Evolution & Learning in Games
Econ 243B

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Lecture 2.
Foundations of
Evolution & Learning in Games II

Outline

In this lecture, we shall:

- Take a first look at local stability. In particular, we shall define an *evolutionary stable state* and explore its relationship to Nash equilibrium.
- Apply what we have learned so far to analyze iterated play of the prisoners' dilemma.

Evolutionary Stable States (ESS)

Maynard Smith and Price (1973) defined the notion of an *evolutionary stable strategy*:

- Their focus was on monomorphic populations: every member plays the same strategy, which can be a mixed strategy.
- We are concerned with a polymorphic population of agents each programmed with a pure strategy.

Evolutionary Stable States (ESS)

Mathematically these problems are identical. For example:

$$\begin{aligned} F_1(x) &= x_1 u(1,1) + x_2 u(1,2) \\ &= u(1,x). \end{aligned}$$

Being randomly matched with a population of agents proportion x_1 of which are programmed with pure strategy 1 and x_2 of which are programmed with pure strategy 2 is the same as playing a single individual who plays strategy 1 with probability x_1 and strategy 2 with probability x_2.

Evolutionary Stable States (ESS)

Hence we can adapt the concept of an evolutionary stable strategy to a population setting:

- The term we shall use is **evolutionary stable state** (ESS).

As we shall eventually see, ESS provides a sufficient condition for local stability under a wide range of evolutionary dynamics, including the replicator dynamic.

Invasion

Let the state be $x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$.

Consider a game F, where $F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ \dots \\ F_n(x) \end{pmatrix}$.

Consider an invasion of mutants who make up a fraction ε of the post-entry population.

The shares of each strategy in the mutant population are represented by $y = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix}$.

Invasion

Therefore, the post-entry population state is:

$$x_\varepsilon = (1-\varepsilon)x + \varepsilon y = \begin{pmatrix} (1-\varepsilon)x_1 + \varepsilon y_1 \\ (1-\varepsilon)x_2 + \varepsilon y_2 \\ \dots \\ (1-\varepsilon)x_n + \varepsilon y_n \end{pmatrix}.$$

The average payoff in the incumbent population in the post-entry state is $x'F\big((1-\varepsilon)x + \varepsilon y\big)$.

The average payoff in the mutant population in the post-entry state is $y'F\big((1-\varepsilon)x + \varepsilon y\big)$.

Uniform Invasion Barrier

The average payoff in the incumbent population is higher if:

$$(y - x)' F\big((1 - \varepsilon)x + \varepsilon y\big) < 0. \qquad (1)$$

State x is said to admit a **uniform invasion barrier** if there exists an $\bar{\varepsilon} > 0$ such that (1) holds for all $y \in X - \{x\}$ and $\varepsilon \in (0, \bar{\varepsilon})$.

That is, for all possible mutations y, as long as the mutant population is less than fraction $\bar{\varepsilon}$ of the postentry population, the incumbent population receives a higher average payoff.

Invasion Barriers

Define the invasion barrier of x against y as:

$$b_x(y) = \inf\big(\{\varepsilon \in (0,1) : (y-x)'F\big((1-\varepsilon)x + \varepsilon y\big) \geq 0\} \cup \{1\}\big). \tag{2}$$

If x admits a **uniform invasion barrier**, then there exists an $\bar{\varepsilon} > 0$ such that $b_x(y) \geq \bar{\varepsilon} > 0$ for all $y \in X - \{x\}$.

ESS

DEFINITION. State $x \in X$ is an **evolutionary stable state** (ESS) of F if there exists a neighborhood O of x such that:

$$(y - x)'F(y) < 0 \quad \text{for all } y \in O - \{x\}. \tag{3}$$

In other words, if x is an ESS, then for any state y sufficiently close to x, a population playing x will receive a larger average payoff in state y than a population playing y (i.e. x is a better reply to y than y is to itself).

Note that this considers invasions of other states y by x rather than invasions of x by other states. Hence it is not clear, at present, why this should be a stability condition.

ESS and Invasion Barriers

Theorem 2.1. State $x \in X$ is an **evolutionary stable state** (ESS) if and only if it admits a uniform invasion barrier.

Thus if x is stable in the face of an arbitrarily large population of entrants who mutate to a nearby state, then it is stable in the face of a sufficiently small population of entrants who mutate to an arbitrary state.

ESS and NE

What is the relationship between ESS and NE?

DEFINITION. Suppose that $x \in X$ is a NE. Then $(y - x)'F(x) \leq 0$ for all $y \in X$.

In addition, suppose there exists a neighborhood of x that does not contain any other NE.

Then x is an **isolated NE.**

Proposition 2.1. Every ESS is an isolated NE.

Proof

Let x be an ESS of F, O be the nhd posited in (3) and $y \in X - \{x\}$ (not necessarily in O).

Then for all $\varepsilon > 0$ sufficiently small, the postentry state $x_\varepsilon = \varepsilon y + (1-\varepsilon)x$ is in O.

This implies that:

$$\begin{aligned} (x_\varepsilon - x)'F(x_\varepsilon) &< 0 \\ (\varepsilon y + (1-\varepsilon)x - x)'F(x_\varepsilon) &< 0 \\ \varepsilon(y - x)'F(x_\varepsilon) &< 0 \\ (y - x)'F(x_\varepsilon) &< 0. \end{aligned} \tag{4}$$

Proof

Taking $\varepsilon \to 0$ yields:

$$(y - x)'F(x) \leq 0.$$

That is, x is a NE.

To establish that x is isolated, note that if $w \in O - \{x\}$ were a NE then $(w - x)'F(w) \geq 0$, contradicting the supposition that x is an ESS [by (3)]. $\square$

The converse of Proposition 2.1 is not true.

- The mixed equilibrium of a two-strategy coordination game is a counterexample.

More on ESS and Nash

Therefore, *ESS is stronger than NE*.

In particular, an ESS satisfies the additional property:

Suppose there exists a state y which is an alternative best reply to x, i.e. $(y - x)'F(x) = 0$.
—Then $(y - x)'F(y) < 0$, i.e. x is a better reply to y than y is to itself.

Therefore:

- A strict NE is an ESS.
- A polymorphic population state (equivalent to a mixed NE) cannot be strict and hence must satisfy the additional property.

More on ESS and Nash

In the case in which agents are matched uniformly at random to play a normal form game (the case we have been focussing on), then it is easy to see why the additional property is required.

Suppose $(y - x)'F(x) = 0$, i.e. y is an alternative best reply to x.

Then:

$$\begin{aligned}
(y-x)'F(\varepsilon y + (1-\varepsilon)x) &= \varepsilon(y-x)'F(y) + (1-\varepsilon)\underbrace{(y-x)'F(x)}_{=0} \\
&= \varepsilon(y-x)'F(y).
\end{aligned} \tag{5}$$

Therefore, $(y - x)'F(y)$ must be negative for (1) to hold and hence, by Theorem 2.1, for x to be an ESS.

Example: Hawk Dove

	Hawk	*Dove*
Hawk	-2, -2	4, 0
Dove	0, 4	0, 0

ESS: $x = (\frac{2}{3}, \frac{1}{3})$. **ESS payoff** = 0.

Example: Hawk Dove

- Consider a mutation y such that $y_1 > x_1 = \frac{2}{3}$.
- Check that $(y-x)'F\big((1-\varepsilon)x+\varepsilon y\big) < 0$ for all such y:

$$(y_1 - x_1)[-2((1-\varepsilon)x_1 + \varepsilon y_1) + 4((1-\varepsilon)(1-x_1) + \varepsilon(1-y_1)].$$

- This equals:

$$(y_1 - x_1)\varepsilon[-2y_1 + 4(1-y_1)]$$

because $-2 \times \frac{2}{3} + 4 \times (1 - \frac{2}{3}) = 0$. This in turn equals:

$$(y_1 - x_1)\varepsilon[4 - 6y_1]$$

which is negative because $y_1 > \frac{2}{3}$ by hypothesis.

- A similar argument can be applied to the case $y_1 < x_1$. Hence x is an ESS.

The Prisoners' Dilemma

	C	D
C	3 3	5 0
D	0 5	1 1

NE/ESS: $x = (0, 1)$.

Therefore, an ESS is not necessarily efficient.

Not Every Game has an ESS

	A	B	C
A	1, 1	2, 0	0, 2
B	0, 2	1, 1	2, 0
C	2, 0	0, 2	1, 1

$x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the unique NE and therefore the only possible ESS.

Not Every Game has an ESS

Note that x is a polymorphic population state (equivalent to a mixed strategy), so any basis vector (pure strategy) is an alternative best reply to x.

Check that the additional property holds: $(e_1 - x)' F(e_1) < 0$, where $e_1 = (1, 0, 0)$, i.e. the pure-strategy A.

This is not the case: $x' F(e_1) = e_1' F(e_1) = 1$.

The Iterated Prisoners' Dilemma

- Two players engage in a series of PD games.
- The engagement ends after the current round with probability $\delta < \frac{1}{2}$. We call this the *stopping probability*.
- The expected number of rounds per engagement is:

$$1 + (1 - \delta) + (1 - \delta)^2 + (1 - \delta)^3 + \ldots = \frac{1}{1 - (1 - \delta)} = \frac{1}{\delta}.$$

- Consider a population in which three strategies are present:
  - C—always cooperate,
  - D—always defect,
  - T—tit-for-tat, i.e. start by cooperating, thenceforth cooperate in period t if partner cooperated in $t - 1$.

Expected Payoffs Within Each Pairing

	C	D	T
C	$\frac{3}{\delta}$	0	$\frac{3}{\delta}$
D	$\frac{5}{\delta}$	$\frac{1}{\delta}$	$4+\frac{1}{\delta}$
T	$\frac{3}{\delta}$	$\frac{1}{\delta}-1$	$\frac{3}{\delta}$

Note:

Payoff from playing T against D is $0+(1-\delta)\frac{1}{\delta}=\frac{1}{\delta}-1$.

Payoff from playing D against T is $5+(1-\delta)\frac{1}{\delta}=4+\frac{1}{\delta}$.

Expected Payoffs Over All Pairings

$$
\begin{aligned}
F_C(x) &= (x_C + x_T)\tfrac{3}{\delta} \\
F_D(x) &= x_C\tfrac{5}{\delta} + x_D\tfrac{1}{\delta} + x_T\left(4 + \tfrac{1}{\delta}\right) \\
F_T(x) &= (x_C + x_T)\tfrac{3}{\delta} + x_D(\tfrac{1}{\delta} - 1)
\end{aligned}
$$

Replicator Dynamics

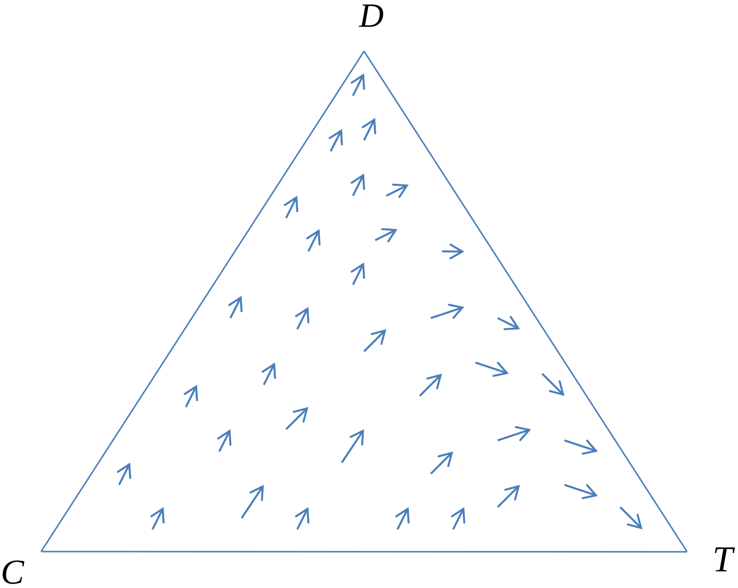
$$\begin{aligned}\frac{d}{dt}\left[\frac{x_T}{x_C}\right] &= \frac{x_T}{x_C}\big(F_T(x) - F_C(x)\big) \\ &= \frac{x_T}{x_C}\big[x_D(\tfrac{1}{\delta} - 1)\big],\end{aligned}$$

which is positive because $\delta < \frac{1}{2}$.

$$\begin{aligned}\frac{d}{dt}\left[\frac{x_T}{x_D}\right] &= \frac{x_T}{x_D}\big(F_T(x) - F_D(x)\big) \\ &= \frac{x_T}{x_D}\big[-x_C\tfrac{2}{\delta} - x_D + \underbrace{x_T(\tfrac{2}{\delta} - 4)}_{>0}\big],\end{aligned}$$

which is positive for x_T sufficiently large.

Vector Field



All-T is not an ESS

Let $x = (x_D, x_C, x_T) = (0, 0, 1)$.

Consider <u>any</u> alternative state y such that $y_D = 0$.

$$
\begin{aligned}
(y-x)'F(y) &= (0 \;\; y_C \;\; y_T - 1) \begin{pmatrix} F_D(y) \\ F_C(y) \\ F_T(y) \end{pmatrix} \\
&= y_C F_C(y) + (y_T - 1) F_T(y) \\
&= [y_C + (y_T - 1)] \tfrac{3}{\delta} \qquad \text{(recall that } y_D = 0\text{)} \\
&= [y_C + (1 - y_C - 1)] \tfrac{3}{\delta} \qquad \text{(because } y_T = 1 - y_C\text{)} \\
&= 0.
\end{aligned}
$$

This violates (3). Hence all-T is not an ESS.

This is a case of evolutionary drift.