Evolution & Learning in Games
Econ 243B

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Lecture 14.
Bargaining Conventions

Bargaining

"Is there any economic activity more basic than two people dividing a pie?"
Tore Ellingsen

- The following bargaining situations can all be modeled as players splitting a pie:
  - A buyer and a seller negotiating the price of a car,
  - Business partners dividing profits from a joint venture,
  - Allies dividing the spoils of war,
  - Spouses allocating household chores.
- Notice how bargaining differs from other trading institutions such as competitive markets.

Bargaining

- Finding a solution to the bargaining problem (i.e. a prediction of play in a bargaining game) was once thought to be an intractable problem.

- Consider the Nash demand game:
  - Two players simultaneously shout out demands s_1 and s_2, respectively.
  - If the demands can be met from the pie, then each gets exactly what he demanded. Otherwise, both players get nothing.

- There is a continuum of Nash equilibria of this game:
  - Suppose the pie is equal to one.
  - Then any pair of demands (s_1, s_2) such that $s_1 + s_2 = 1$ is a Nash equilibrium (*note*: there are others).

Nash Bargaining (Nash 1950)

The Environment

- Consider a pie of size one.
- Two players $i = 1, 2$.
- Player i's share is $x_i \in [0, 1]$.
- Player 1's utility function is $u : [0, 1] \to \mathbb{R}$ and player 2's is $v : [0, 1] \to \mathbb{R}$.

Nash Bargaining

The Bargaining Set

- $U = \{(u(x_1), v(x_2)) : x_1 + x_2 \leq 1 \text{ and } x_1, x_2 \geq 0\}$.
- Disagreement point $d = (d_1, d_2)$.
- A bargaining problem is a pair (U, d).

Nash's Axioms

Let F be a function that assigns a unique outcome $F(U, d) \in U$ to every bargaining problem (U, d).

1. Weak Pareto efficiency (*WPAR*). If $(u, v) = F(U, d)$, then there is no $(u', v') \in U$ such that $u' > u$ and $v' > v$.

2. Symmetry (*SYM*). (U, d) is a symmetric problem if $d_1 = d_2$ and $(u, v) \in U \Leftrightarrow (v, u) \in U$. If (U, d) is a symmetric problem and $(u, v) = F(U, d)$ then $u = v$.

Nash's Axioms

3. Invariance to equivalent payoff representations (*INV*). Given $\alpha_i > 0$ and β_i let:

$$\begin{aligned} u' &= \alpha_1 u + \beta_1, \;\; v' = \alpha_2 v + \beta_2, \\ U' &= \{(\alpha_1 u + \beta_1, \alpha_2 v + \beta_2) : (u, v) \in U\} \\ d' &= (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2). \end{aligned}$$

Then $(u, v) = F(U, d) \;\Leftrightarrow\; (\alpha_1 u + \beta_1, \alpha_2 v + \beta_2) = F(U', d')$.

4. Independence of Irrelevant Alternatives (*IIA*). If $U' \subseteq U$, $d' = d$ and $F(U, d) \in U'$ then $F(U, d) = F(U', d')$.

The Nash Bargaining Solution

- There is a unique solution to the bargaining problem (U, d) that satisfies Nash's four axioms:

$$\underset{(x_1,x_2)}{\arg\max}\ \big(u(x_1) - d_1\big)\big(v(x_2) - d_2\big)$$
$$\text{s.t. } (u, v) \in U \text{ and } u_1 \geq d_1, u_2 \geq d_2.$$

- In the symmetric case ($u = v$ and $d_1 = d_2$), the NBS is $x_1^* = x_2^* = \frac{1}{2}$.

The Nash Bargaining Solution

- Let $d_1 = d_2 = 0$. Then the NBS is:

$$\arg\max_{x \in [0,1]} u(x)v(1-x).$$

The Nash Bargaining Solution

Bargaining Power

- If in addition we drop the symmetry axiom, solutions to the bargaining problem that satisfy the remaining three axioms are of the form:

$$\operatorname*{arg\,max}_{x \in [0,1]} u(x)^a v(1-x)^b.$$

- a and b can be interpreted as levels of *bargaining power*.

- The first-order condition which yields the **asymmetric NBS** is:

$$a\frac{u'(x^*)}{u(x^*)} = b\frac{v'(1-x^*)}{v(1-x^*)}.$$

Strategic Bargaining (Rubinstein 1982)

Alternating Offers—a noncooperative approach

- Suppose that players 1 and 2 have discount factors a and b respectively.
- Player 1 begins by proposing a split. Player 2 can accept or reject. If she rejects, she makes a counteroffer ... This goes on until an offer is accepted.
- There is a unique subgame perfect equilibrium of this game.
- As the time between rounds $\rightarrow 0$, equilibrium shares converge to the asymmetric NBS above.
- Here one's bargaining power is determined by one's patience (i.e. discount factor).

Evolutionary Bargaining (Young 1993, JET)

- Consider two disjoint populations, rows and columns (e.g. buyers and sellers; workers and bosses) of equal size N.
- Every period, two players (one from each population) are matched to play a Nash demand game.
- If their two demands (x_1, x_2) sum to one or less, each player receives her demand, and the associated utility $u(x_1)$ or $v(x_2)$. Both players get zero otherwise.
- The utility functions u and v are strictly increasing, concave and continuously differentiable (C^1 not required in the paper).

Evolutionary Bargaining

- We assume that only one of the matched players each period revises her strategy. This revising player is chosen at random.
- To keep the strategy set finite, consider a discretized set of demands $\Delta \equiv \{\delta, 2\delta, 3\delta, ..., 1\}$.
- Every division $(x, 1-x)$, such that $0 < x < 1$, constitutes a *strict Nash equilibrium* of the demand game. We shall call such a division a bargaining **norm**.

Evolutionary Bargaining

Adaptive Play Protocol

- The history of play is a vector $h^t = \big((x_1^{t-m}, x_2^{t-m}), \ldots (x_1^{t-1}, x_2^{t-1})\big)$, where x_1^{t-1} is the most recent demand made by a row player and m is the memory length.

- A **convention** is a history of the form $h^* = \big((x, 1-x), \ldots (x, 1-x)\big)$, i.e. m instances of a norm.

- A revising row player at time t draws a random sample of size am (an integer) from the m previous plays by members of the column population, i.e. from h^t.

- A revising column player at time t draws a random sample of size bm from the m previous plays by members of the row population.

Evolutionary Bargaining

Adaptive Play Protocol

- The revising player computes the frequency $p(x)$ of each demand x in her sample.
- With high probability $1 - \varepsilon$, a revising player best responds to her sample.
- With low probability ε, she chooses a demand x within δ of a best response to her sample (*note*: 'local errors' not required in paper).

Evolutionary Bargaining

Proposition 13.1 The unperturbed process converges almost surely to a convention from any initial state and locks in.

Proof.
For convergence, we need to show there is a positive probability path from any state to a convention:

- With positive probability, the next m revisions are by row players and that each revising row player draws the same sample, playing the same best response x.
- There is also a positive probability that the subsequent m revisions are by column players. They must each draw a sample consisting solely of demands equal to x and choose the best response $1 - x$.

Evolutionary Bargaining

Proof.

Second, each convention is an absorbing state of the unperturbed process:

- To see this, consider m instances of the norm $(x, 1-x)$.
- All possible samples for a row player consist of am plays of $1-x$ to which the unique BR is x.
- Similarly for column players.
- Thus the convention is perpetuated.

Evolutionary Bargaining

Which bargaining norms are stochastically stable?

- To answer this question, we need to analyze transitions between conventions under the perturbed dynamic.
- Suppose that the process is in convention $(x, 1 - x)$.
- All transitions are 'local'. What is the probability of a 'downward' transition to convention $(x - \delta, 1 - x + \delta)$?

Evolutionary Bargaining

- Suppose there are i consecutive plays of $1 - x + \delta$ by column players, so that the next revising row player can draw a sample with i instances of $1 - x + \delta$.
- By reducing her demand to $x - \delta$, the row player gets $u(x - \delta)$ for certain.
- By retaining her demand x, the row player estimates that she gets $u(x)$ with prob. $1 - \frac{i}{am}$, and zero otherwise.

Evolutionary Bargaining

- Hence row players retain their demand if:

$$(1 - \tfrac{i}{am})u(x) \geq u(x - \delta).$$

- The critical value of i is:

$$i^*(x) = am\frac{u(x) - u(x - \delta)}{u(x)}.$$

Evolutionary Bargaining

- Similarly, a column player retains her demand of $1-x$ (rather than increasing it to $1-x+\delta$) when drawing a sample of j instances of $x-\delta$ if:

$$v(1-x) \geq \tfrac{j}{bm} v(1-x+\delta).$$

- The critical value of j is:

$$j^*(x) = bm\frac{v(1-x)}{v(1-x+\delta)}.$$

Evolutionary Bargaining

- Therefore it takes a minimum of $\lceil i^*(s) \rceil \wedge \lceil j^*(s) \rceil$ to induce a downward transition.
- When δ is sufficiently small, the first term is the smaller of the two (as loss from sticking to x is large relative to loss of δ from reducing demand, for δ small).
- Therefore the resistance of a downward transition is:

$$r(x, x - \delta) = \left\lceil am \frac{u(x) - u(x - \delta)}{u(x)} \right\rceil.$$

- Similarly, the resistance of an upward transition is:

$$r(x, x + \delta) = \left\lceil bm \frac{v(1 - x) - v(1 - x - \delta)}{v(1 - x)} \right\rceil.$$

Evolutionary Bargaining

- For δ sufficiently small, the resistances are well approximated by:

$$r(x, x - \delta) \approx \left\lceil am \frac{\delta u'(x)}{u(x)} \right\rceil.$$

$$r(x, x + \delta) \approx \left\lceil bm \frac{\delta v'(1 - x)}{v(1 - x)} \right\rceil.$$

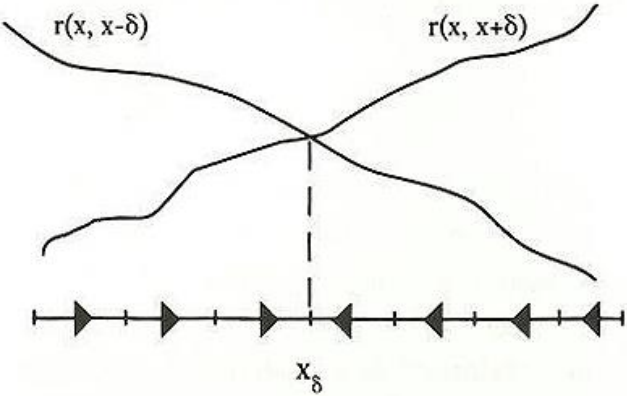
Evolutionary Bargaining

- Define the function:

$$f_\delta(x) = \mathsf{min}\{r(x, x+\delta), r(x, x-\delta)\}.$$

- $r(x, x+\delta)$ is an increasing function of x, whereas $r(x, x-\delta)$ is a decreasing function of x.
- Therefore, $f_\delta(x)$ is unimodal (see figure).
- Let x_δ be a maximizer of $f_\delta(x)$.

Spanning Tree



Evolutionary Bargaining

- The tree rooted at x_δ is the least resistant rooted tree.
- Hence the stochastically stable state(s) correspond to the convention(s) that maximize(s) $f_\delta(x)$.
- When δ is small and m is large relative to δ (so that there are no integer issues), any maximum of $f_\delta(x)$ lies close to the point x^* at which the two curves intersect:

$$a\frac{u'(x^*)}{u(x^*)} = b\frac{v'(1-x^*)}{v(1-x^*)}.$$

Evolutionary Bargaining

- Recall that this is simply the first-order condition that defines the asymmetric Nash bargaining solution:

$$a\frac{u'(x^*)}{u(x^*)} = b\frac{v'(1-x^*)}{v(1-x^*)}.$$

- Here one's bargaining power is determined by one's sample size.

Theorem 13.2 Consider random matching from two populations to play the discrete Nash demand game using the adaptive play protocol with memory m and sample sizes am and bm. As δ becomes small, the stochastically stable division(s) converge to the asymmetric Nash bargaining solution.