Evolution & Learning in Games
Econ 243B

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Lecture 3.
Population Games I:
Introduction and Potential Games

Population Games

1. Number of agents is large,
2. Individual agents are small,
3. Anonymous interaction,
4. The number of roles is finite,
   - each agent is a member of one of a finite number of populations.
   - members of a population have identical strategy sets and payoff functions.
5. Payoffs are continuous in the population state.
   - sometimes require stronger assumption, e.g. F is continuously differentiable (C^1).

Population Games

Theorem 3.1. Every population game admits at least one Nash equilibrium.

—can be proved in the usual way using Kakutani's fixed point theorem.

Population Games

- So far, we have studied random matching in normal-form games.

- In such games, payoffs depend *linearly* on the population state:

$$
\begin{aligned}
F_i(x) &= x_1 u(i,1) + x_2 u(i,2) + \ldots x_n u(i,n) \\
&= \sum_{j=1}^{n} x_j F_i(e_j),
\end{aligned}
$$

  where e_j is the j-th basis vector.

- However, this is a rather special form of interaction in large populations:
  - There are many contexts in which agents' payoffs depend 'directly' on the strategies of all other players.

Example: Congestion Games

- Two towns A and B are connected by a network of *links*.
- There is a single population of agents, each of whom needs to commute from A (where he lives) to B (where he works).
- An agent must choose a *path* connecting the two towns.
- His payoff is decreasing in the delay on the path, which is the sum of the delays on its constituent links.
- The delay on a link depends on the number of agents using that link.
- In such games, payoffs can depend *non-linearly* on the population state.

Example: Congestion Games

Formally (and more generally):

- Define a collection of *facilities* L (e.g. links in a highway network).
- Every strategy $i \in S$ requires the use of some collection of facilities $L_i \subseteq L$ (e.g. links in route i).
- The set $\rho(\ell) = \{i \in S : \ell \in L_i\}$ contains the strategies that require facility ℓ.

Example: Congestion Games

- Each facility has a *cost function* $c_\ell : \mathbb{R}_+ \to \mathbb{R}$.
- Its argument argument is the facility's *utilization level* μ_ℓ which equals the mass of agents using the facility:

$$\mu_\ell(x) = \sum_{i \in \rho(\ell)} x_i.$$

- Payoffs are:

$$F_i(x) = -\sum_{\ell \in L_i} c_\ell\big(\mu_\ell(x)\big).$$

- The cost function is increasing and may be convex.

Example: Congestion Games

- It is straightforward to extend the description of the game to multiple populations:
  - There are many towns connected by a highway network.
  - For each ordered pair of towns, there is a population of agents, each of whom needs to commute from the first (where he lives) to the second (where he works).
- It turns out that congestion games belong to an important class of games known as *potential games*. In addition, every potential game can be written as a congestion game (Monderer and Shapley 1996).

Potential Games

- Potential games are games in which all relevant information about payoffs (i.e. relevant to agents' incentives to deviate from a given state) can be summarized by a single scalar-valued function.
  - The important point is that the same function applies to all agents.
- This is called the game's *potential function*.
- In such a game, instead of keeping track of the payoff to each strategy i in each state x, one need only keep track of the (scalar-valued) potential of each state x.

Potential Games: Normal-Form

- Let us break for the moment from our focus on population games with a continuum of players.
- Consider a normal-form game, with a finite number of strategies, m players indexed by p, and payoffs $U = (u_1, u_2, ..., u_m)$.

Potential Games: Normal-Form

- U is a (weighted) potential game if there exists real numbers $\lambda_1, \lambda_2, \ldots \lambda_m$ and a potential function $f : S \to R$ satisfying:

$$\lambda_p \left[u_p(s'_p, s_{-p}) - u_p(s) \right] = f(s'_p, s_{-p}) - f(s)$$

  for all strategy profiles $s \in S$, strategies $s' \in S^p$ for player p, and players $p = 1, 2, \ldots m$.

- That is, utility functions can be rescaled so that the gain from unilaterally deviating (for the deviator) is equal to the change in potential.

Potential Games: Normal-Form

Consider the symmetric normal-form game:

Stag Hunt

	H	S
H	a, a	c, d
S	d, c	b, b

An (exact) potential function $f(s)$ for this game is:

	H	S
H	$a - d$	0
S	0	$b - c$

Potential Games: Normal-Form

- Every weighted (normal-form) potential game has at least one NE in pure strategies.
- Consider the global maximum potential $f(s^*)$, with maximizer s^* (we know there exists such a maximizer).
- By definition, no unilateral deviation can increase potential.
- Hence there are no profitable unilateral deviations from $s^* \Rightarrow s^*$ is a NE.
- By the same reasoning, local maxima of the potential function are also NE.
- In addition, there exists a finite path of unilateral deviations from any state to a Nash equilibrium.

Nash Equilibria of Potential Games

Theorem 3.2. The Nash equilibria of a potential game can be characterized in terms of its potential function f as follows:

(i) If f is concave on X, then the set of Nash equilibria of F, $NE(F)$, is the convex set of maximizers of f on X.

(ii) If f is strictly concave on X, then $NE(F)$ is a singleton containing the unique maximizer of f on X.

Back to Population Games

- Recall that the domain of F is $X = \{x \in \mathbb{R}^n_+ : \sum_{i \in S} x_i = 1\}$, i.e. the simplex or set of population states.

- Define the **tangent space** of X as:

$$TX = \{z \in \mathbb{R}^n : \sum_{i \in S} z_i = 0\}.$$

- TX is the set of *displacement vectors*, such as $z = e_j - e_i$, where e_i is the i-th basis vector.

- Changes in the population state are represented by elements of TX.

Definitions

- The **orthogonal projection** of $\mathbb{R}^n_+$ onto TX is defined by:

$$\Phi = I - \frac{1}{n}\mathbf{1}\mathbf{1}'$$

an $n \times n$ matrix.

Definitions

- The projected payoff vector is:

$$\Phi F(x) = \begin{pmatrix} F_1(x) - \frac{1}{n}\sum_{j\in S} F_j(x) \\ F_2(x) - \frac{1}{n}\sum_{j\in S} F_j(x) \\ \dots \\ F_n(x) - \frac{1}{n}\sum_{j\in S} F_j(x) \end{pmatrix}$$

- $\Phi F(x)$ represents relative payoffs (i.e. payoffs relative to the *unweighted* average); it preserves differences between components of $F(x)$, but normalizes their sum to zero.

- For incentives to switch strategies (and hence for the purpose of identifying Nash equilibria), all we need to know are payoff *differences*.

Potential Games

Definition. Let $F : X \to \mathbb{R}^n$ be a population game. F is a potential game if it admits a potential function, a C^1 function $f : X \to \mathbb{R}$ that satisfies:

$$\nabla f(x) = \Phi F(x) \qquad \text{for all } x \in X.$$

- The gradient vector $\nabla f(x)$ is an element of TX, because the potential function f has domain X.

Potential Games

- Suppose that a small group of agents switch from strategy i to j, a switch represented by the displacement vector $z = e_j - e_i$.

- The marginal impact on potential is:

$$\frac{\partial f}{\partial z}(x) = \nabla f(x)' z = \frac{\partial f}{\partial x_j}(x) - \frac{\partial f}{\partial x_i}(x).$$

- According to the definition of a potential game, the marginal impact on potential equals:

$$\left[\Phi F(x)\right]' z = F_j(x) - F_i(x),$$

  the difference in payoffs between strategies j and i.

- Hence *profitable strategy revisions increase potential*.

Potential Games

Theorem 3.3. Suppose the population game F is C^1. Then F is a potential game if and only if it satisfies the following symmetry condition:

$$\frac{\partial(F_j - F_i)}{\partial(e_\ell - e_k)}(x) = \frac{\partial(F_\ell - F_k)}{\partial(e_j - e_i)}(x) \quad \text{for all } i, j, k, \ell \in S, x \in X. \tag{1}$$

- That is, the change in the payoff to strategy j relative to the payoff to strategy i as agents switch from strategy k to ℓ

  equals

  the change in the payoff to strategy ℓ relative to the payoff to strategy k as agents switch from strategy i to j.

Potential Games: Examples

Matching in Symmetric Normal-Form Games

- Suppose that agents are randomly matched from a *single population* to play a two-player strategic form game A, where $A_{ij} = u(i, j)$.
- The game is symmetric if the matrix A is symmetric, i.e. $A_{ij} = A_{ji}$ for all i, j [or equivalently if $u(i, j) = u(j, i)$ for all i, j].
- This generates the population game $F(x) = Ax$.

Proposition 3.1. Consider the population game $F(x) = Ax$. Suppose A is symmetric. Then F is a potential game.

Potential Games: Examples

Matching in Symmetric Normal-Form Games

Proof. Recall that F is a potential game iff (1) holds. Given random matching in a normal-form game, we have:

$$\frac{\partial(F_j - F_i)}{\partial(e_\ell - e_k)}(x) = u(j,\ell) - u(j,k) - u(i,\ell) + u(i,k),$$

and

$$\frac{\partial(F_\ell - F_k)}{\partial(e_j - e_i)}(x) = u(\ell,j) - u(\ell,i) - u(k,j) + u(k,i),$$

which are equal if A is symmetric, i.e. if $u(a,b) = u(b,a)$ for all $a, b \in S$. □

Potential Games: Examples

Matching in Symmetric Normal-Form Games

Proposition 3.2. Consider the population game $F(x) = Ax$, where $A = C + \mathbf{1}r'$. Suppose C is symmetric and $r \in \mathbb{R}^n$ is an arbitrary vector (which represents a separable component of i's payoff, which depends solely on what j does). Then F is a potential game.

Exercise: Prove this.