Evolution & Learning in Games
Econ 243B

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Lecture 4.
Population Games II:
Stable Games and Supermodular Games

Population Games

Let us consider two more classes of population games, for which there exists a substantial body of results:

- Stable Games,
- Supermodular Games.

Stable Games

- Stables games, as we shall see, are games whose Nash equilibria form a single convex set, for example:

    1. Two-player zero-sum games,
    2. Games with an interior evolutionary stable state,
    3. Potential games with concave potential functions.

Stable Games

Definition. The population game $F : X \to \mathbb{R}^n$ is a **stable game** if:

$$(y - x)'\big(F(y) - F(x)\big) \leq 0 \qquad \text{for all } x, y \in X \tag{1}$$

If the inequality in condition (1) holds strictly whenever $x \neq y$, F is a strictly stable game, whereas if this inequality always binds, F is a null stable game.

Examples

(a) (Perturbed) Concave Potential Games.

- $F : X \to \mathbb{R}^n$ is a concave potential game if it is a potential game whose potential function $f : X \to \mathbb{R}$ is concave.

- Since (i) $y - x \in TX$, (ii) the orthogonal projection matrix Φ is symmetric, and (iii) $\nabla f = \Phi F$:

$$\begin{aligned}
(y-x)'\big(F(y)-F(x)\big) &= \big(\Phi(y-x)\big)'\big(F(y)-F(x)\big) \\
&= (y-x)'\big(\Phi F(y)-\Phi F(x)\big) \\
&= (y-x)'\big(\nabla f(y)-\nabla f(x)\big) \\
&\leq 0.
\end{aligned}$$

- Therefore, F is a stable game.

- If the inequality is satisfied strictly, then it will continue to be satisfied if the payoff function is slightly perturbed.

Stable Games

When payoffs are differentiable, stable games can be characterized in terms of their derivative matrices $DF(x)$ on $TX \times TX$.

Theorem 4.1. Suppose that the population game F is C^1. Then F is a stable game if and only if it satisfies **self-defeating externalities**, that is:

$$DF(x) \text{ is negative semidefinite with respect to } TX \text{ for all } x \in X. \tag{2}$$

Stable Games

- Condition (2) requires:

$$z'DF(x)z \leq 0 \qquad \text{for all } z \in TX, x \in X,$$

- or equivalently:

$$\sum_{i \in S} z_i \frac{\partial F_i}{\partial z}(x) \leq 0 \qquad \text{for all } z \in TX, x \in X. \tag{3}$$

- $\frac{\partial F_i}{\partial z}(x)$ is the marginal effect that the displacement z has on the payoffs of agents currently using strategy i.
- The expression in (3) is the weighted sum of these effects, where the weights are changes in the use of each strategy.

Self-Defeating Externalities

The intuition behind self-defeating externalities is as follows:

- Begin in state x.
- Let a small group of agents switch strategies (the strategies that these agents abandon may be different, as may be the strategies which they switch to).
- The change in the state is represented by the displacement vector z.

Self-Defeating Externalities

- A game exhibits self defeating externalities if any displacement z (weakly) improves the payoffs of the abandoned strategies relative to the strategies to which agents have switched to.

- For example, if $z = e_2 - e_1$, i.e. all switches are from strategy 1 to 2, then the expression in (3) becomes:

$$-\frac{\partial F_1}{\partial z}(x) + \frac{\partial F_2}{\partial z}(x) + 0 + 0 + \ldots 0 \leq 0,$$

or:

$$\frac{\partial F_1}{\partial z}(x) \geq \frac{\partial F_2}{\partial z}(x).$$

Examples

(b) Hawk-Dove

- Consider random matching within a single population to play a Hawk-Dove game:

$$F^{HD} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_H \\ x_D \end{pmatrix} = \begin{pmatrix} -x_H + 2x_D \\ x_D \end{pmatrix}.$$

- $DF(x) = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}.$

Examples

- Note that:

$$z'DF(x)z = z' \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} z.$$

- Let $z' = (z_1 \quad -z_1)$.

- Then $z'DF(x)z$ equals:

$$\begin{pmatrix} -z_1 & 2z_1 - z_1 \end{pmatrix} \begin{pmatrix} z_1 \\ -z_1 \end{pmatrix} = -2z_1^2.$$

- Therefore, $z'DF(x)z < 0$ for all $z \in TX - \{\mathbf{0}\}$.

- By Theorem 4.1 then, Hawk-Dove is a strictly stable game.

Examples

(c) Random matching in two-strategy games.

- Consider the following game played by members of a single population:

  $$F(x) = Ax = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

- $DF(x) = A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$

Examples

- Note that:

$$z'DF(x)z = z' \begin{pmatrix} a & b \\ c & d \end{pmatrix} z.$$

- This equals:

$$\begin{pmatrix} z_1(a-c) & z_1(b-d) \end{pmatrix} \begin{pmatrix} z_1 \\ -z_1 \end{pmatrix} = z_1^2\big[(a-c)-(b-d)\big].$$

- Therefore, $z'DF(x)z \leq 0$ for all $z \in TX$ if and only if $a - c \leq b - d$.

- Hence random matching to play the Prisoners' dilemma is a stable game; random matching to play a coordination game is not.

Examples

(d) Matching in Symmetric Normal-Form Games with an Interior Evolutionarily or Neutrally Stable State

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric normal-form game. The population game is $F(x) = Ax$.

Recall that state $x \in X$ is an **evolutionarily stable state** (ESS) of A if:

$$(x - y)'F(x) \geq 0 \qquad \text{for all } y \in X;$$

and

$$[(x - y)'F(x) = 0 \;\text{ and }\; y \neq x] \quad \text{implies that} \quad (x - y)'F(y) > 0.$$

Examples

Given that $F(x) = Ax$, the two conditions can be rewritten for this game as:

$$(x - y)'Ax \geq 0 \qquad \text{for all } y \in X; \tag{4}$$

and

$$[(x - y)'Ax = 0 \ \text{ and } \ y \neq x] \quad \text{implies that} \quad (x - y)'Ay > 0. \tag{5}$$

Examples

- Condition (4) says that x is a symmetric Nash equilibrium of the normal-form game A.

- Condition (5) says that if y is an alternative best reply to x, then x is a better reply to y than y is to itself.

- Suppose that x is an interior ESS.

- Since x is an interior Nash equilibrium [by (4)], all strategies $y \in X$ must yield the same payoff, i.e. $(x-y)'Ax = 0$.

- Subtract (5) from $(x-y)'Ax = 0$ to get $(x-y)'A(x-y) < 0$.

Examples

- Since x is in the interior of X, every displacement vector $z \in TX$ can be expressed as $x - y$ for some choice of $y \in X$.
- Therefore, $z'Az < 0$ for all $z \in TX - \{\mathbf{0}\}$.
- That is, F is a *strictly* stable game.

Examples

- Let us weaken (5) to:

$$[(x-y)'Ax = 0 \;\text{ and }\; y \neq x] \quad \text{imply that} \quad (x-y)'Ay \geq 0. \tag{6}$$

- A state that satisfies (4) and (6) is called a **neutrally stable state** (NSS).
- The same method can be used to show that every game F that admits an interior NSS is a stable game.

Examples

(e) Random Matching in Rock-Paper-Scissors

Suppose a win is worth $w > 0$, a loss $-l < 0$ and a draw 0. Then:

$$A = \begin{pmatrix} 0 & -l & w \\ w & 0 & -l \\ -l & w & 0 \end{pmatrix}.$$

When $w = l$, let us call the game (standard) *RPS*. When $w > l$, we call it *good RPS* and when $w < l$ we call it *bad RPS*.

For all cases, the unique Nash equilibrium of A is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, an interior equilibrium.

Examples

- Rather than analyzing A, let us define $d = w - l$ and analyze:

$$\widehat{A} = A + A' = \begin{pmatrix} 0 & d & d \\ d & 0 & d \\ d & d & 0 \end{pmatrix}.$$

- Note that $z'Az = \frac{1}{2}z'(A + A')z$. Therefore, F is a stable game if $z'\widehat{A}z \leq 0$ for all $z \in TX$.

$$z'\widehat{A}z = -d\big[(z_1 + z_2)^2 + z_1^2 + z_2^2\big].$$

- Therefore, F is a stable game if and only if $d \geq 0$: *good RPS* is strictly stable, *standard RPS* is null stable, and *bad RPS* is unstable.

Examples

(f) Matching in Symmetric Zero-Sum Games

- A symmetric two-player normal-form game A is symmetric zero-sum if A is skew-symmetric, that is if $A_{ji} = -A_{ij}$ for all $i, j \in X$.

- Payoffs in the *single* population game are $F(x) = Ax$. Then $z'DF(x)z = z'Az$ equals:

$$
\begin{aligned}
z_1 \sum_{j=1}^{n} z_j A_{1j} + z_2 \sum_{j=1}^{n} z_j A_{2j} + \ldots &= \sum_{i=1}^{n} \sum_{j=1}^{n} z_i z_j A_{ij} \\
&= \sum_{(i,j) \in X \times X} z_i z_j (A_{ij} + A_{ji}) \\
&= 0,
\end{aligned}
$$

for all $z \in \mathbb{R}^n$. Therefore, F is a null stable game.

Examples

(g) Matching in Standard Zero-Sum Games

- When the game agents are matched to play is not symmetric, then the payoffs to an agent in a match depends on the role to which he is assigned (e.g. row or column player).
  - This takes us to the multipopulation case.
- In a two-player game, let the payoffs to the player assigned to role k be represented by the matrix U^k, $k = 1, 2$.
- A general two-player normal-form game $U = (U^1, U^2)$ is zero-sum if $U^2 = -U^1$, so that the two players' payoffs always sum to zero.

Examples

- Consider a population game with two populations. A pair of agents is selected at random, one from each population and matched to play the game U. The agent from population 1 takes role 1.
- The population game can be written:

$$
\begin{aligned}
F(x^1, x^2) &= \begin{bmatrix} 0 & U^1 \\ (U^2)' & 0 \end{bmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \\
&= \begin{bmatrix} 0 & U^1 \\ -(U^1)' & 0 \end{bmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}.
\end{aligned}
$$

Examples

- Let z be a vector in $\mathbb{R}^n = \mathbb{R}^{n^1+n^2}$. Then:

$$\begin{aligned} z'DF(x)z &= \begin{pmatrix} (z^1)' & (z^2)' \end{pmatrix} \begin{bmatrix} 0 & U^1 \\ -(U^1)' & 0 \end{bmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \\ &= (z^1)'U^1z^2 - (z^2)'(U^1)'z^1 \\ &= 0. \end{aligned}$$

- Therefore, F is a null stable game.

Multi-Population Stable Games

The following is a more general result on multi-population stable games:

Proposition 4.1. Suppose F is a C^1 stable game without own-population interactions, i.e. $F^p(x)$ is independent of x^p for each population p. Then F is a null stable game.

Global Stability

Definition. Let $x \in X$ be a **globally neutrally stable state** (GNSS). Then:

$$(y - x)' F(y) \leq 0 \qquad \text{for all } y \in X.$$

Let $x \in X$ be a **globally evolutionarily stable state** (GESS). Then:

$$(y - x)' F(y) < 0 \qquad \text{for all } y \in X - \{x\}.$$

- The latter says that x can strictly invade every other state; alternatively, is a better reply to y than y is to itself.
- This is the same as the condition for an ESS except it applies here globally to all other states $y \in X - \{x\}$, not just to states in the nhd of x.

Global Stability & Invasion

- Let $GNSS(F)$ and $GESS(F)$ be the sets of globally neutrally stable states and globally evolutionarily stable states, respectively.
- $GNSS(F)$ is the set of states that can weakly invade every other state.

  —$GNSS(F)$ is a convex set. Why?
- $GESS(F)$ is the set of states that can strictly invade every other state.

Global Stability & Nash Equilibrium

Proposition 4.2. (i) If $x \in GNSS(F)$, then $x \in NE(F)$.

(ii) If $x \in GESS(F)$, then $NE(F) = \{x\}$, i.e. if a $GESS$ exists, it is a unique NE.

Proof. (i) Let $x \in GNSS(F)$. Define $x_\varepsilon = \varepsilon y + (1 - \varepsilon)x$, for some $y \neq x$.

Since x is a $GNSS$, $(x - x_\varepsilon)' F(x_\varepsilon) \geq 0$ for all $\varepsilon \in (0, 1]$.

Simplifying and dividing by ε, we have $(x - y)' F(x_\varepsilon) \geq 0$ for all $\varepsilon \in (0, 1]$.

Taking $\varepsilon \to 0$ yields $(x - y)' F(x) \geq 0$, i.e. $x \in NE(F)$.

Global Stability & Nash Equilibrium

Proof Continued. (ii) By the same argument we can establish that if x is a $GESS$, then x is a Nash equilibrium.

To establish that $NE(F) = \{x\}$, note that if $x \in GESS(F)$, then $(x - y)'F(y) > 0$ for all $y \neq x$.

This means that no $y \neq x$ can be a NE. $\square$

Global Stability & Nash Equilibrium in Stable Games

Theorem 4.2. (i) If F is a stable game, then $NE(F) = GNSS(F)$, which is convex.

(ii) In addition, if F is strictly stable at some $x \in NE(F)$ (i.e. if $(y-x)'\big(F(y)-F(x)\big) < 0$ for all $y \neq x$), then $NE(F) = GESS(F) = \{x\}$.

Proof. (i) Consider an arbitrary $y \neq x$. Since F is stable:

$$(y-x)'\big(F(y)-F(x)\big) \leq 0. \tag{7}$$

Since $x \in NE(F)$:

$$(y-x)'F(x) \leq 0.$$

Global Stability & Nash Equilibrium in Stable Games

Proof Continued. Adding the two inequalities:

$$(y - x)' F(y) \leq 0. \tag{8}$$

This applies for all $y \neq x$. Hence x is a *GNSS*.

(ii) If F is strictly stable at x, then (7) holds strictly. This implies that (8) holds strictly.

Hence x is a *GESS* of F and (by Proposition 4.2) the unique Nash equilibrium of F. □

Supermodular Games

Supermodular games are games that exhibit **strategic complementarities**.

- *Informally*: If one player 'increases' his strategy, then this increases the marginal return to other players from increasing their strategies.

Theorem 4.3. Suppose F is a supermodular game. Then F has a minimal and a maximal Nash equilibrium, $\underline{x}^*$ and $\overline{x}^*$ respectively.

Supermodular games also have attractive *comparative statics* properties.

Supermodular Population Games

Definition. The C^1 population game F is **supermodular** if:

$$\frac{\partial\left(F_{i+1}^{p}-F_{i}^{p}\right)}{\partial\left(e_{j+1}^{q}-e_{j}^{q}\right)}(x) \geq 0$$

for all $i < n^p$, $j < n^q$, populations $p, q \in \mathcal{P}$, and $x \in X$.

—This is simply a more formal definition of **strategic complementarities**.

(For a definition that does not require F to be C^1, see Sandholm p. 94.)

Examples

(a) Bertrand Oligopoly with Differentiated Products

- A population of firms produce at zero marginal cost and compete in prices $S = \{1, \ldots, n\}$.

- Suppose the demand faced by a firm that charges price i, $q_i(x)$, satisfies:

$$\frac{\partial q_i}{\partial(e_{j+1} - e_j)}(x) \geq 0 \qquad \text{and} \qquad \frac{\partial(q_{k+1} - q_k)}{\partial(e_{j+1} - e_j)}(x) \geq 0$$

  for all $i \leq n$, $j, k < n$.

- That is, the demand faced by a firm increases when competitors raise their prices and this effect does not diminish when the firm itself charges higher prices (i.e. the loss in demand from raising one's price is lower when other players raise their price).

Examples

- The payoff to a firm that charges price i when the distribution is x is $F_i(x) = iq_i(x)$. Hence:

$$\frac{\partial(F_{i+1} - F_i)}{\partial(e_{j+1} - e_j)}(x) = (i+1)\frac{\partial q_{i+1}}{\partial(e_{j+1} - e_j)}(x) - i\frac{\partial q_i}{\partial(e_{j+1} - e_j)}(x)$$
$$= i\frac{\partial(q_{i+1} - q_i)}{\partial(e_{j+1} - e_j)}(x) + \frac{\partial q_{i+1}}{\partial(e_{j+1} - e_j)}(x) \geq 0.$$

- Therefore, F is a supermodular game.