Bruce Francis and $H_{\infty}$ Control Theory

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Pramod P. Khargonekar  
University of California, Irvine
From the Preface to the book

“My aim in this book is to give an elementary treatment of linear control theory with an $H_\infty$ optimality criterion ...”

“Only one problem is solved in this book: how to design a controller which minimizes ...”

“It is a pleasure to express my gratitude to George Zames, Bill Helton, and John Doyle.”
Background and Context

Zames (1981) - “Feedback and optimal sensitivity ...”

Zames and Francis (1983), Francis and Zames (1983, 1984) - $H_\infty$ formalism and solution to the scalar case

Francis, Helton, Zames (1984) - Multivariable case solution

Doyle (1984) - Honeywell-ONR workshop lecture notes

Francis and Doyle (1986) - Linear control theory with an $H_\infty$ optimality criterion
Outline and Structure of the Book

Eight chapters, 132 pages

Background mathematics: Chapters 2, 5, 7

Background control: Chapters 3, 4

Scalar case solution: Chapter 6

Multivariable case: Chapter 8
Sets up the standard problem: minimize the norm of the closed loop transfer function $T_{zw}$.

Sets up stability of the closed-loop: all closed-loop transfer functions in Figure 3.2 are stable.
Two special cases:

A model matching problem: minimize the norm of

\[ T_1 - T_2 QT_3 \]

A tracking problem: minimize the norm of \( T_{zw} \)

where

\[ z = \begin{bmatrix} r - v \\ \rho u \end{bmatrix} \]
YJBK Parametrization via Coprime Factorization

\[ G = NM^{-1} = \tilde{M}^{-1}\tilde{N} \]

\[
\begin{bmatrix}
\tilde{X} & -\tilde{Y} \\
-\tilde{N} & \tilde{M}
\end{bmatrix}
\begin{bmatrix}
M & Y \\
N & X
\end{bmatrix} = I
\]
Lemma 1. The following are equivalent statements about $K$

(i) $K$ stabilizes $G$,

(ii) \[
\begin{bmatrix}
M & U \\
N & V
\end{bmatrix}^{-1} \in \text{RH}_\infty,
\]

(iii) \[
\begin{bmatrix}
\check{V} & -\check{U} \\
-\check{N} & \check{M}
\end{bmatrix}^{-1} \in \text{RH}_\infty.
\]

The main result of this chapter is the following.

Theorem 1. The set of all (proper real-rational) $K$'s stabilizing $G$ is parametrized by the formulas

\[
K = (Y-MQ)(X-NQ)^{-1}
\]

(2)

\[
= (\check{X} - Q\check{N})^{-1}(\check{Y} - Q\check{M})
\]

(3)

$Q \in \text{RH}_\infty$.

Proof. Let's first prove equality (3). Let $Q \in \text{RH}_\infty$. From (1) we have

\[
\begin{bmatrix}
I & Q \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\check{X} & -\check{Y} \\
-\check{N} & \check{M}
\end{bmatrix}
\begin{bmatrix}
M & Y \\
N & X
\end{bmatrix}
\begin{bmatrix}
I & -Q \\
0 & I
\end{bmatrix} = I
\]

so that

\[
\begin{bmatrix}
\check{X} - Q\check{N} & -(\check{Y} - Q\check{M}) \\
-\check{N} & \check{M}
\end{bmatrix}
\begin{bmatrix}
M & Y-MQ \\
N & X-NQ
\end{bmatrix} = I.
\]

(4)

Equating the (1,2)-blocks on each side in (4) gives

\[
(\check{X} - Q\check{N})(Y-MQ) = (\check{Y} - Q\check{M})(X-NQ),
\]

which is equivalent to (3).
Chapter 5: Hankel Operators and Nehari's Theorem

Notation

\[
\Pi_1 : L_2 \to H_2^\perp \quad \text{Orthogonal projection}
\]
\[
F \in L_\infty
\]
\[
\Lambda_F (g) = Fg, g \in L_2 \quad \text{Toeplitz operator}
\]
\[
\Gamma_F = \Pi_1 \Lambda_F \mid H_2 \quad \text{Hankel operator}
\]

Theorem: There exists a closest \( H_\infty \) matrix \( X \) to a given \( L_\infty \) matrix \( R \) and

\[
\| (R - X) \| = \| \Gamma_R \|
\]
Model Matching Problem – Scalar Case

Minimize norm of \( T_1 - T_2Q \)

Inner function: \( T(s) \) is inner if \( T(-s)T(s) = 1 \)

Outer function: \( T(s) \) is outer if it has no zeros in \( Re(s) > 0 \)

Inner-Outer Factorization: \( T(s) = T_i(s)T_o(s) \)

If \( T \) has no zeros on the imaginary axis then inverse of \( T_o \) is also in \( \mathbb{H}_\infty \)

\[
R := T_{2i}^{-1}T_1 \quad \quad X := T_{2o}Q
\]
Scalar Case Solution via Nehari Theorem

Theorem 1. The infimal model-matching error $\alpha$ equals $\|\Gamma_R\|$, the unique optimal $X$ equals $R - \alpha f / g$, and, for the optimal $Q$, $T_1 - T_2 Q$ is all-pass.

Proof. From Nehari's theorem there exists a function $X$ in $H_\infty$ such that

$$\|R - X\|_\infty = \|\Gamma_R\|.$$ (18)

It is claimed that

$$(R - X)g = \Gamma_R g.$$ (19)

To prove this, define $h := (R - X)g$ and look at the $L_2$-norm of $h - \Gamma_R g$:

$$\|h - \Gamma_R g\|_2^2 = \langle h - \Gamma_R g, h - \Gamma_R g \rangle.$$
Book

Elegant

Self contained

Focused

State-of-the-art

Sparingly written

Ideal for new graduate students
Remembering Bruce Francis: A Great Scholar and a True Gentleman

PRAMOD P. KHARGONEKAR
Thank you, Bruce, for being a role model and an inspiration.