Evolution & Learning in Games
Econ 243B

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Lecture 5.
Revision Protocols and Evolutionary Dynamics

Population Games played by Boundedly Rational Agents

- We have introduced the idea of a population game, a game played by a large population (or populations) of agents.
- We shall now study the evolutionary dynamics that arise when such games are played by agents who use simple, myopic rules for revising their strategies.

Revision Protocols

- **Revision protocols** in evolutionary game theory exhibit two forms of *bounded rationality*:

  1. ***Inertia***: agents do not consider revising their strategies at every point in time.

  2. ***Myopia***: agents' choices depend on current behavior and payoffs, not on beliefs about the future trajectory of play.

- In large populations, interactions tend to be *anonymous*—the identities of the agents with whom one is interacting do not matter. Hence considerations such as *reputation* and other sources of forward-looking behavior in repeated games are far less relevant here.

- When agents revise their strategies slowly, then strategies that are appealing today will continue to be so for some time.

Revision Protocols

- Let $F : X \to \mathbb{R}^n$ be a population game with pure strategy sets $(S^1, \dots S^P)$, one for each population $p \in \mathcal{P}$.
- Each population is large but finite.

Definition. A revision protocol ρ^p is a map $\rho^p : \mathbb{R}^{n^p} \times X^p \to \mathbb{R}_+^{n^p \times n^p}$. The scalar $\rho^p_{ij}(\pi^p, x^p)$ is called the *conditional switch rate* from strategy $i \in S^p$ to strategy $j \in S^p$ given payoff vector π^p and population state x^p.

- A population game F, a population size N and a revision protocol $\rho = (\rho^1, ...\rho^P)$ defines a continuous-time evolutionary process on the state space $\mathcal{X}^N$.
- In the single population case, $\mathcal{X}^N = \{x \in X : Nx \in \mathbb{Z}^n\}$, i.e. a discrete grid embedded in the original state space X.

Revision Opportunities

- Let each agent be equipped with a (Poisson) alarm clock. When the alarm clock rings the agent has one opportunity to revise his strategy.

  In particular, assume that the time between rings of agents' alarm clocks is independent and follows a rate 1 exponential distribution. Then the number of rings during time interval $[0, t]$ follows a Poisson distribution with mean t.

- If an i player in population p is faced with a revision opportunity, he switches to strategy j with probability $\rho_{ij}^p(\pi^p, x^p)$, which is a function only of the current payoff vector π^p and the current strategy distribution x^p in population p (alone).

Mean Dynamics

- When each agent uses such a revision protocol, the state x follows a stochastic process $\{X_t^N\}$ on the state space $\mathcal{X}^N$.
- We shall now derive a deterministic process—the **mean dynamic**—which describes the expected motion of $\{X_t^N\}$.
- Later we shall show that the mean dynamic provides a very good approximation to $\{X_t^N\}$ when the time horizon is finite and the population size is large.

Mean Dynamics

- Recall that the mean number of rings of one agent's alarm clock during time interval $[0, t]$ is t.

- Given the current state x, the number of revision opportunities received by agents currently playing i over the next dt time units (dt small) is approximately:

$$N x_i dt.$$

  This is an approximation because x_i may change between time 0 and dt, but this change is likely to be small if dt is small.

Mean Dynamics

- Therefore, the number of switches $i \to j$ in the next dt time units is approximately:

$$Nx_i\rho_{ij}dt.$$

- Hence, the expected change in the use of strategy i is:

$$N\left(\sum_{j\in S} x_j\rho_{ji} - x_i\sum_{j\in S}\rho_{ij}\right)dt,$$

i.e. the ‘inflow’ minus the ‘outflow’.

Mean Dynamics

- Dividing by N (to get the expected change in the proportion of agents) and eliminating the time differential dt yields:

$$\dot{x}_i = \sum_{j \in S} x_j \rho_{ji} - x_i \sum_{j \in S} \rho_{ij},$$

  the **mean dynamic** corresponding to the revision protocol ρ and population game F.

- For the general multipopulation case we write:

$$\dot{x}_i^p = \sum_{j \in S^p} x_j^p \rho_{ji}^p - x_i^p \sum_{j \in S^p} \rho_{ij}^p.$$

Classes of Protocols

1. **Imitative Protocols**, e.g. imitation of more successful strategies.

2. **Direct Protocols**, e.g. myopic best responses.

We shall now consider examples of the two, focussing on the single population case.

Imitative Protocols & Dynamics

- Imitative protocols take the following form:

$$\rho_{ij}(\pi, x) = x_j r_{ij}(\pi, x).$$

- That is, when faced with a revision opportunity an i player is randomly exposed to an opponent and observes his strategy, say j. He then switches to j with probability proportional to r_{ij}.

- Note that for any agent to switch to strategy j, x_j must be greater than zero.

Examples

(a) Pairwise Proportional Imitation

- An agent is paired at random with an opponent, and imitates the opponent only if the opponent's payoff is higher than his own, doing so with a probability proportional to the payoff difference:

$$\rho_{ij}(\pi, x) = x_j \left[\pi_j - \pi_i\right]_+ .$$

Examples

- This generates the following mean dynamic:

$$
\begin{aligned}
\dot{x}_i &= \sum_{j \in S} x_j \rho_{ji}\big(F(x), x\big) - x_i \sum_{j \in S} \rho_{ij}\big(F(x), x\big) \\
&= \sum_{j \in S} x_j x_i \big[F_i(x) - F_j(x)\big]_+ - x_i \sum_{j \in S} x_j \big[F_j(x) - F_i(x)\big]_+ \\
&= x_i \sum_{j \in S} x_j \big[F_i(x) - F_j(x)\big] \qquad (1) \\
&= x_i \Big(F_i(x) - \sum_{j \in S} x_j F_j(x)\Big) \\
&\equiv x_i \big(F_i(x) - \overline{F}(x)\big).
\end{aligned}
$$

This is the **replicator dynamic**.

Examples

(b) Imitation Driven by Dissatisfaction

- Given a revision opportunity, an agent abandons his strategy with a probability that is linearly decreasing in his current payoff, and then imitates the strategy of a randomly chosen opponent:

$$\rho_{ij}(\pi, x) = (K - \pi_i)x_j,$$

where K can be thought of as an *aspiration level*; K must be sufficiently large for switch rates to always be positive.

Examples

- The mean dynamic is:

$$\begin{aligned}\dot{x}_i &= \sum_{j\in S} x_j \rho_{ji}\big(F(x), x\big) - x_i \sum_{j\in S} \rho_{ij}\big(F(x), x\big) \\ &= \sum_{j\in S} x_j \big(K - F_j(x)\big) x_i - x_i \big(K - F_i(x)\big) \\ &= x_i \left(K - \sum_{j\in S} x_j F_j(x) - K + F_i(x)\right) \\ &\equiv x_i \big(F_i(x) - \overline{F}(x)\big).\end{aligned} \tag{2}$$

Once again, the replicator dynamic.

Examples

(c) Imitation of Success

- Given a revision opportunity, a player is exposed to a randomly chosen opponent and imitates his strategy, say j with a probability that is linearly increasing in the current payoff to strategy j:

$$\rho_{ij}(\pi, x) = x_j(\pi_j - K),$$

where again K can be thought of as an *aspiration level*; K must be smaller than any feasible payoff for switch rates to always be positive.

Examples

- The mean dynamic is:

$$
\begin{aligned}
\dot{x}_i &= \sum_{j \in S} x_j \rho_{ji}\big(F(x), x\big) - x_i \sum_{j \in S} \rho_{ij}\big(F(x), x\big) \\
&= \sum_{j \in S} x_j x_i \big(F_i(x) - K\big) - x_i \sum_{j \in S} x_j \big(F_j(x) - K\big) \\
&= x_i \big(F_i(x) - K\big) + x_i K - x_i \sum_{j \in S} x_j F_j(x) \\
&\equiv x_i \big(F_i(x) - \overline{F}(x)\big).
\end{aligned}
\tag{3}
$$

Yet again, the replicator dynamic.

- Of course, as we shall verify later, there are imitative protocols that do not generate the replicator dynamic.

Direct Protocols & Dynamics

- Direct Protocols are ones in which agents can switch to a strategy directly, without having to observe the strategy of another player.

- This requires awareness of the full set of available strategies S.

Examples

(a) Logit Choice

- Suppose choices are made according to the logit choice protocol:

$$\rho_{ij}(\pi) = \frac{exp(\eta^{-1}\pi_j)}{\sum_{k\in S} exp(\eta^{-1}\pi_k)},$$

where η is the noise level.

- As $\eta \to \infty$, the choice probabilities tend to be uniform.
- As $\eta \to 0$, choices tend to best responses.

Examples

- The mean dynamic is:

$$\begin{aligned}\dot{x}_i &= \sum_{j\in S} x_j \rho_{ji}\big(F(x), x\big) - x_i \sum_{j\in S} \rho_{ij}\big(F(x), x\big) \\ &= \sum_{j\in S} x_j \frac{exp(\eta^{-1}F_i(x))}{\sum_{k\in S} exp(\eta^{-1}F_k(x))} - x_i \sum_{j\in S} \frac{exp(\eta^{-1}F_j(x))}{\sum_{k\in S} exp(\eta^{-1}F_k(x))} \\ &= \frac{exp(\eta^{-1}F_i(x))}{\sum_{k\in S} exp(\eta^{-1}F_k(x))} - x_i.\end{aligned} \tag{4}$$

This is the **logit dynamic** with noise level η.

Examples

(b) Comparison to the Average Payoff

- Suppose that when faced with a revision opportunity, an agent chooses a strategy at random. If that strategy's payoff is above the average payoff, then he switches to it with probability proportional to that strategy's excess payoff (payoff above average):

$$\rho_{ij}(\pi) = \left[\pi_j - \sum_{k \in S} x_k \pi_k\right]_+ .$$

Examples

- The mean dynamic is:

$$
\begin{aligned}
\dot{x}_i &= \sum_{j \in S} x_j \rho_{ji}\big(F(x), x\big) - x_i \sum_{j \in S} \rho_{ij}\big(F(x), x\big) \\
&= \sum_{j \in S} x_j \Big[F_i(x) - \sum_{k \in S} x_k F_k(x)\Big]_+ - x_i \sum_{j \in S} \Big[F_j(x) - \sum_{k \in S} x_k F_k(x)\Big]_+ \\
&= \Big[F_i(x) - \sum_{k \in S} x_k F_k(x)\Big]_+ - x_i \sum_{j \in S} \Big[F_j(x) - \sum_{k \in S} x_k F_k(x)_k\Big]_+ ,
\end{aligned}
\tag{5}
$$

which is called the **Brown–von Neumann–Nash dynamic**.

Examples

(c) Pairwise Comparisons

- Suppose that when faced with a revision opportunity, an agent chooses a strategy at random. If that strategy's payoff is higher than his current strategy's payoff, then he switches to it with probability proportional to the difference between the two payoffs:

$$\rho_{ij}(\pi) = \left[\pi_j - \pi_i\right]_+ .$$

Examples

- The mean dynamic is:

$$\begin{aligned}\dot{x}_i &= \sum_{j \in S} x_j \rho_{ji}\big(F(x), x\big) - x_i \sum_{j \in S} \rho_{ij}\big(F(x), x\big) \\ &= \sum_{j \in S} x_j \big[F_i(x) - F_j(x)\big]_+ - x_i \sum_{j \in S} \big[F_j(x) - F_i(x)\big]_+\end{aligned} \tag{6}$$

which is called the **Smith dynamic**.

Evolutionary Dynamics—Some General results

Definition. Fix an open set $O \subseteq \mathbb{R}^n$. The function $f : O \to \mathbb{R}^m$ is *Lipschitz continuous* if there exists a scalar K such that:

$$|f(x) - f(y)| \leq K|x - y| \qquad \text{for all } x, y \in O.$$

For example:

- The continuous but not (everywhere) differentiable function $f(x) = |x|$ is Lipschitz continuous.
- The continuous but not (everywhere) differentiable function $f(x) = 0$ for $x < 0$ and $f(x) = \sqrt{x}$ for $x \geq 0$ is not Lipschitz continuous.
  - Its right-hand slope at $x = 0$ is $+\infty$ and hence no K can be found that meets the above condition.

Evolutionary Dynamics—Some General results

Theorem 5.1. If F is Lipschitz continuous, then for any evolutionary dynamic and from any initial condition $\xi \in X$, there exists at least one trajectory of the dynamic $\{x_t\}_{t \geq 0}$ with $x_0 = \xi$. In addition, for every trajectory, $x_t \in X$ for all $t > 0$. That is, the set of trajectories exhibits *existence* and *forward invariance*.

- However, this does not guarantee uniqueness.
- Note: uniqueness requires that for each initial state $\xi \in X$, there exists exactly one solution $\{x_t\}_{t \geq 0}$ to the system of differential equations with $x_0 = \xi$.

Evolutionary Dynamics—Some General results

- Let the dynamic be characterized by the vector field $V_F : X \to \mathbb{R}^n$, where $\dot{x} = V_F(x)$.

Theorem 5.2. Suppose that V_F is Lipschitz continuous and $V_F(x)$ is contained in the tangent cone $TX(x)$, that is the set of directions of motion that do not point out of X. Then the set of trajectories of the evolutionary dynamic exhibits existence, forward invariance, *uniqueness* and *Lipschitz continuity*.

- The latter states that for each t, $x_t = x_t(\xi)$ is a Lipschitz continuous function of ξ.

Evolutionary Dynamics—Some General results

Theorem 5.3. Suppose that the revision protocol ρ is Lipschitz continuous. Then the **mean dynamic** satisfies existence, forward invariance, uniqueness and Lipschitz continuity.