

Evolution & Learning in Games

Econ 243B

Jean-Paul Carvalho

Lecture 13. Stochastic Stability

Analyzing Large-Dimensional Markov Processes

- ▶ We have seen that stationary distributions for reversible Markov processes can be computed.
- ▶ We are still left with the problem that the stationary distribution might spread weight widely over the state space.
- ▶ In perturbed dynamics with noise level ε , we can deal with this by taking $\varepsilon \rightarrow 0$.
- ▶ This leads us to the concept of **stochastic stability** which enables us to make sharp predictions about the long-run behavior of a Markov process, independently of initial conditions.

Stochastic Stability

- ▶ When the noise level ε is positive and the population size N is finite, individual behavior gives rise to an irreducible (and aperiodic) Markov process on a finite state space.
- ▶ We know that there exists a unique stationary distribution $\mu^{\varepsilon, N}$ of such a process.
- ▶ However, this stationary distribution puts positive weight on every state.
- ▶ As we take the limit as $\varepsilon \rightarrow 0$ (the small noise limit) or as $N \rightarrow \infty$ (the large population limit) or both, the stationary distribution concentrates weight on a small set of states (often a single state).

Stochastic Stability

- ▶ There are different ways of defining **stochastic stability** depending on which limits are taken and in what order.
- ▶ The results can differ depending on the definition used.
- ▶ We shall focus on the small noise limit:

Definition: *Stochastic Stability* (Foster and Young 1990). A state x is stochastically stable if $\lim_{\epsilon \rightarrow 0} \mu^{\epsilon, N}(x) > 0$.

- ▶ In other words, for ϵ small, the process spends virtually all the time (as $t \rightarrow \infty$) in the stochastically stable set of states.

Risk dominance

- ▶ Consider the following general symmetric normal-form game:

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

- ▶ Let x equal the weight on strategy 1. The mixed equilibrium is $x^* = \frac{d-c}{(a-b)+(d-c)}$.
- ▶ $(0, 0)$ *risk dominates* $(1, 1)$ if the product of the deviation losses is at least as high for $(0, 0)$, and *vice versa* (Harsanyi and Selten 1988).
- ▶ In a symmetric game this means: $(a - b) \geq (d - c)$, which implies that $x^* \leq \frac{1}{2}$.
- ▶ If the inequalities are strict then $(0, 0)$ strictly risk dominates state $(1, 1)$.

Two-Strategy Coordination Games

- ▶ Consider the following symmetric normal-form *pure* coordination game:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

- ▶ Note that:
 - ▶ Strategy 1 is a best response if and only if $x \geq x^* = \frac{2}{5}$, where x^* is the mixed Nash equilibrium.
 - ▶ Strategy 0 is a best response if and only if $x \leq \frac{2}{5}$.
 - ▶ As $x^* < \frac{1}{2}$, $(1, 1)$ is strictly risk dominant (here it is both the payoff dominant and risk dominant equilibrium).

Two-Strategy Coordination Games

Theorem 12.1. In two-strategy coordination games, for N sufficiently large, the $BRM(\varepsilon)$ revision protocol selects the state in which everyone plays the risk-dominant Nash equilibrium as the unique stochastically stable state.

Proof

- ▶ When individuals are randomly matched to play a symmetric coordination game, the only possible recurrence classes of the unperturbed $BRM(\varepsilon)$ dynamic are the two coordination equilibria (which are absorbing states).
- ▶ Let state 0 be the “All 0” coordination equilibrium and state 1 be the “All 1” coordination equilibrium.
- ▶ If $\frac{\mu_1^\varepsilon}{\mu_0^\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$, then state 0 is stochastically stable.
- ▶ If $\frac{\mu_1^\varepsilon}{\mu_0^\varepsilon} \rightarrow \infty$ as $\varepsilon \rightarrow 0$, then state 1 is stochastically stable.

Proof

- Recall from Theorem 11.5 that:

$$\frac{\mu_1}{\mu_0} = \prod_{j=1}^N \frac{N-j+1}{j} \cdot \frac{\rho_{01}(F(\frac{j-1}{N}), \frac{j-1}{N})}{\rho_{10}(F(\frac{j}{N}), \frac{j}{N})}. \quad (1)$$

- Note that for $x > x^*$, strategy 1 is a best response and is chosen with prob. $1 - \frac{\varepsilon}{2}$. For $x < x^*$, strategy 0 is a best response.
- Substituting into (1), we get the expression for the $BRM(\varepsilon)$ dynamic:

$$\frac{\mu_1}{\mu_0} = \prod_{j=\lceil Nx^* \rceil}^N \frac{N-j+1}{j} \cdot \frac{1 - \frac{\varepsilon}{2}}{\frac{\varepsilon}{2}} \prod_{j=1}^{\lfloor Nx^* \rfloor} \frac{N-j+1}{j} \cdot \frac{\frac{\varepsilon}{2}}{1 - \frac{\varepsilon}{2}}. \quad (2)$$

Proof

- Rearranging:

$$\frac{\mu_1}{\mu_0} = \left(\frac{1 - \frac{\varepsilon}{2}}{\frac{\varepsilon}{2}} \right)^{N - \lceil Nx^* \rceil + 1} \left(\frac{\frac{\varepsilon}{2}}{1 - \frac{\varepsilon}{2}} \right)^{\lfloor Nx^* \rfloor} \prod_{j=1}^N \frac{N - j + 1}{j}. \quad (3)$$

Recognizing that $\lceil Nx^* \rceil = \lfloor Nx^* \rfloor + 1$, so that $N - \lceil Nx^* \rceil + 1 = N - \lfloor Nx^* \rfloor$, we can write:

$$\frac{\mu_1}{\mu_0} = \left(\frac{1 - \frac{\varepsilon}{2}}{\frac{\varepsilon}{2}} \right)^{N - 2\lfloor Nx^* \rfloor} \prod_{j=1}^N \frac{N - j + 1}{j}, \quad (4)$$

which is proportional to:

$$\left(\frac{1 - \frac{\varepsilon}{2}}{\frac{\varepsilon}{2}} \right)^{N - 2\lfloor Nx^* \rfloor}. \quad (5)$$

Proof

- ▶ As $\varepsilon \rightarrow 0$:

$$\left(\frac{1 - \frac{\varepsilon}{2}}{\frac{\varepsilon}{2}} \right) \rightarrow \infty. \quad (6)$$

- ▶ Therefore $\frac{\mu_1}{\mu_0} \rightarrow \infty$ if $N > 2 \lfloor Nx^* \rfloor$ and $\frac{\mu_1}{\mu_0} \rightarrow 0$ if $N < 2 \lfloor Nx^* \rfloor$.

- ▶ For N sufficiently large, this is equivalent to:

- ▶ $\frac{\mu_1}{\mu_0} \rightarrow \infty$ if $x^* < \frac{1}{2}$ —“All 1” is the unique stochastically stable state,
- ▶ $\frac{\mu_1}{\mu_0} \rightarrow 0$ if $x^* > \frac{1}{2}$ —“All 0” is the unique stochastically stable state.

Pure Coordination

Corollary 12.1. Under the $BRM(\varepsilon)$ revision protocol, $\lim_{\varepsilon \rightarrow 0} \mu_1^\varepsilon = 1$ in the pure coordination game, i.e. “All 1” is the unique stochastically stable state.

Impure Coordination

- ▶ Now consider the following symmetric normal-form coordination game called **Stag Hunt**:

$$A = \begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix}.$$

- ▶ Now $x^* = \frac{2}{3} > \frac{1}{2}$ is the mixed Nash equilibrium.
Therefore, $(0, 0)$ is risk-dominant, but Pareto inefficient.

Corollary 12.2. Under the $BRM(\varepsilon)$ revision protocol, $\lim_{\varepsilon \rightarrow 0} \mu_0^\varepsilon = 1$ in the stag hunt game, i.e. “All 0” (all hare) is the unique stochastically stable state.

Stochastic Stability

- ▶ Thus far, we have learned how to identify stochastically stable states for *reversible Markov processes*, for which a stationary distribution can be computed.
- ▶ Here we shall broaden the scope of our stochastic stability analysis and in doing so we shall require new concepts and techniques.

Regular Perturbed Markov Process

Definition. For each $\varepsilon \in [0, \varepsilon^*]$, let $\{X_t^\varepsilon\}$ be a Markov process. $\{X_t^\varepsilon\}$ is a *regular perturbed Markov process* if it is (a) irreducible for all $\varepsilon \in (0, \varepsilon^*]$ and (b) if for every $x, x' \in X$, $P_{xx'}^\varepsilon$ approaches $P_{xx'}^0$ at an exponential rate, i.e.

$$\lim_{\varepsilon \rightarrow 0} P_{xx'}^\varepsilon = P_{xx'}^0,$$

and

if $P_{xx'}^\varepsilon > 0$ for some $\varepsilon > 0$, then $0 < \lim_{\varepsilon \rightarrow 0} P_{xx'}^\varepsilon / \varepsilon^{r(x, x')} < \infty$

for some $r(x, x') \geq 0$.

Resistances

- ▶ The real number $r(x, x')$ is called the **resistance** of the transition $x \rightarrow x'$.
- ▶ There cannot be two distinct exponents that satisfy the last condition, therefore $r(x, x')$ is uniquely identified.
- ▶ $P_{xx'}^0 > 0$ if and only if $r(x, x') = 0$, i.e. transitions that occur under the unperturbed process have zero resistance.
- ▶ It is straightforward to show that the BRM(ε) and logit choice protocols generate regular Perturbed Markov processes.

Computing Stochastically Stable States

- ▶ Let E_1, E_2, \dots, E_K be recurrence classes of the unperturbed process ($\varepsilon = 0$).
- ▶ For each pair of distinct recurrence classes E_j and E_k a jk -path is a sequence of states $\zeta = (x_1, x_2, \dots, x_q)$ that begins in E_j and ends in E_k .
- ▶ The resistance is the sum of the resistances of its edges, i.e. $r(\zeta) = r(x_1, x_2) + r(x_2, x_3) + \dots + r(x_{q-1}, x_q)$.
- ▶ Let $r_{jk} = \min r(\zeta)$ be the least resistance over all jk paths.
- ▶ $r_{jk} > 0$ for all $j \neq k$ by the definition of a recurrence class.

Computing Stochastically Stable States

- ▶ Construct a complete directed graph of K vertices, one for each recurrence class.
- ▶ The weight on the directed edge $j \rightarrow k$ is r_{jk} .
- ▶ A tree rooted at vertex j (a j -tree) is a set of $K - 1$ directed edges such that *from every vertex different to j there is a unique directed path in the tree to j .*
- ▶ The resistance of a **rooted tree** is the sum of the resistance of its edges.
- ▶ The **stochastic potential** γ_j of E_j is the minimum resistance over all trees rooted at j .

Stochastic Potential & Stochastic Stability

Theorem 12.2. Let $\{X_t^\varepsilon\}$ be a regular perturbed Markov process and let μ^ε be the unique stationary distribution of $\{X_t^\varepsilon\}$ for each $\varepsilon > 0$.

Then $\lim_{\varepsilon \rightarrow 0} \mu^\varepsilon = \mu^0$ exists, and μ^0 is a stationary distribution of the unperturbed process $\{X_t^0\}$.

The stochastically stable states are precisely those states that are contained in the recurrence classes of $\{X_t^0\}$ having minimum stochastic potential.

Coordination Games Again

- ▶ Consider a 2×2 coordination game. There are two recurrence classes E_0 ('all 0') and E_1 ('all 1')
- ▶ With a certain sequence of nonbest responses, the perturbed process can transit between recurrence classes.
- ▶ Let $r(E_0, E_1)$ be the *resistance* of the transition $E_0 \rightarrow E_1$.
- ▶ Under the $BRM(\varepsilon)$ protocol, this is the minimum number of errors required for the process to transit from state E_0 to state E_1 .

Coordination Games Again

- ▶ The corresponding trees are:

- ▶ Therefore, for N sufficiently large, we see that E_1 is stochastically stable if and only if it is risk dominant.

Absorbing States and Limit Cycles

- ▶ What if the recurrence classes of the process include not only absorbing states but limit cycles?
- ▶ Consider the following (two-population) game:
 - ▶ Under the best response dynamic, the game produces a single absorbing state (D, D) and a single best-response cycle $CA \rightarrow CB \rightarrow AB \rightarrow AC \rightarrow BC \rightarrow BA \rightarrow CA$.

Absorbing States and Limit Cycles

- ▶ The resistances of the transitions between (D, D) and the cycle are calculated as the path of least resistance from (D, D) to some state in the cycle and *vice versa*.

- ▶ For N large enough $\lceil \frac{1}{6}N \rceil < \lceil \frac{11}{14}N \rceil$ and hence the limit cycle is stochastically stable.

- ▶ Intuitively, as $\varepsilon \rightarrow 0$ it is far easier to enter the cycle from the sink (D, D) than it is to exit from the cycle to the sink.

A Sufficient Condition for Stochastic Stability

- ▶ When there are many recurrence classes of the unperturbed Markov process $\{X_t^0\}$, the spanning tree method of finding a stochastically stable state can *sometimes* be difficult to implement.
- ▶ Consider the following alternative method introduced by (Ellison 1993) that provides a sufficient condition for stochastic stability (the spanning tree method provides both a necessary and sufficient condition).
- ▶ Let the unperturbed process have distinct recurrence classes E_1, E_2, \dots, E_K .
- ▶ Once it hits one of these recurrence classes, the unperturbed process locks in to that class.

The Radius-Coradius Theorem

- ▶ However, with a certain sequence of nonbest responses, the perturbed process can transit between recurrence classes.
- ▶ Once again, let $r(E, E_\ell)$ be the *resistance* of the transition $E \rightarrow E_\ell$.
- ▶ Define the **radius** of recurrence class E by:

$$R(E) = \min_{\ell} r(E, E_{\ell})$$

and the coradius of E by:

$$CR(E) = \max_{\ell} r(E_{\ell}, E)$$

The Radius-Coradius Theorem

Theorem 12.3. If $R(E) > CR(E)$, then E is the unique stochastically stable class of $\{X_t^\varepsilon\}$.

The intuition for this is as follows:

- ▶ A lower bound on the probability of exiting recurrence class E is on the order of $\varepsilon^{R(E)}$.
- ▶ An upper bound on the probability of entering recurrence class E is on the order of $\varepsilon^{CR(E)}$.
- ▶ As $\varepsilon \rightarrow 0$, $\varepsilon^{R(E)-CR(E)} \rightarrow 0$ iff $R(E) > CR(E)$.
- ▶ In this case, the proportion of time spent in E is far, far greater than the proportion of time spent outside of E as the noise level becomes vanishingly small.