Evolution & Learning in Games
Econ 243B

Jean-Paul Carvalho

Lecture 14.
Applications of Stochastic Stability

Applications

In this lecture, we shall apply the stochastic stability framework to:

- Schelling's (1971) residential segregation model,
- An evolutionary model of bargaining.

Schelling Residential Segregation

- Consider N agents located at one of N spots on a circle.
- There are two types of agents—As and Bs.
- Each pair of agents is selected at random to consider trading places.
- An agent is *discontent* if both of his two immediate neighbors are of a different type to himself; otherwise he is *content*.

Preferences and Equilibrium

- An *equilibrium configuration* is one in which there is no pair such that one (or both) agents are currently discontent and would both be content after they trade places.
  - Think of an agent who strictly gains from a trade compensating an agent who does not to cover his relocation cost.
  - Such Pareto improving trades are *advantageous*; all other trades are *disadvantageous*.
- There are many possible equilibrium configurations: any configuration in which everyone lives next to at least one person of their own type—no one is "isolated"—is an equilibrium configuration.
- Which ones are stochastically stable?

Revision Protocol

- We assume that an advantageous trade occurs with high probability, approaching one as $\varepsilon \to 0$.
- Disadvantageous trades occur with low probability, which decreases exponentially with the partners' net loss in utility. Specifically, suppose there exist real numbers $0 < a < b < c$ such that the probability of a disadvantageous trade is:
  - ε^a if the sum of the gains to trade, net of relocation costs, is zero;
  - ε^b if both parties were content before and one party becomes content after;
  - ε^c if both were content before and both are discontent after.

Absorbing States

- The resulting stochastic process is a regular perturbed Markov process.
- The absorbing states of the unperturbed process are the ones in which no player is isolated.
- Let us begin by showing that these are the only recurrence classes of the process.
- Consider a nonabsorbing state. There must be at least one agent who is discontent.

Global Convergence

- Consider a discontent agent i and wlog suppose that i is a type A.
- Go clockwise around the circle until one comes to the next type A agent, whom we shall call i'.
- The individual just before i' must be a type B agent, whom we shall call j.
- If i and j trade places then they both will be content.

Global Convergence

- There is a positive probability that at any point in time i and j are selected and trade places, leaving fewer discontent agents.
- Iterating this process, we find that from any nonabsorbing state there is a positive probability of transiting to an absorbing state in a finite number of periods.
- Therefore, the absorbing states are the only recurrent states; in particular there are no *limit cycles*.

Stochastic Stability

- Denote the set of absorbing states by $X^* = X^s \cup X^{ns}$, where X^s is the set of *segregated* absorbing states.

- We claim that:

  (i) for every $x \in X^{ns}$, every x-tree has at least one edge with resistance b or c (which are greater than a).

  (ii) for every $x \in X^s$, there exists an x-tree in which every edge has resistance a.

Schelling Residential Segregation

- In this case the stochastic potential of each segregated absorbing state is $a(|X^*| - 1)$ (because each tree rooted at such a state has $|X^*| - 1$ edges).
- The stochastic potential of each nonsegregated absorbing state is at least $a(|X^*| - 2) + b$, which is strictly larger because $b > a$.
- Therefore, by Theorem 12.2, *a state is stochastically stable if and only if it is a segregated absorbing state.*

Stochastic Stability

- Let us establish claim (i).
- Any tree rooted at a state $x \in X^{ns}$ contains at least one edge that is directed from a segregated absorbing state to a nonsegregated absorbing state.
- Any such edge has resistance of at least b, because any trade that breaks up a segregated state must create at least one discontented (isolated) agent.

Stochastic Stability

- Now let us establish claim (ii).
- Consider a state $x \in X^s$. From each absorbing state $x' \neq x$, we shall construct a sequence of absorbing states $x' = x^1, x^2, ..., x^k = x$ such that $r(x^{j-1}, x^j) = a$ for $1 < j \leq k$.
- The construction is done so that the union of the directed edges on all these paths forms an x-tree.

Stochastic Stability

- **Case 1:** x' is also segregated, i.e. consists of two contiguous groups.
- Working clockwise, let the first member of the A-group trade places with the first member of the B-group.
- Both are content before and after, so the trade occurs with probability proportional to ε^a.
- The trade shifts the A-group and B-group by one position clockwise around the circle.
- Hence in N steps, we can reach any segregated absorbing state and in particular x (through a sequence of absorbing states) with each transition having resistance a.

Stochastic Stability

- **Case 2:** x' is not segregated.
- Since x' is absorbing, this means that that there is at least one contiguous group of As consisting of at least two players and at least one contiguous group of Bs consisting of at least two players.
- Pick such a group of As, denoted by $\mathbf{A}$. Working clockwise, denote the first such group of Bs by $\mathbf{B}$ and the next group of As by $\mathbf{A}'$.
- Let the first player in $\mathbf{A}$ trade places with the first player in $\mathbf{B}$.
- Both are content before and after, so the trade occurs with probability proportional to ε^a.

Stochastic Stability

- This reduces by one the number of players between $\mathbf{A}$ and $\mathbf{A}'$.
- This either results in a new absorbing state, or else one (discontent) B type remains between $\mathbf{A}$ and $\mathbf{A}'$.
- This B player can trade with the first player in $\mathbf{A}$, a move that has zero resistance.
- Repeating this process we can join groups $\mathbf{A}$ and $\mathbf{A}'$.

Stochastic Stability

- Repeating for other contiguous groups of Bs that separate distinct groups of A types, we can reach a segregated absorbing state, with each transition between absorbing states having resistance at most a.

- From there, the argument in case 1 implies that we can transit to x through a sequence of absorbing states, with each transition between absorbing states having resistance at most a.

- There are no cycles, because the number of distinct groups never increases.

- The union of all these paths forms an x tree whose edges each have the minimum resistance $a < b$.

- This completes the proof.

Bargaining

- Finding a solution to the bargaining problem (i.e. a prediction of play in a bargaining game) was once thought to be an intractable problem.

- Consider the Nash demand game:
  - Two players simultaneously shout out demands s_1 and s_2, respectively.
  - If the demands can be met from the pie, then each gets exactly what he demanded. Otherwise, both players get nothing.

- There is a continuum of Nash equilibria of this game:
  - Suppose the pie is equal to one.
  - Then any pair of demands (s_1, s_2) such that $s_1 + s_2 = 1$ is a Nash equilibrium (*note*: there are others).

Nash Bargaining (Nash 1950)

The Environment

- Consider a pie of size one.
- Two players $i = 1, 2$.
- Player i's share is $s_i \in [0, 1]$.
- Player 1's utility function is $u : [0, 1] \rightarrow \mathbb{R}$ and player 2's is $v : [0, 1] \rightarrow \mathbb{R}$.

Nash Bargaining

The Bargaining Set

- $U = \{(u(s_1), v(s_2)) : s_1 + s_2 \leq 1 \text{ and } s_1, s_2 \geq 0\}$.
- Disagreement point $d = (d_1, d_2)$.
- A bargaining problem is a pair (U, d).

Nash Bargaining

The Nash Bargaining Solution—an axiomatic approach

- There is a unique solution to the bargaining problem (U, d) that satisfies Nash's four axioms:

$$\arg\max_{(s_1, s_2)} \left(u(s_1) - d_1\right)\left(v(s_2) - d_2\right)$$
$$\text{s.t. } u \in U \text{ and } u_1 \geq d_1, u_2 \geq d_2.$$

- In the symmetric case ($u = v$ and $d_1 = d_2$), the NBS is $s_1^* = s_2^* = \frac{1}{2}$.

Nash Bargaining

The Nash Bargaining Solution—an axiomatic approach

- Let $d_1 = d_2 = 0$. Then the NBS is:

$$\arg\max_{s \in [0,1]} u(s)v(1-s).$$

Nash Bargaining

Bargaining Power

- If in addition we drop the symmetry axiom, solutions to the bargaining problem that satisfy the remaining three axioms are of the form:

$$\arg\max_{s \in [0,1]} u(s)^a v(1-s)^b.$$

- a and b can be interpreted as levels of *bargaining power*.
- The first-order condition which yields the **asymmetric NBS** is:

$$a\frac{u'(s)}{u(s)} = b\frac{v'(1-s)}{v(1-s)}.$$

Strategic Bargaining (Rubinstein 1982)

Alternating Offers—a noncooperative approach

- Suppose that players 1 and 2 have discount factors a and b respectively.
- Player 1 begins by proposing a split. Player 2 can accept or reject. If she rejects, she makes a counteroffer ... This goes on until an offer is accepted.
- There is a unique subgame perfect equilibrium of this game.
- As the time between rounds $\to 0$, equilibrium shares converge to the asymmetric NBS above.
- Here one's bargaining power is determined by one's patience (i.e. discount factor).

Evolutionary Bargaining (Young 1993, JET)

- Consider two disjoint populations, rows and columns (e.g. buyers and sellers; workers and bosses) of equal size N.
- Players are continuously paired to play a Nash demand game.
- If the two demands (s_1, s_2) sum to one or less, each player receives his demand, and the associated utility $u(s_1)$ or $v(s_2)$. Both players get zero otherwise.
- The utility functions u and v are strictly increasing, concave and continuously differentiable (C^1 not required in the paper).

Evolutionary Bargaining

- Each player has an independent Poisson alarm clock whose rings signal opportunities to revise her strategy.
- To keep the strategy set finite, consider a discretized set of demands $\Delta \equiv \{\delta, 2\delta, 3\delta, ..., 1\}$.
- Every division $(s, 1-s)$, such that $0 < s < 1$, constitutes a *strict Nash equilibrium* of the demand game. We shall call such a division a bargaining **norm**.

Evolutionary Bargaining

Adaptive Play Protocol

- The history of play is a vector $h^t = \left((s_1^{t-m}, s_2^{t-m}), \ldots (s_1^{t-1}, s_2^{t-1})\right)$, where s_1^{t-1} is the most recent demand made by a row player and m is the memory length.
- A **convention** is a history of the form $h^* = \left((s, 1-s), \ldots (s, 1-s)\right)$, i.e. m instances of a norm.
- A revising row player at time t draws a random sample of size am (an integer) from the m previous plays by members of the column population, i.e. from h^t.
- A revising column player at time t draws a random sample of size bm from the m previous plays by members of the row population.

Evolutionary Bargaining

Adaptive Play Protocol

- The revising player computes the frequency $p(s)$ of each demand s in her sample.
- With high probability $1 - \varepsilon$, a revising player best responds to her sample.
- With low probability ε, she chooses a demand s within δ of a best response to her sample (*note*: 'local errors' not required in paper).

Evolutionary Bargaining

Lemma 13.1 The unperturbed process converges almost surely to a convention from any initial state.

Proof.

- We claim that there exists a probability at least $p > 0$ of transiting from any state to a convention in at most T periods.
- The probability of not transiting to a convention in λT periods is then at most $(1-p)^{\lambda}$ which goes to 0 as $\lambda \to \infty$.
- This would establish the lemma, because each convention is an absorbing state of the unperturbed process.
  - To see this, consider m instances of the norm $(s, 1-s)$.
  - All possible samples for a row player (for example) consist of am plays of $1-s$. The unique BR is s. Thus the convention is perpetuated.

Evolutionary Bargaining

Proof.

- Now to establish the claim, note that there is a positive probability that the next m revisions are by row players and that each revising row player draws the same sample, playing the same best response s.

- There is also a positive probability that the subsequent m revisions are by column players. They must each draw a sample consisting solely of demands equal to s and choose the best response $1 - s$.

- This establishes the claim. □

Evolutionary Bargaining

Which bargaining norms are stochastically stable?

- To answer this question, we need to analyze transitions between conventions under the perturbed dynamic.
- Suppose that the process is in convention $(s, 1-s)$.
- All transitions are 'local'. What is the probability of a 'downward' transition to convention $(s-\delta, 1-s+\delta)$?

Evolutionary Bargaining

- Suppose there are i consecutive plays of $1 - s + \delta$ by column players, so that the next revising row player can draw a sample with i instances of $1 - s + \delta$.
- By reducing her demand to $s - \delta$, the row player gets $u(s - \delta)$ for certain.
- By retaining her demand s, the row player estimates that she gets $u(s)$ with prob. $1 - \frac{i}{am}$, and zero otherwise.

Evolutionary Bargaining

- Hence row players retain their demand if:

$$(1 - \tfrac{i}{am})u(s) \geq u(s - \delta).$$

- The critical value of i is:

$$i^*(s) = am\frac{u(s) - u(s - \delta)}{u(s)}.$$

Evolutionary Bargaining

- Similarly, a column player retains her demand of $1 - s$ (rather than increasing it to $1 - s + \delta$) when drawing a sample of j instances of $s - \delta$ if:

$$v(1-s) \geq \tfrac{j}{bm} v(1-s+\delta).$$

- The critical value of j is:

$$j^*(s) = bm \frac{v(1-s)}{v(1-s+\delta)}.$$

Evolutionary Bargaining

- Therefore it takes a minimum of $\lceil i^*(s) \rceil \wedge \lceil j^*(s) \rceil$ to induce a downward transition.

- When δ is sufficiently small, the first term is the smaller of the two.

- Therefore the resistance of a downward transition is:

$$r(s, s-\delta) = \left\lceil am \frac{u(s) - u(s-\delta)}{u(s)} \right\rceil.$$

- Similarly, the resistance of an upward transition is:

$$r(s, s+\delta) = \left\lceil bm \frac{v(1-s) - v(1-s-\delta)}{v(1-s)} \right\rceil.$$

Evolutionary Bargaining

- For δ sufficiently small, the resistances are well approximated by:

$$r(s, s-\delta) \approx \left\lceil am \frac{\delta u'(s)}{u(s)} \right\rceil.$$

$$r(s, s+\delta) \approx \left\lceil bm \frac{\delta v'(1-s)}{v(1-s)} \right\rceil.$$

Evolutionary Bargaining

- Define the function:

$$f_\delta(s) = \min\{r(s, s+\delta), r(s, s-\delta)\}.$$

- $r(s, s+\delta)$ is an increasing function of s, whereas $r(s, s-\delta)$ is a decreasing function of s.

- Therefore, $f_\delta(s)$ is unimodal (see figure).

- Let s_δ be a maximizer of $f_\delta(s)$.

Evolutionary Bargaining

- The tree rooted at s_δ is the least resistant rooted tree.
- Hence the stochastically stable state(s) correspond to the convention(s) that maximize(s) $f_\delta(s)$.
- When δ is small and m is large relative to δ (so that there are no integer issues), any maximum of $f_\delta(s)$ lies close to the point s^* at which the two curves intersect:

$$a\frac{u'(s^*)}{u(s^*)} = b\frac{v'(1-s^*)}{v(1-s^*)}.$$

Evolutionary Bargaining

- Recall that this is simply the first-order condition that defines the asymmetric Nash bargaining solution:

$$a\frac{u'(s)}{u(s)} = b\frac{v'(1-s)}{v(1-s)}.$$

- Here one's bargaining power is determined by one's sample size.

Theorem 13.1 Consider random matching from two populations to play the discrete Nash demand game using the adaptive play protocol with memory m and sample sizes am and bm. As δ becomes small, the stochastically stable division(s) converge to the asymmetric Nash bargaining solution.