Evolution & Learning in Games
Econ 243B

Jean-Paul Carvalho

Lecture 7.
Global Convergence of Evolutionary Dynamics I

Global Convergence

- We have examined the connection between the rest points of various dynamics and Nash equilibria of the underlying game.
- Now we shall study the limiting behavior of various evolutionary dynamics when set in motion from arbitrary initial conditions.
- In particular, we shall derive conditions on games and dynamics under which behavior converges to equilibrium from all (or almost all) initial states.
- Our focus will be on potential, stable and supermodular games, though we will also touch upon dominance solvable games.
- Positive correlation (PC) will play some role in our out-of-equilibrium analysis.

Limit Sets

- Let us characterize the limiting behavior of deterministic dynamics as follows.

- The ω-limit of trajectory $\{x_t\}_{t\geq 0}$ is the set of all points that the trajectory approaches arbitrarily closely infinitely often:

$$\omega(\{x_t\}) = \Big\{ y \in X : \text{there exists } \{t_k\}_{k=1}^{\infty} \text{ with } \lim_{k\to\infty} t_k = \infty \text{ such that } \lim_{k\to\infty} x_{t_k} = y \Big\}.$$

- If $\omega(\{x_t\}) = x^*$, a singleton, then x^* is a called an **absorbing state**.

- Otherwise, $\omega(\{x_t\})$ is called a **recurrence class** or **ω-limit set** of the dynamic.

Limit Sets

- For dynamics that admit a unique forward solution trajectory from each initial condition, $\omega(\xi)$ denotes the ω-limit set of the trajectory starting from state ξ.

- The set of all ω-limit points of all solution trajectories is:

$$\Omega(V_F) = \bigcup_{\xi \in X} \omega(\xi).$$

- The notion of recurrence (or the set of recurrence classes) of deterministic dynamic is captured by $\Omega(V_F)$.

Stability Concepts

- Let $A \subseteq X$ be a closed set, and call $O \subseteq X$ a neighborhood of A if it is open relative to X and contains A.
- A is **Lyapunov stable** if for every neighborhood O of A, there exists a neighborhood O' of A such that every solution $\{x_t\}$ that starts in O' is contained in O, that is, $x_0 \in O'$ implies that $x_t \in O$ for all $t \geq 0$.
- Intuitively, this requires that all solutions that start near A, stay near A at all points in time.
- Any displacement from A does not lead the process to go 'very far' from A *at any point in time*.

Stability Concepts

- A is attracting if there is a neighborhood Y of A such that every solution that starts in Y converges to A, that is, $x_0 \in Y$ implies $\omega(\{x_t\}) \subseteq A$.
- A is globally attracting if it is attracting with $Y = X$.
- Intuitively, this requires that given any displacement from A, the process returns to A in the limit.

Stability Concepts

- A is **asymptotically stable** if it is Lyapunov stable and attracting.
- A is **globally asymptotically stable** if it is Lyapunov stable and globally attracting.
- Intuitively, this requires that given any displacement from A, the process never travels 'very far' from A <u>and</u> returns to A in the limit.

Lyapunov Functions

The most common method for proving global convergence in dynamical systems is by constructing a **strict Lyapunov function**:

- A scalar-valued function.
- The value of the function changes monotonically along every solution trajectory.
- For many ODEs, the existence of a Lyapunov function is a necessary and sufficient condition for stability.
- The Lyapunov function allows us to (partially) characterize the evolution of play without requiring explicit solutions to the differential equation (or inclusion).

Definition. The C^1 function $L : X \to \mathbb{R}$ is a (decreasing) strict Lyapunov function for the differential equation $\dot{x} = V_F(x)$ if $\dot{L}(x) = \nabla L(x)' V_F(x) \leq 0$ for all $x \in X$, with equality only at rest points of V_F.

Lyapunov Functions and Stability

Theorem 7.1. (*Lyapunov Stability*) Let $A \subseteq X$ be closed and let $Y \subseteq X$ be a neighborhood of A. Let $L : Y \to \mathbb{R}_+$ be Lipschitz continuous with $L^{-1}(0) = A$. If each solution $\{x_t\}$ of V_F satisfies $\dot{L}(x_t) \leq 0$ for almost all $t \geq 0$, then A is Lyapunov stable under V_F.

Theorem 7.2. (*Asymptotic Stability*) Let $A \subseteq X$ be closed and let $Y \subseteq X$ be a neighborhood of A. Let $L : Y \to \mathbb{R}_+$ be C^1 with $L^{-1}(0) = A$. If each solution $\{x_t\}$ of V_F satisfies $\dot{L}(x_t) < 0$ for all $x \in Y - A$, then A is asymptotically stable under V_F. If in addition, $Y = X$, then A is globally asymptotically stable under V_F.

Potential Games

- Let us turn to global convergence in potential games.

- In potential games, the natural candidate for a strict (increasing) Lyapunov function is the potential function.

- Recall that in a potential game $F : X \to \mathbb{R}^n$, the potential function $f : X \to \mathbb{R}$ summarizes all information about incentives:

$$\nabla f(x) = \mathbf{\Phi} F(x) \qquad \text{for all } x \in X.$$

Lyapunov Function for Potential Games

Lemma 7.1. Let F be a potential game with potential function f. Suppose the evolutionary dynamic $\dot{x} = V_F(x)$ satisfies positive correlation (PC). Then f is a strict Lyapunov function for V_F.

Proof. $\dot{f}(x) = \nabla f(x)'\dot{x} = (\mathbf{\Phi} F(x))' V_F(x) = F(x)' V_F(x)$.

The result then follows immediately from PC. □

Convergence in Potential Games

Theorem 7.3. Let F be a potential game, and let $\dot{x} = V_F(x)$ be an evolutionary dynamic for F that admits a unique forward solution from each initial condition and that satisfies PC. Then $\Omega(V_F) = RP(V_F)$.

For example, if V_F is an imitative dynamic, then $\Omega(V_F) = RE(F)$, the set of restricted equilibria of F.

Convergence in Potential Games

- What about convergence of the *best response* dynamic?
- Recall that the best response dynamic is:

$$\dot{x} \in M\big(F(x)\big) - x, \text{where } M(\pi) = \arg\max_{i \in S} \pi_i.$$

- To state the appropriate result one must account for the fact that the dynamic is multivalued.

Convergence in Potential Games

Theorem 7.4. Let F be a potential game with potential function f, and let $\dot{x} \in V_F(x)$ be the best response dynamic for F. Then:

$$\frac{\partial f}{\partial z}(x) = max_{j \in S} \widehat{F}_j(x) \qquad \text{for all } z \in V_F(x), x \in X.$$

Therefore, every solution trajectory $\{x_t\}$ of V_F satisfies $\omega(\{x_t\}) \subseteq NE(F)$. That is, the set of Nash equilibria of F is globally asymptotically stable.

Stable Games

- Recall that the population game F is stable if it satisfies:

$$(y - x)'\big(F(y) - F(x)\big) \leq 0 \qquad \text{for all } x, y \in X.$$

- When F is C^1 this is equivalent to *self-defeating externalities*:

$$z'DF(x)z \leq 0 \qquad \text{for all } z \in TX, x \in X.$$

- The set of Nash equilibria of a stable game is convex and usually a singleton.
- Uniqueness itself does not guarantee convergence (as we shall see later).

Lyapunov Functions for Stable Games

- Once again, convergence proofs rely upon construction of a Lyapunov function.
- But unlike potential games, there is no natural candidate for a Lyapunov function; a distinct one must be constructed for each dynamic.
- We shall now write the Lyapunov function as decreasing over time.

Definition. A C^1 function L is a (decreasing) strict Lyapunov function for the dynamic $\dot{x} = V_F(x)$ if $\dot{L}(x) \leq 0$ for all $x \in X$, with equality only at rest points of V_F.

Replicator Dynamics in Stable Games

- For convergence of the replicator dynamics, we need to confine attention to strictly stable games.

- We also need to restrict attention to a subset of all initial conditions $\xi \in X$, because if ξ places no mass on a strategy in the support of a Nash equilibrium x^*, then the dynamic cannot converge to x^* from ξ.

- Let the support of x be $S(x) = \{i \in S : x_i > 0\}$. Then $X_y = \{x \in X : S(y) \subseteq S(x)\}$ is the set of states in X whose supports contain the support of y.

Replicator Dynamics in Stable Games

- The Lyapunov function (in the single-population case) is $h_y : X_y \to \mathbb{R}$ where:

$$h_y(x) = \sum_{i \in S(y)} y_i \log \frac{y_i}{x_i}.$$

h_y is known as the relative entropy of y given x.

Replicator Dynamics in Stable Games

Theorem 7.5. Let F be a strictly stable game with unique Nash equilibrium x^*, and let $\dot{x} = V_F(x)$ be the replicator dynamic for F.

Then h_{x^*} is non-negative, $h_{x^*}^{-1}(0) = \{x^*\}$ and $h_{x^*}(x)$ approaches infinity whenever x approaches $X - X_{x^*}$.

Moreover, $\dot{h}_{x^*}(x) \leq 0$, with equality only when $x = x^*$. Therefore, x^* is globally asymptotically stable with respect to X_{x^*}.

If F is simply a stable game, then x^* is Lyapunov stable.

Best Response Dynamics in Stable Games

- Recall that the best response dynamic is:

$$\dot{x} \in M\big(\hat{F}(x)\big) - x,$$

where:

$$M(\hat{\pi}) = \arg\max_{y \in X} y' \hat{\pi},$$

i.e. the set of maximizers of (excess) payoffs.

Best Response Dynamics in Stable Games

Theorem 7.6. Let F be a C^1 stable game, and let $\dot{x} \in V_F(x)$ be the best response dynamic for F. Define the Lipschitz continuous function $G : X \to \mathbb{R}_+$ by:

$$G(x) = max_{i \in S} \hat{F}_i(x),$$

which is non-negative and satisfies $G^{-1}(0) = NE(F)$.

Moreover, if $\{x_t\}_{t \geq 0}$ is a solution to V_F then $\dot{G}(x_t) \leq -G(x_t)$ for almost all $t \geq 0$, and so $NE(F)$ is globally asymptotically stable under V_F.