Evolution & Learning in Games
Econ 243B

Jean-Paul Carvalho

Lecture 8.
Global Convergence of Evolutionary Dynamics II

Supermodular Games

- Supermodular games are games that exhibit **strategic complementarities**.
- The strategy set $S = \{1, 2, ..., n\}$ is naturally endowed with a linear order.
- In a supermodular game, higher choices by one's opponent makes it relatively more attractive for one to make a high choice.

Two-player Normal Form Games

- Consider the (possibly asymmetric) two-player normal form game $U = (U^1, U^2)$.
- Let F be the population game generated by random matching of two populations to play U.
- The best response dynamic for F takes the following form:

$$\begin{aligned} \dot{x}^1 \in M^1\big(F^1(x)\big) - x^1 &= M^1\big(U^1 x^2\big) - x^1 \\ \dot{x}^2 \in M^2\big(F^2(x)\big) - x^2 &= M^2\big((U^2)' x^1\big) - x^2 \end{aligned} \qquad (1)$$

Supermodularity

- The game $U = (U^1, U^2)$ is **supermodular** if:

$$\begin{aligned} U^1_{i+1,j+1} - U^1_{i,j+1} &\geq U^1_{i+1,j} - U^1_{i,j} \quad \text{and} \\ U^2_{i+1,j+1} - U^2_{i+1,j} &\geq U^2_{i,j+1} - U^2_{i,j} \end{aligned} \tag{2}$$

  for all $i < n^1$ and $j < n^2$.

- If (2) holds, then the population game F induced by U is supermodular as well.

- If the inequality in (2) always holds strictly, then U is strictly supermodular.

Two More Conditions

- The convergence results also require **strictly diminishing returns**: the gain from increasing one's strategy, holding the opponent's strategy fixed, is decreasing (i.e. payoffs are concave in one's own strategy):

$$
\begin{aligned}
U^1_{i+2,j} - U^1_{i+1,j} &< U^1_{i+1,j} - U^1_{i,j} \quad \text{for all } i \leq n^1 - 2, j \in S^2 \\
U^2_{i,j+2} - U^2_{i,j+1} &< U^2_{i,j+1} - U^2_{i,j} \quad \text{for all } i \in S^1, j \leq n^2 - 2.
\end{aligned} \tag{3}
$$

- The final condition is **nondegeneracy**: U is nondegenerate if for each fixed pure strategy of the opponent, a player is not indifferent among any of his pure strategies.

Simple Solutions

Definition. A particular solution trajectory $\{x_t\}_{t\geq 0}$ of (1) is **simple** if the set of times at which it is not differentiable has no accumulation (or limit) point, and if at other times, all elements of $M^p(x_t)$ are pure (i.e. vertices of X^p).

- For such a solution trajectory, one can list the sequence of times $\{t_k\}$ at which the solution is not differentiable (these are the times at which the target state for at least one population changes.)

- During each open interval of times $I_k = (t_{k-1}, t_k)$, revising agents only use the pure strategies $i_k \in S^1$ and $j_k \in S^2$, respectively.

- Strategies i_k and j_k are called the interval k selections for populations 1 and 2.

Convergence of Best Response Dynamic

Theorem 8.1. Suppose that F is generated by matching in a two-player normal form game U that is strictly supermodular, exhibits strictly diminishing returns, and is nondegenerate. Then every simple solution trajectory of the best response dynamic (1) converges to a pure Nash equilibrium.

Convergence of Best Response Dynamic

Proof. We claim that the sequence of times $\{t_k\}$ is finite. (See Sandholm p. 252-3 for a proof.)

Let the final element be t_K and let i^* and j^* be the selections made by revising agents after time t_K, so that $\{x_t\}$ converges to $x^* = (e^1_{i^*}, e^2_{j^*})$ (i.e. the population 1 state is e_{i^*} and the population 2 two state is e_{j^*}).

For this to occur under the best response dynamic, the pure state x^* must be in the best response set $M(x_t)$ for all $t \geq t_K$.

Payoffs are continuous, so $x^* \in M(x^*)$, i.e. x^* is a NE. Therefore, the best response dynamic converges to a Nash equilibrium. □

More on Supermodularity

- Before stating a convergence result in a more general class of supermodular games, we need to introduce some further definitions.

- Define the $n^p \times (n^p - 1)$ matrix:

$$\tilde{\Sigma} = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 1 & -1 & \ddots & \vdots \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & -1 \\ 0 & \ddots & 0 & 1 \end{pmatrix}.$$

- Also define the block diagonal matrix $\tilde{\boldsymbol{\Sigma}} = diag\left(\tilde{\Sigma}, ..., \tilde{\Sigma}\right)$.

More on Supermodularity

- If the population game F is C^1, then F is *supermodular* if and only if:

$$\tilde{\mathbf{\Sigma}}' DF(x) \tilde{\mathbf{\Sigma}} \geq \mathbf{0} \qquad \text{for all } x \in X.$$

- This is equivalent to:

$$\frac{\partial \left(F^p_{i+1} - F^p_i \right)}{\partial \left(e^q_{j+1} - e^q_j \right)}(x) \geq 0$$

  for all $i < n^p$, $j < n^q$, populations $p, q \in \mathcal{P}$, and $x \in X$.

- This is a more general characterization of strategic complementarities.

- F is **irreducible** if each column of $\tilde{\mathbf{\Sigma}}' DF(x) \tilde{\mathbf{\Sigma}}$ contains at least one positive element.

Stochastically Perturbed Best Response Dynamics

- To derive a convergence result without such strict assumptions (i.e. strict supermodularity and diminishing returns), let us turn to perturbed best response dynamics.

- Recall from Theorem 4.3 that the set of Nash equilibria of a supermodular game has a minimal element $\underline{x}^*$ and a maximal element $\overline{x}^*$.

- Let $\underline{x}^p = (m^p, 0, ..., 0)$, where m^p is the mass of agents comprising population p. The minimal state in $X = \Pi_{p \in \mathcal{P}} X^p$ is $\underline{x} = (\underline{x}^1, ..., \underline{x}^P)$.

- Similarly, let $\overline{x}^p = (0, 0, ..., m^p)$. The maximal state in $X = \Pi_{p \in \mathcal{P}} X^p$ is $\overline{x} = (\overline{x}^1, ..., \overline{x}^P)$.

Stochastically Perturbed Best Response Dynamics

Theorem 8.2. Let F be a C^1 irreducible supermodular game, and let $\dot{x} = V_{F,\eta}(x)$ be the logit dynamic for F. Then:

(i) States $\underline{x}^* \equiv \omega(\underline{x})$ and $\overline{x}^* \equiv \omega(\overline{x})$ exist and are the minimal and maximal elements of the set of logit equilibria $logit(F, \eta)$, respectively; moreover $[\underline{x}^*, \overline{x}^*]$ contains all ω-limit points of $V_{F,\eta}(x)$ and is globally asymptotically stable.

(ii) Solutions to $\dot{x} = V_{F,\eta}(x)$ from an open, dense, full measure set of initial conditions in X converge to states in $logit(F, \eta)$.

Dominated Strategies

- In population game F, strategy $i \in S^p$ is strictly dominated if there exists a strategy $j \in S^p$ such that $F_j(x) > F_i(x)$ for all $x \in X$.

- Under the best response dynamic, agents always switch to optimal strategies. Strictly dominated strategies are never optimal, so they cannot persist.

- Let $\{x_t\}$ be a solution trajectory of the best response dynamic for population game F in which strategy $i \in S^p$ is strictly dominated. Then $\lim_{t \to \infty} (x_t)_i^p = 0$.

- In fact, because i is never a best response, $(\dot{x}_t)_i^p = -(x_t)_i^p$. Therefore, $(x_t)_i^p = (x_0)_i^p e^{-t}$, so that the mass of agents playing the dominated strategy not only converges to zero, but also does so exponentially quickly.

Iteratively Dominated Strategies

- Note that once a strictly dominated strategy is deleted from the game, more strictly dominated strategies may present themselves.
- For each population, there is always a nonempty set of strategies that survive *iterated deletion of strictly dominated strategies*.
- If each of these sets is a singleton, then the game is said to be **dominance solvable**.
- The **dominance solution** of F is the pure state in which each agent plays her population's sole surviving strategy (the game's unique Nash equilibrium).

Iteratively Dominated Strategies & the Best Response Dynamic

Theorem 12.3. Let $\{x_t\}$ be a solution trajectory of the best response dynamic for population game F, in which strategy $i \in S^p$ does not survive *iterative elimination of strictly dominated strategies*. Then $\lim_{t\to\infty}(x_t)_i^p = 0$. In particular, if F is dominance solvable, then all solutions of the best response dynamic converge to the dominance solution.

Iteratively Dominated Strategies & Imitative Dynamics

Theorem 12.4. Let $\{x_t\}$ be an interior solution trajectory of an imitative dynamic for population game F, in which strategy $i \in S^p$ does not survive iterative elimination of strictly dominated strategies. Then $\lim_{t\to\infty}(x_t)_i^p = 0$. In particular, if F is dominance solvable, then all interior solutions of an imitative dynamic converge to the dominance solution.