (Lambert 1915, §92.14).[35] Because the intuition of the figure plays no role in the proofs themselves, the function of the individual constructed figure is merely to show that the concept is free from contradiction. Since one instance of a concept suffices to show that it is free from contradiction, Lambert does not require that the individual drawn figure be produced from a construction procedure that is general *in Kant's sense*. In Lambert's case, then, just as with Leibniz, we see that Kant's view that mathematics is rational cognition from the construction of concepts made him more, not less, sceptical of his contemporaries' treatment of parallel lines.

Lambert and Kant then disagree fundamentally about the kind of generality that a constructed figure must possess in order for a definition to be real. This disagreement about real definitions should not however obscure their agreement about postulates. Both think of postulates as practical indemonstrable propositions, and both believe (against Leibniz and Wolff) that they are not containment truths (Laywine 1998, 2010). Both further distinguish postulates as practical from axioms as theoretical (Heis, forthcoming, §3). Kant further believes that definitions and practical propositions that assert that a certain construction is possible are virtually interchangeable. In the case of basic concepts such as $<$circle$>$, this is the claim (discussed above) that Euclid's third postulate is simply a practical corollary of Euclid's definition of a circle. Although Lambert denies this, this is again a reflection of his different notion of what makes a real definition. A real definition in mathematics for Lambert need not express a construction procedure (it need not be genetic), and so the very tight connection that Kant sees between definitions and practical propositions like postulates is severed.

There is, though, one significant difference between Kant and Lambert's conceptions of postulates, a difference that rests on their differing view of mathematical proof. Lambert argues that the practicality of postulates rests on their fundamental role in establishing that certain combinations of simple concepts are possible, and he criticizes other philosophers who emphasize the practicality of postulates as confusing a superficial property with what is really essential (1915, §48ff., 1771, §18–§20). Euclid, in showing what can be done constructively, was thereby showing what combinations of concepts are possible. And since the possibility of compound, but not simple concepts, needs to be either self-evident or proven, there need to be principles that express the possible combinations of simple concepts. This is the deeper characterization of postulates, which is not explicitly expressed by simply saying that postulates are practical indemonstrable propositions.

Kant, however, does not thematize the difference between simple and compound concepts as Lambert does. And for Kant, it really is essential that the postulates express not just which combinations of concepts are possible, but also what constructions can be executed. After all, since for Kant proofs require the construction of concepts, it would simply not be enough to know that, say, equilateral triangles are possible. To prove anything about equilateral triangles, the geometer needs to produce a representative instance and so needs to know how to actually carry out the construction. Kant's conception of mathematical proof, then, leads him to emphasize the practicality of postulates – an emphasis that Lambert, whose conception of proof is different, thinks confuses the accidental with the essential.

6. The immediacy condition on real definitions

In Section 1 of the paper, I argued that Kant's conception of mathematical methodology required that possessing a concept and knowing its definition are coeval, that every definition exhibits the method for constructing instances of the defined concepts, and that this constructive procedure must be sufficient for producing any and all instances of the concepts. These requirements are sufficient for showing why Kant rejected both the Leibniz/Wolff definition of *parallel lines* in terms of equidistance (without a proof of the axiom of parallels, it is not clear that we can produce any instances of the concept) and the Euclidean definition in terms of non-intersection (this definition does not exhibit any method for constructing parallels, and moreover without a proof of the axiom of parallels, we cannot know that the construction procedure given by Euclid in I.31 is sufficient to produce any and all instances of the concept). If the argument of the paper has made these claims plausible, then the paper has succeeded.

In Section 1, though, I argued that Kant is also committed to a yet stronger claim, that the definition not only must express a construction procedure for the defined concept, but also that the definition must make it *immediately certain* that that construction procedure is possible. Although this stronger claim is not necessary for my argument in the paper, I do believe that Kant was committed to this stronger view, and given the interest of the topic, I would like to end my paper with a defence of the stronger claim.

To begin with, it is incontestable that Kant maintained the immediacy condition for those definitions for which postulates are practical corollaries, since he says so explicitly at Ak 11:53 and A234/B287, in both cases taking as his example the concept <circle>. In note 12, I called such concepts 'basic', and distinguished them from concepts whose construction procedures are not licensed by postulates. A concept of this kind, such as <equilateral triangle> or <parallel lines>, I call 'complex'. The interesting question, then, is whether the immediacy condition should be applied to complex concepts as well.

An interpretation of Kant that maintains the immediacy condition only for basic concepts and not for complex ones would attribute to Kant a principled distinction among concepts akin to Lambert's distinction between *Grundbegriffe* (axiomatic concepts), whose possibility is immediately certain, and *Lehrbegriffe* (derived concepts), whose possibility is proved subsequently to grasping them.[36] On this view, Kant (like Lambert) would allow that one and the same definition would be nominal when it is first grasped, and then would later become real when a proof is provided for it.

I do not believe that Kant can be assimilated to Lambert in this way. For one thing, the plain reading of Kant's texts provides no basis for it. After all, Kant writes:

> [T]he object that [mathematics] thinks it also exhibits *a priori* in intuition, and this [object] can surely contain neither more nor less than the concept, since through the definition of the concept the object is originally given. (A713/B741)

> A mathematical definition must 'at the same time [zugleich] contain in itself the construction of the concept'. (Reflexion 6)

> The definition, as always in geometry, is at the same time the construction of the concept. (19 May 1789, Letter to Reinhold, Ak 11:42)

There is no suggestion in these passages that the immediacy condition is restricted to some concepts and not others. In fact, in Reflexion 6, Kant's claim comes between a discussion of the definition of *circle* and the definition of *parallel lines* – one basic and one complex concept – and Kant in both cases is making the same point. Moreover, I hope to have shown that the case of the definition of *parallel lines* was precisely where the immediacy condition had become an issue in eighteenth-century Germany, and it was the case that Lambert used to illustrate the importance of introducing a difference in kind between those concepts whose possibility was immediately obvious from those concepts whose possibility had to be proven subsequently to their being grasped. In fact, Kant goes on in the very next sentence to endorse the argument (Euclid's definition is not real, because it does not contain its own construction immediately) that Lambert used his distinction to deny. (This is a nice illustration of the advantage at looking at the stances that Kant and his contemporaries took on specific concrete cases. If we just look at frequently discussed passages like Ak 11:53 and A234/B287, where Kant's example is circles, we would never realize that Kant's immediacy condition is quite strong and quite surprising in other cases.) Moreover, I do not think that we can simply attribute to Kant a principled distinction between basic and complex concepts on the grounds that it is obviously the right thing to do and so Kant would have acknowledged it even if he did not do so explicitly. After all, neither Leibniz nor Wolff drew such a distinction – for them, properly formulated mathematical definitions of all concepts satisfy the immediacy condition – and Lambert would not have emphasized it so strongly if he thought it was obvious.

But I think that the argument that Kant held the immediacy condition even for complex concepts goes beyond marshalling passages. Kant's theory of mathematical proof seems to commit him to it. As I argued above, a proof that a concept can be constructed would have to be a mathematical proof, and so – on Kant's view – would already require a construction. A principled distinction between concepts like Lambert's would require a conception of mathematical proof like Lambert's, where we can prove (merely in words) what combinations of concepts are possible.

Still, though, I do believe that Kant's strictures on mathematical definitions are too strong and cannot be made to work in all cases. This is because Kant's position commits him to its being possible to suitably formulate definitions of geometrical concepts so that what were traditionally called 'problems' become obvious. Thus, although Kant clearly does not believe (as Wolff and Leibniz do) that *all* of the axioms and propositions of geometry are derivable from definitions alone, Kant is committed to some seemingly substantive propositions being immediately derivable from definitions. For instance, of the 48 propositions of *Elements* I, there are three propositions that assert that a complex concept is constructible: I.1: to construct an equilateral triangle on a given straight line; I.11: to draw a straight line at right angles to a given straight line from a given point on it; and I.31: to draw a straight line through a given point parallel to a given straight line. In the case of I.31, there was a definition of *parallel lines* well known in the eighteenth century that does satisfy the immediacy condition. Though there is no evidence that Kant was aware of this, Borelli defined parallel lines as:

> any two straights AC, BD, which toward the same parts stand at right angles to a certain straight AB. (Saccheri 1920, 89)

As Saccheri admits, this definition is 'set forth *by a state* (as he says) possible and most evident', since the definition in essence bids the geometer to take the constructive procedure for <perpendicular line>, which Euclid presents in I.11, and repeat it. As Kant would put it: one cannot think <straight line> without drawing straight lines in thought (in accord with Euclid's first two postulates); one cannot think <perpendicular lines> without erecting one straight line from a point of a second straight (in accord with Euclid I.11); and one cannot think Borelli's definition of *parallel lines* without erecting two perpendiculars on a given straight line. Each of the component concepts in Borelli's definition contains its own construction, and the construction procedure for the definition is patent from the way the component concepts are put together. What's more, the case of Borelli demonstrates very clearly that endorsing the immediacy condition even for complex concepts, as I think Kant does, was hardly unprecedented in the early modern period.

What of Euclid I.1?[37] In Euclid I.1, we construct an equilateral triangle on a given base AB by drawing a circle of radius AB centred on A, and then drawing a second circle of radius BA centred on B. An intersection point of the two circles, say C, forms the equilateral triangle ABC. Perhaps surprisingly, on Kant's view it is plausible that the possibility of this construction procedure is evident from thinking the definition of *equilateral triangles* as a trilateral figure whose three sides are all equal. In Section 2, we saw that Kant defended the reality of Euclid's definition of circle (in Kant's terms, a line in a plane every point of which is the same distance from a single one) on the grounds that a distance is just a line segment and one cannot think a line without drawing it. So to think of a line everywhere equidistant from a given point is just to rotate in thought a given line around that point. If Kant's story about circles is granted to him, then it is easily extended to equilateral triangles. To think of an equilateral triangle on the base AB is just to think of that base AB rotated around A and rotated around B until a plane figure is formed. If equidistant points from a given point can only be thought by describing a circle, then to think of three equidistant points is just to think of the intersection of two circles. And so the possibility of I.1 is immediately patent from the definition of *equilateral triangle*.

Even if this story about I.1 is granted, one might still be suspicious that it would work for every case in Euclid's *Elements*. I agree, but I do not think that Kant was committed to that. After all, Kant was perfectly willing to argue that Euclid's definition of parallel lines was illegitimate. He was certainly not alone in that assessment. Indeed, it was the near universal practice of Kant's contemporary geometers to rework Euclid's *Elements* fundamentally in an attempt to get the definitions, axioms, postulates and proofs that would satisfy some ideal of the proper mathematical method. A defence of a certain philosophy of geometry – whether it was given by Kästner, Schultz, Lambert, Leibniz or Wolff – was a normative claim about how geometry should be done, and an expression of a reform campaign for reworking geometry, including its definitions. The point was not to match Euclid exactly – it was to improve the foundations of geometry. And we should expect that many of these reform programmes, despite being philosophically attractive, were ultimately unworkable. (This was certainly the case for Leibniz's and Wolff's philosophies of geometry.)

To confess my own view: I think that the suspicion that a Kantian reconstruction of propositions like *Elements* I.1 is trying to pull a rabbit out of a hat is rooted in a deep problem with Kant's account. The claim that the possibility of a construction procedure is immediately certain from a definition is only as clear as the notion of *immediacy* itself. For Wolff, there is a clear, non-vague notion of immediacy: a proposition is an immediate consequence of a definition if it can be derived from that definition in one syllogism. When Kant rejects the Wolffian account and asserts that mathematical knowledge is rational knowledge from the construction of concepts, he is thereby committed to a new kind of immediacy, *intuitive immediacy*. This is the sense in which a mathematical axiom is supposed to be immediately certain for Kant, and the sense in which a construction procedure is supposed to be immediately certain given the correctly formulated definition. But I worry that this notion of immediacy is vague, and it is this vagueness that makes itself felt when looking at particular cases. Of course, in this respect Kant was no worse than Lambert, who has more or less no positive story of the immediate certainty of axioms and postulates, and Wolff, whose criterion of immediacy is clear but entirely unworkable. But these are large issues and extend far outside the scope of this paper.

Notes

1. Of the many passages where Kant aligns himself with Euclid, see for instance, Reflexion 11, where Kant criticizes Wolff for not conducting his proof of Euclid I.29 ‘in the Euclidean way’ (Reflexion 11, 14:52). Wolff aligns himself with Euclid (against his early modern critics, like Ramus) at *Preliminary Discourse* (§131) (in Wolff 1740). Lambert aligns himself with Euclid (against Wolff, whom he thinks distorts Euclid) in Lambert (1895a, 1915, §§78–§§82). Of course, none of these thinkers hold Euclid above criticism: each rejects Euclid’s axiom of parallels, for example, and Wolff, Leibniz and Kant all reject Euclid’s definition of <parallel lines>.

   Works by Wolff and Lambert will be cited by paragraph number (‘§’). Citations of works of Kant besides the *Critique of Pure Reason* are according to the German Academy (‘Ak’) edition pagination in Kant (1902). I also cite Kant’s reflexions by number. For the *Critique of Pure Reason*, I follow the common practice of citing the original page numbers in the first (‘A’) or second (‘B’) edition of 1781 and 1787. Passages from Kant’s *JL* (edited by Kant’s student Jäsche and published under Kant’s name in 1800) are also cited by paragraph number (‘§’). I use the translation in Kant (1992) both for *JL* and for Kant’s other logic lectures. Passages from Kant’s correspondence are cited by Ak page number and by date; translations are from Kant (1967).

   On Euclid’s influence on Kant, see Shabel (2003) and Friedman (1992); on Lambert’s self-conscious Euclideanness, see Laywine (2001, 2010) and Dunlop (2009).
2. There are notable exceptions: on Kant’s notes on parallels, see Adickes (1911) and Webb (2006); on Leibniz, see De Risi (2007); on Adickes’s and Webb’s readings of Kant, see note 17.
3. I discuss Kant’s theory of real definitions also in Heis (forthcoming, §I). Two paragraphs in this section – and one paragraph from §2 – appear also slightly modified in that paper.
4. I follow the common practice of referring to concepts in angled brackets and words in quotation marks.
5. Kant is speaking loosely when he says that one cannot *think* a circle without describing it. Indeed, he is at pains elsewhere to insist that it is possible to think of a mathematical concept without carrying out its construction, as for instance a philosopher would if she were trying to prove Euclid I.32 in a merely discursive way: A716/B744. Kant is being more precise when, in a passage parallel to B154, he says: ‘in order to *cognize* something in space, e.g., a line, I must *draw* it’ (B138). Kant’s point is that one cannot employ the concept <circle> in the uniquely mathematical way – that is, in the way that leads to rational *cognition* – without carrying out the construction and describing a circle.
6. Kant characterizes ‘real definitions’ in multiple ways, all of which turn out to be equivalent. In the first *Critique* (A241-2), Kant says real definitions ‘make distinct

[the concept's] objective reality', which is equivalent to the 'real possibility' of the concept (see e.g. A220/B268). Elsewhere, he says that real definitions contain the 'essence of the thing' (Vienna Logic, Ak 24:918), which is equivalent to the 'first inner principle of all that belongs to the possibility of a thing' (Kant 2004, Ak 4:467, 2002, Ak 8:229). That mathematical definitions are real, see A242; Blomberg Logic, Ak 24:268; Dohna-Wundlacken Logic, Ak 24:760; *JL* §106, note 2, Ak 9:144; Reflexion 3000, Ak 16:609.

7. On genetic definitions, see also Reflexion 3001, Ak 16:609. That mathematical definitions are genetic: Reflexion 3002, Ak 16:609.
8. Dunlop (2012) makes the same point in her discussion of real definitions in geometry.
9. 26 May 1789, Letter to Herz, Ak 11:53. I quote and discuss this letter in Section 2. In the letter, Kant is referring specifically to the concept <circle>. I take Kant to be further committed to the immediacy condition for all geometrical definitions, not just for geometrical definitions of basic concepts such as <circle>. This further commitment is quite strong, and my reading will require some defense. I return to this issue in the closing section of the paper.
10. On postulates, see also Reflexion 3133, Ak 16:673; Heschel Logic 87 (Kant 1992, 381); 'Über Kästner's Abhandlungen', Ak 20:410-1; Letter to Schultz, 25 November 1788, Ak 10:556.
11. A24, A47/B65, A163/B204, A239-40/B299, A300/B356. Though Kant does not say explicitly in *JL* §35 that axioms are theoretical and never practical, I believe that this was his view. See Heis (forthcoming, §3).
12. By 'basic', I mean those concepts whose definitions have postulates as corollaries. Definitions of complex concepts that are composed from basic concepts are genetic, but their corollaries are not postulates, but problems. The concept <triangle> is complex in this sense, since it is defined as 'a figure enclosed in three straight lines' (Wolff 1716, 1417) and is therefore composed from the basic concept <straight line>.
13. For similar definitions of *circle*, see Wolff (1741, Geometriae, §37, 1710, Introduction §4, 1710, Geometrie, Definition 5). That all radii in a circle are equal is Axiom 3 in Wolff (1710).
14. It is obvious that Leibniz, Wolff and Lambert each rejected Euclid's axiom because each tried to prove it. The argument that Kant also would have rejected Euclid's axiom would require a fuller defense, which I hope to present on another occasion.
15. For a more extended interpretation and discussion of these notes, see (Heis forthcoming).
16. 'Rectae, quae se invicem ubique habent eodem modo' (Leibniz 1858, 201, Definition XXXIV, §3). This definition actually is more abstract than the definition suggested in *New Essays*. Literally, it says that parallel lines are 'straight lines that everywhere have the same situations with respect to one another'. In §9, he notes that this is equivalent to saying that parallels are equidistant straight lines, but declines to use the definition in terms of distance because he does not yet have a definition of the minimal curve from a straight line to a straight line. But we can ignore that subtlety in this paper, as interesting as it may be.

    Leibniz almost surely got the idea of defining parallelism in terms of equidistance from Clavius (whose edition of Euclid Leibniz used) and Borelli, whose work he studied closely (see De Risi 2007, 80). On Clavius and Borelli, see Saccheri (1920, 87–91) and Heath's commentary in Euclid (1925, 194).
17. My reading of these notes differs from that proposed by Adickes, who does not see Kant criticizing Euclid's definition (Ak 14:31), and that proposed by Webb (2006,

230–232), who reads Kant as endorsing a 'proof' of Euclid's axiom using Leibniz's definition. For a fuller defence of my reading, see Heis (forthcoming).

18. On real definitions, see Leibniz (1996, III.iii.15, 19); 'Meditations on Knowledge, Truth, and Ideas' (Leibniz 1989, 25–26); 'On Universal Synthesis' (Leibniz 1970, 230–231); Letter to Tschirnhaus (Leibniz 1970, 194); *Discourse on Metaphysics* (Leibniz 1989, §24).

    In this paper, I feel free to cite passages from Leibnizian works that were unknown in the eighteenth century. Although this could be dangerous in contextual histories like this one, in this paper, it will do no damage. All of the doctrines I am ascribing to Leibniz are clearly expressed in *New Essays* and 'Meditations' – both works that Kant and his contemporaries knew very well.
19. *Discourse on Metaphysics* (Leibniz 1989, §24). See also Leibniz (1996, III.iii.18) and *Discourse* (Leibniz 1989, §26, 1970, 230).
20. On the following page, Leibniz provides an additional consideration, not found in Kant: that the same concept can have two real definitions, as – for example – an ellipse can be generated either by sectioning a cone or tracing a curve with a thread whose ends are fixed on the foci. He suggests, however, that there will still be one unique most perfect real definition.
21. See Leibniz (1996, IV.xii.6), supplement to a Letter to Huygens (Leibniz 1970, 250), 'On Contingency' (Leibniz 1989, 28) and 'On Freedom' (Leibniz 1989, 96).
22. I think this point helps explain another fundamental difference between Kant's and Leibniz's notion of real definition. Leibniz (1996, III.iii.18; 1989, 26) allows for *a posteriori* real definitions, since we can know, through experiencing an actual instance, that a concept is possible (*Discourse on Metaphysics* [Leibniz 1989, §24]). But Kant strongly denies that there can be real definitions of empirical concepts, partly because we can never be sure through experience that we have successfully identified marks that will pick out *all* instances of a concept (A727-8/B755-6).
23. A similar complaint against Wolff is levelled by Lambert in 1771 (§11) and in 3 February 1766, Letter to Kant, Ak 10:64.
24. My understanding of Lambert owes much to Laywine (2001, 2010) and Dunlop (2009).
25. On necessity of simple concepts, see Lambert (1764, §653–§654, 1915, §36). Lambert alleges that Wolff did not fully appreciate the role of simple concepts (1771, §11–§18, 1915, §26). For Lambert, Leibniz was more cognizant of the importance of simples than Wolff. Still, though, he lacked a sure criterion for distinguishing simples from compounds, and lacked principles that would license (or preclude) simple combinations of simples (Lambert, 1771, §7–§8). From Lambert's point of view, it is striking that Leibniz (1989, 26) will claim that 'the possibility of a thing is known *a priori* when we resolve it into its requisites, that is, into other notions known to be possible, and we know that there is nothing incompatible among them' (see also *Discourse* [Leibniz 1989, §24]). But how do we know when the resolution has reached 'simple, primitive notions understood in themselves' (Leibniz 1970, 231)? And how would we know that there is nothing incompatible among these simples? Certainly not by their definitions.
26. I do not think, though, that Kant would have agreed with Wolff that axioms are corollaries of definitions. Kant's position is therefore intermediate between Leibniz's and Lambert's. I cannot, however, defend this reading here. See Heis (forthcoming, §3).
27. See Lambert (1764, §650): 'the composition of individual marks is a means of attaining concepts and one can proceed arbitrarily insofar as the possibility of such a concept can be proven *later* (§65ff.). Now as long as the possibility has *not yet* been proved, the concept remains hypothetical' (my emphasis). Lambert says that these

hypothetical concepts (which presumably have only nominal definitions) can later become derived concepts [Lehrbegriffe] (which would then have real definitions).

28. That Euclid's definition can be proved without the axiom of parallels (Lambert 1895b, §8); that Euclid's definition is real (Lambert 1915, §81, 1895a, §7–§8, 1895b, §3, §7, §10) and that Euclid's definition is therefore preferable (Lambert 1915, §79, 1895b, §4–§10, 1771, §12, §23–§24).
29. These are Lambert's examples from (1764, part 1, §63).
30. 'The proof of a derivative concept *a priori* depends on its mode of genesis from axiomatic concepts. An a posteriori proof, however, depends on the mode of genesis of the thing' (Lambert 1915, §92.9–§10). A derivative concept is a concept whose possibility has been proved from the axioms and postulates. This proof is a priori in the sense that it is a logical proof – done purely syllogistically – from what is conceptually prior: axioms and postulates that exhibit the immediately certain (im) possibilities of combining simples.

    Lambert's claim that genetic definitions are appropriate only for a posteriori concepts gives an interesting set of contrasting positions. For Wolff, both empirical and mathematical concepts can have genetic definitions; for Kant, only mathematical concepts do; for Lambert, only empirical concepts do.
31. On <equilateral triangle>, see Lambert (1771, §20, 1915, §79, 1895a, #4); on <triangle>, see Lambert (1915, §79, 1895a, #6); on <parallel lines>, see Lambert (1895b, §3, 1895a, #7).
32. According to Webb (2006, 219), Kant's view 'fits Lambert like a glove'. A similar comment (including speculation about the role of Lambert's 3 February 1766 letter to Kant in shaping Kant's thinking) appears in Hintikka (1969, 43–44).
33. For a penetrating discussion of Lambert's argument, see Dunlop (2009, §5).
34. As an anonymous referee pointed out, it is noteworthy that in the passage quoted from Lambert (1771, §12), Lambert attributes the generality of the figure to the *postulates*, and does not mention the axioms. I have been claiming that for Kant, an instance of the concept <parallel lines> constructed according to Euclid I.31 would be general in Kant's sense, only if we are justified in believing Euclid's *axiom* of parallels. This difference is, I believe, revealing. For Lambert, postulates assert that two or more simple concepts <A> ... <N> can be combined (i.e. that there are instances of <A and ... and N>), while an axiom asserts that two or more simple concepts cannot be combined (i.e. there are no instances of <A and ... and N>). If all that matters for making a definition real is that there provably are instances of the definiendum (as I claim is the case for Lambert), then in general the postulates will be sufficient to show that a definition is real. Axioms, on the other hand, would in general be necessary to show that there are no instances of a concept other than those that meet some condition. For example, we would need axioms to show that though there are instances of the concept <non-intersecting coplanar straights>, the concept <cut by a transversal making interior angles less than two rights> cannot be combined with <non-intersecting coplanar straights>. Postulates would be in general sufficient to show that *all* straights constructed according to the procedure described in I.31 are parallels. Axioms would be necessary to show that *all and only* straights constructed in that way are parallel. Lambert's real definitions need the first condition. Kant's real definitions require the second.
35. See also Lambert (1895b, §11), where he compares giving a proof with solving an algebraic equation, and denies that the 'representation of the thing' can play any role in the proof.
36. That the possibility of only derivative concepts requires proof, whereas axiomatic concepts do not (Lambert 1915, §45, 57, 66, 92.9, 1764, §652). That the possibility

of concepts such as <straight line> does not need to be demonstrated (Lambert 1918, §21).

37. Thanks to Alison Laywine for urging me to make sense of I.1 on my interpretation.

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