Evolution & Learning in Games
Econ 243B

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Lecture 9.
Local Stability

Local Stability

- Where global convergence does not occur (or cannot be proved), we can at least say something about the local stability of the rest points of an evolutionary dynamic.
- We can immediately state some results about local stability under imitative dynamics and in potential games.
- We will then explore the relationship between evolutionary stable states (in a multiple population setting) and locally stable states.
- Finally, we shall examine two methods of analyzing local stability, via:
  - Lyapunov functions,
  - Linearization of dynamics.

Non-Nash Rest Points of Imitative Dynamics

We can now formalize the argument that such rest points of imitative dynamics are not plausible predictions of play.

Theorem 9.1. Let V_F be an imitative dynamic for population game F, and let $\hat{x}$ be a non-Nash rest point of V_F. Then $\hat{x}$ is not Lyapunov stable under V_F, and no interior solution trajectory of V_F converges to $\hat{x}$.

Non-Nash Rest Points of Imitative Dynamics

- Recall that imitative dynamics exhibit *strictly monotone percentage growth rates*:

$$G_i^p(x) > G_j^p(x) \quad \text{if and only if} \quad F_i^p(x) > F_j^p(x),$$

  where $G_i^p(x)$ is the percentage growth rate of strategy i in population p in state x.

- The result follows from this fact.

Local Stability in Potential Games

- In Lecture 7, we used the fact that the potential function is a strict Lyapunov function for any evolutionary dynamic satisfying PC to prove global convergence (to rest points of V_F).
- This fact is also important to local stability.
- $A \subseteq X$ is a **local maximizer set** of the potential function f if:
  - A is connected,
  - f is constant on A, and
  - there exists a neighborhood O of A such that $f(x) > f(y)$ for all $x \in A$ and all $y \in O - A$.

Local Stability in Potential Games

For a potential game, a local maximizer set A consists entirely of Nash equilibria.

Theorem 9.2. Let F be a potential game with potential function f, let V_F be an *imitative* dynamic operating on F, and suppose that $A \subseteq NE(F)$ is a local maximizer set of f.

Then A is Lyapunov stable under V_F.

Local Stability in Potential Games

$A \subseteq NE(F)$ is **isolated** if there is a neighborhood of A that does not contain any Nash equilibria other than A.

Theorem 9.3. Let F be a potential game with potential function f, let V_F be the best response dynamic, and let $A \subseteq NE(F)$ be smoothly connected. Then A is an isolated local maximizer set of f if and only if A is asymptotically stable under V_F.

Evolutionarily Stable States

- We have already introduced the notion of evolutionarily stable states (ESS) in a single population setting.

- Suppose x is an ESS. Consider a fraction ε of mutants who switch to $y \neq x$. Then the average post-entry payoff in the incumbent population is higher than that in the mutant population, for ε sufficiently small.

- We showed that this is equivalent to:

  Suppose x is an ESS. Consider a fraction ε of mutants who switch to y. Then the average post-entry payoff in the incumbent population is higher than that in the mutant population, for y sufficiently close to x.

Evolutionarily Stable States

- Thus an ESS is defined with respect to population averages and explicitly it says nothing about dynamics.

- We shall now extend the ESS concept to a multipopulation setting and relate it to the local stability of evolutionary dynamics.

Taylor ESS

Definition. If F is a game played by $p \geq 1$ populations, we call $x \in X$ a **Taylor ESS** of F if:

There is a neighborhood O of x such that $(y - x)'F(y) < 0$ for all $y \in O - \{x\}$.

This is the same as the statement for single-population games, except F can now be a multipopulation game.

Note that in the multipopulation setting:

$$X = \prod_{p \in \mathcal{P}} X^p = \{x = (x^1, ..., x^p) : x^p \in X^p\}.$$

Taylor ESS

Once again, we have the result:

Theorem 9.4. Suppose that F is Lipschitz continuous. Then x is a Taylor ESS if and only if:

x is a Nash equilibrium: $(y-x)'F(x) \leq 0$ for all $y \in X$, and

There is a neighborhood O of x such that for all $y \in O - \{x\}$, $(y-x)'F(x) = 0$ implies that $(y-x)'F(y) < 0$.

Regular Taylor ESS

- For some local stability results we require a strengthening of the Nash equilibrium condition.
- In a **quasistrict equilibrium** x, all strategies in use earn the same payoff, a payoff that is strictly greater than that of each unused strategy.
- This is a generalization of strict equilibrium, which in addition requires x to be a pure state.
- The second part of the Taylor ESS condition is also strengthened, replacing the inequality with a differential version.

Regular Taylor ESS

Definition. We call x a **regular Taylor ESS** if and only if:

x is a quasistrict Nash equilibrium: $F_i^p(x) = \bar{F}^p(x) > F_j^p(x)$ when $x_i^p > 0, x_j^p = 0$, and

For all $y \in X - \{x\}$, $(y - x)'F(x) = 0$ implies that $(y - x)'DF(x)(y - x) < 0$.

Note: every regular Taylor ESS is a Taylor ESS.

Local Stability via Lyapunov Functions

We can use Lyapunov functions to prove the following theorems which establish the connection between ESS and local stability:

Theorem 9.5. Let x^* be a Taylor ESS of F. Then x^* is asymptotically stable under the replicator dynamic for F.

Theorem 9.6. Let x^* be a regular Taylor ESS of F. Then x^* is asymptotically stable under the best response dynamic for F.

Local Stability via Lyapunov Functions

Theorem 9.7. Let x^* be a regular Taylor ESS of F. Then for some neighborhood O of x^* and each small enough $\eta > 0$, there is a unique $logit(\eta)$ equilibrium $\tilde{x}^\eta$ in O, and this equilibrium is asymptotically stable under the $logit(\eta)$ dynamic. Finally, $\tilde{x}^\eta$ varies continuously in η, and $\lim_{\eta \to 0} \tilde{x}^\eta = x^*$.

Linearization of Dynamics

- Another technique for establishing local stability of a rest point is to linearize the dynamic around the rest point.
- This requires the dynamic to be smooth around the rest point, but does not require the guesswork of finding a Lyapunov function.
- If a rest point is found to be stable under the linearized dynamic, then it is **linearly stable**.
- Linearization will also be used to prove that a rest point is unstable, and we shall use this in the next lecture to study nonconvergence of evolutionary dynamics.

Linear Approximation

The linear (first-order Taylor) approximation to a function F around point a is:

$$F(a+h) \approx F(a) + DF(a)h.$$

Let $o(|h|)$ be the remainder, the difference between the two sides:

$$o(|h|) \equiv F(a+h) - F(a) - DF(a)h.$$

Linear Approximation

Suppose F is a function of one variable. Then:

$$\frac{o(|h|)}{h} = \frac{F(a+h) - F(a)}{h} - F'(a) \to 0 \quad as \quad h \to 0,$$

by the definition of the derivative $F'(a)$.

The approximation gets better as h gets smaller and it gets better at an order of magnitude smaller than h.

Eigenvalues & Eigenvectors

Let A be an $n \times n$ matrix. A non-zero vector v is an **eigenvector** of A if it satisfies:

$$Av = \lambda v,$$

for some scalar λ called an **eigenvalue** of A.

Note that:

$$Av = \lambda v \Longrightarrow (A - \lambda I)v = 0 \Longrightarrow |A - \lambda I| = 0.$$

Therefore, an eigenvalue of A is a number λ which when subtracted from each of the diagonal entries of A converts A into a *singular* matrix.

Eigenvalues & Eigenvectors

EXAMPLE: $A = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$

$$|A - \lambda I| = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2).$$

Therefore, A has two eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$.

Linearization of Dynamics

- A single-population dynamic $\dot{x} = V(x)$, which we shall refer to as (D), describes the evolution of the population state through the simplex X.

- Near x^*, the dynamic (D) can typically be well approximated by the linear dynamic:

$$\dot{z} = DV(x^*)z, \tag{L}$$

  where (L) is a dynamic on the tangent space TX.

- (L) approximates the motion of deviations from x^* following a small displacement z.

Linearization of Dynamics

- Consider the linear mapping which maps each displacement vector $z \in TX$ into a new tangent vector $DV(x^*)z \in TX$.
- The scalar $\lambda = a + \mathrm{i}b$ is an eigenvalue of this map if $DV(x^*)z = \lambda z$.
- If all eigenvalues of this map have negative real part, then the rest point x^* is **linearly stable** under (D).

Linearization of Dynamics

Theorem 9.8. Let x^* be a regular Taylor ESS of F. Then x^* is linearly stable under the replicator dynamic.

Theorem 9.9. Let $x^* \in int(X)$ be a regular Taylor ESS of F. Then for some neighborhood O of x^* and all $\eta > 0$ less than some threshold $\hat{\eta}$, there is a unique and linearly stable $logit(\eta)$ equilibrium $\tilde{x}^\eta$ in O.