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MEASURES OF RISK EQUITY

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This article covers measures of fairness or equity for situations involving risk, as represented with a probability distribution over outcomes, which often include adverse health or safety outcomes. For an introduction to fairness, the reader is referred to Keller *et al.* [1]. Such risk equity measures may be used in societal utility functions to identify preferred policies from the perspective of what is good for society as a whole. The aim of this article is to present enough detail on this topic so that a person can understand the basic concepts and find further references to get more details; this is not a complete compilation of all the research in this field.

Suppose two policies are being considered to improve public safety, and only one policy can be funded.

Safety Policy Scenario

Policy A: Install emergency phones along roadsides; this will save 10 lives annually.

Policy B: Install a flood warning system, with a 50% chance of saving 20 lives annually (when flood occurs and assume one flood only in that year) and a 50% chance of saving no lives (when no flood occurs).

One way of evaluating such alternative societal risk policy options might be to calculate expected fatalities, then choose the policy with the maximum number of lives saved. However, in this case, the expected number of lives saved is 10 for both policies.

To go beyond using the objective of maximizing expected number of lives saved to make this choice, one could construct a societal utility function over the number of lives. The utility of policy A is $u(\text{A}) = u(10$ lives saved). The utility of policy B is $u(\text{B}) = 0.50u(20 \text{ lives saved}) + 0.50u(0 \text{ lives saved})$. A risk-averse utility function would favor the choice of policy A, with the sure number of 10 lives saved. A risk-prone utility function would favor the choice of policy B.

While specifying a risk-averse or risk-prone utility function in this scenario will lead to a clear choice in this case, more generally one may wish to consider how these fatalities are distributed among the members or groups of society. In policy A, 10 lives are saved each year, from separate roadside incidents. In policy B, if there is a flood during a year, 20 lives will be saved from that one flood, and in nonflood years, no lives will be saved.

Keeney and Winkler [2] point out that

> In evaluating public risks, it may be useful to separate three issues:
>
> 1. The undesirability of fatalities, aside from equity considerations.
> 2. *Ex post* equity, meaning the equity associated with the fatalities that actually occur.
> 3. *Ex ante* equity, meaning the equity of the process and risks which eventually lead to the fatalities.

See the following sections titled "*Ex Ante* Equity," "*Ex Post* Equity," "Catastrophe Avoidance," and "Envy-Free Allocation," for details. The section titled "Decision Models Containing Equity Measures" presents equity models and applications. For other papers on the use of utility functions to represent preferences regarding societal equity, see Keeney [3–5], Broome [6], Fishburn [7], and Sarin [8].

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EQUITY AXIOMS

Individual Equity

Suppose there are two people at potential risk. (We often begin to examine equity measures by simplified scenarios involving just two people, and then extend the concepts to n people.) The concept of *individual equity* is that *society is indifferent if the two people switch their risk of being the sole person to die* and everything else stays the same.

This preference is contained in Fishburn's [7] *individual equity axiom E1*. To set up the notation for his axioms, which are stated for just two people, denoted as 1 and 2, he first identifies the four basic consequences and the chances of each:

p chance that (Person 1 lives, Person 2 dies),

q chance that (Person 1 dies, Person 2 lives),

r chance that (Person 1 lives, Person 2 lives) = both live, and

$1-(p+q+r)$ = #chance that (Person 1 dies, Person 2 dies) = both die.

We will arbitrarily scale the societal utility function with two end points: Suppose that the certainty (i.e., the probability $r = 1$) that both live has utility 2 and the certainty (i.e., the probability $1-(p+q+r) = 1$) that both die has utility 0. Denote the utility of person 1 living alone as a_1 and the utility of person 2 living alone as a_2, with both a_1 and a_2 between 0 and 2.

Having set the utility of both people dying as 0 (the minimum possible utility) and the utility of both people living as 2 (the maximum possible utility), there are only two remaining utilities to specify. They are

a_1 = the utility of person 1 living and person 2 dying

a_2 = the utility of person 1 dying and person 2 living.

These utilities (a_1 and a_2) might be equal to each other or unequal. They might both equal 1, or both be under 1 or above 1, and so on. Different equity axioms lead to specific possible ranges of values for the utilities of these two intermediate outcomes.

For each decision scenario, we want to represent the societal utility for a policy which will result in the four basic consequences with some probability of each. This societal utility can be conveniently written, using the probabilities of the four basic consequences, and suppressing the detailed description of the consequences, as

$$u(p,q,r,1-(p+q+r)).$$

(We will use this short-hand notation whenever we present Fishburn's axioms in the sections on equity axioms.)

So, we compute the vonNeumann–Morgenstern expected utility of a policy as follows:

$$\begin{aligned}
u(\text{policy}) &= u(p,q,r,1-(p+q+r)) \\
&= pu(\text{Person 1 lives, Person 2 dies}) \\
&\quad + qu(\text{Person 1 dies, Person 2 lives}) \\
&\quad + ru(\text{Person 1 lives, Person 2 lives}) \\
&\quad + (1-(p+q+r))u(\text{Person 1 dies,} \\
&\qquad \text{Person 2 dies}) \\
&= p(a_1) + q(a_2) + r(2) + (1-(p+q+r))0 \\
&= a_1 p + a_2 q + 2r
\end{aligned}$$

Different equity axioms result in different possible values of u(Person 1 lives, Person 2 dies) $= a_1$ and u(Person 1 dies, Person 2 lives) $= a_2$.

Using our notation, the assumption of *individual equity* is that

$$u(p,q,r,\#) = u(q,p,r,\#),$$

where # represents the probability of the fourth consequence = 1 − the sum of the first three probabilities, which, on the left side of the equation is $1-(p+q+r)$.

Under this individual equity axiom, the utility of the left-hand side is $a_1p + a_2q + 2r$. The utility of the right-hand side is $a_1q + a_2p + 2r$. They are equal, so

$$a_1p + a_2q + 2r = a_1q + a_2p + 2r.$$

Subtracting $2r$ from both sides,

$$a_1 p + a_2 q = a_1 q + a_2 p.$$

Rearranging, we get

$$a_1(p - q) = a_2(p - q),$$

and thus

$$a_1 = a_2.$$

Ex Ante Equity

Ex ante equity can be seen "before the fact" of a risk event being resolved. A person who prefers *ex ante* equity will choose an equal distribution of risk, where risk is defined by the probability of an individual becoming a fatality, over an unequal distribution.

Consider the scenario below.

Ex Ante Equity Scenario

Alternative A: 50% chance that Person 1 lives, Person 2 dies
50% chance that Person 1 dies, Person 2 lives.

Alternative B: Person 1 lives,
Person 2 dies.

In Alternative B, one person dies for sure, and we know that it will be person 2. In Alternative A, one person dies for sure, also. So, if a societal decision maker wishes to use a von Neumann–Morgenstern utility function defined over number of fatalities (only), then the model will be indifferent between the two alternatives.

Before the uncertainty in Alternative A is resolved, each person has an equal 0.5 chance of death, so they share in facing the risk. Alternative B has an unequal distribution, with person 1 having a probability of 0 of death and person 2 having a probability of 1. A decision maker who prefers Alternative A, because it "is fairer since each person has an equal chance of surviving," is demonstrating a preference for *ex ante* equity.

This seemingly reasonable assumption has been debated in the literature and has caused some to conclude that the decision analytic approach (when considering only the number of fatalities rather than the distribution of risk) cannot satisfactorily address this equity issue; see Broome [6].

Risk-Sharing Equity. To generalize the notion of *ex ante* equity from the scenario above, recall Fishburn's notation introduced in the section titled "Individual Equity," where p and q refer to the probabilities:

p chance that (Person 1 lives, Person 2 dies)

q chance that (Person 1 dies, Person 2 lives).

According to Fishburn's [7] *risk-sharing equity axiom E2*,

$$\text{if } p + q = p^* + q^* \text{ and } |p - q| < |p^* - q^*| \text{ then}$$
$$u(p, q, r, \#) > u(p^*, q^*, r, \#).$$

In the scenario above, $p = q = 0.5$ in Alternative A, and in Alternative B we have $p^* = 1$ and $q^* = 0$, so according to the risk-sharing equity axiom, $u(\text{A}) > u(\text{B})$.

Independence Risk Equity. Next, we consider a preference for equal *independent* probabilities of death. The *independent risk equity axiom E3* by Fishburn [7] considers the two person's *ex ante* independent probabilities of living (rather than dying), and states it is better if the probabilities are closer together (rather than farther apart). Let α be the probability that person 1 lives and β be the probability that person 2 lives. Each can live alone or with the other person.

Consider Option 1 with the following probabilities and outcomes:

$\alpha(1 - \beta)$ p chance that Person 1 lives,
Person 2 dies,

$(1 - \alpha)\beta$ q chance that Person 1 dies, Person 2 lives,

$\alpha\beta$ r chance that Person 1 lives, Person 2 lives, i.e. both live, or

$\#$ $1 - (p + q + r)$ chance that Person 1 dies, Person 2 dies, i.e. both die.

Person 1 can live alone with the probability $\alpha(1 - \beta)$, or can live with Person 2 with the probability $\alpha\beta$. So, person 1 has a probability of living $= \alpha = \alpha(1 - \beta) + \alpha\beta$ and Person 2 has a β probability of living, where $\beta = (1 - \alpha)\beta + \alpha\beta$.

Option 2 has the same structure, but α and β are replaced with α^* and β^*.

Independence Risk Equity Axiom.

$$\text{If } \alpha + \beta = \alpha^* + \beta^* \text{ and}$$
$$|\alpha - \beta| < |\alpha^* - \beta^*|, \text{ then}$$
$$u(\text{Option 1}) = u(\alpha(1 - \beta), (1 - \alpha)\beta, \alpha\beta, \#)$$
$$> u(\text{Option 2}) =$$
$$u(\alpha^*(1 - \beta^*), \quad (1 - \alpha^*)\beta^*, \alpha^*\beta^*, \#).$$

So, a more even division of the unconditional probabilities of living is preferred. This is the same as Keeney's [5] risk equity assumption under independence between the unconditional events "1 dies" and "2 dies." Under this axiom, u(Person 1 lives, Person 2 dies) $= a_1 = u$(Person 1 dies, Person 2 lives) $= a_2 < 1$.

Consider the scenario below given by Keller and Sarin [9] to Americans and by Bian and Keller [10] to Chinese.

Serum Distribution Scenario. There are 100 islanders who are susceptible to a specific fatal disease which has recently appeared on the mainland. Scientists have identified a kind of serum which has the potential of protecting people from contracting the disease. *Unfortunately, there is not enough serum available to give all the susceptible islanders a high enough dose to successfully prevent the disease.* Action must be taken immediately to protect the public health. *All susceptible people must be injected with the serum within 24 hours, or each will have a 15% chance of contracting the disease and eventually dying.* There is no time to acquire more serum. There are only 3000 milligrams of the serum available. As the public health officer, it is your job to choose between the following options. Circle your choice.

A. Give the same low dose of 30 milligrams of serum to all 100 susceptible islanders. 50 of those susceptible are northerners, 50 are southerners. Each susceptible person will have an independent 10% chance of dying. The expected number of deaths is 10. (Considered more fair and chosen by most Americans and Chinese)

B. Divide up the available serum among the 50 northerners who are susceptible to the disease. Thus, these people will receive a higher 60 milligram dose. Each of the 50 will now have an independent 5% chance of dying. Since the 50 susceptible southerners will receive none of the serum, each will still have a 15% chance of dying by contracting this disease. The expected number of deaths is 10.

Preference for equal independent probabilities of death in option A reveals a preference for independence risk equity. It is clear from the survey (in which half the subjects reported which option was fairer and the other half reported their chosen action) that the subjects' preference for option A has resulted from it being a fairer choice; however, technically speaking, such a choice can be explained by assuming that their utility functions defined over number of fatalities are risk prone. See the section titled "Catastrophe Avoidance" for a discussion of utility function shapes.

Measure of *Ex Ante* Equity Φ. There are different ways to measure *ex ante* equity quantitatively. A specific measure of *ex ante* equity Φ is one that penalizes unevenness across groups in terms of the average probability of death $\bar{p}_i$ for a person in group i, $i = 1$ to m [9]. Presuming there are roughly equal numbers of people in each group, a special form for Φ is defined by

$$\Phi = -\sum_{i=1}^{m} \bar{p}_i - \bar{p} \,,$$

where the average of the probabilities of death for the m groups is $\bar{p} = \sum_{i=1}^{m} \bar{p}_i / m$. For the simple *ex ante* equity scenario above at the beginning of the section titled "*Ex Ante* Equity," there are two groups, each having one person in it.

For Alternative A,

$$\Phi = -(\,0.5 - 0.5 + 0.5 - 0.5\,) = 0,$$

For Alternative B,

$$\Phi = -(\,1 - 0.5 + 0 - 0.5\,) = -1.$$

So, Alternative A has the higher (and presumably better) *ex ante* equity. See the section titled "Fair-Risk Model, Incorporating *Ex Post* and *Ex Ante* Equity Preferences" for a societal utility function which includes Φ.

For another example of preference for *ex ante* equity, see the rescuer at risk scenario in ***Fairness and Equity in Societal Decision Analysis***.

Ex Post Equity

A person who prefers *ex post* equity will choose an equal *ex post* distribution of final outcomes for the affected people, instead of an unequal distribution [7].

Consider the scenario below. First note that, in terms of *ex ante* equity, both alternatives are equal, since both lead to an *ex ante* probability of 0.5 that Person 1 dies and 0.5 that Person 2 dies. Thus, a decision maker cannot distinguish between the two alternatives in terms of *ex ante* equity. Now examine the alternatives below in terms of *ex post* equity.

Ex Post Equity Scenario

Alternative C: 50% chance that Person 1 lives, Person 2 lives
50% chance that Person 1 dies, Person 2 dies

Alternative A: 50% chance that Person 1 lives, Person 2 dies
50% chance that Person 1 dies, Person 2 lives

After the uncertainty is resolved in Alternative C, the *ex post* outcome is either both people will be dead or both people will be alive. In Alternative A, the *ex post* outcome will be that one person is alive and one is dead. So, Alternative C will be chosen by a person preferring *ex post* equity.

Common-Fate Equity. Common-fate equity is a typical kind of *ex post* equity. Note that in Alternative C above, after the fact of the risk event occurring (or not) the two people will share a common fate (either both living or both dying), no matter what happens. In Alternative A, they will have different fates, no matter what happens. In many situations, people will prefer to share a common fate. But, in some situations, such as when top executives of a company or parents of young children take air flights, they may take different flights to avoid the common fate of dying in the same airplane crash. In both of these situations, they are thinking of their responsibilities to their company or family.

Fishburn's [7] *common-fate equity* axiom E4 states

$$u(p - \delta, q - \delta, r + \delta, \#) > u(p, q, r, \#).$$

In this axiom, it is an improvement to move the probability up that the two people share common fates (which occurs in consequence 3: both living, and consequence 4: both dying) and move the probability down that they have unequal fates (in consequence 1: only Person 1 lives and consequence 2: only Person 2 lives). Under this axiom, u(Person 1 lives, Person 2 dies) + u(Person 1 dies, Person 2 lives) = $a_1 + a_2 < 2$.

Gajdos *et al.* [11] present a new method for modeling preference for (or against) what they call "shared destinies," which weakens a necessary and sufficient condition by Fishburn and Straffin [12] for two societal risk distributions to be judged to be indifferent whenever their associated distributions of risk of death for individuals and for the number of fatalities are the same.

For an example of common-fate equity, see the miner location scenario discussed in Keller and Sarin [9]; see also Section 3 in ***Impossibility Theorems and Voting Paradoxes in Collective Choice Theory***.

Also, in the section titled "Catastrophe Avoidance," Alternative D has common-fate equity, with either all 100 dying or all 100 living.

Common-Fate Independence. Fishburn's [7] *common-fate independence* axiom E5 states

$$\text{If } p + r = p^* + r^* \text{ and } q + r = q^* + r^*,$$
$$\text{then } u(p, q, r, \#) = u(p^*, q^*, r^*, \#).$$

In this axiom, it does not matter how a person dies, as long as the probability of dying remains the same. As long as a person's probability of dying is constant, the probability can be split in any division between dying alone and dying together. This is a different kind of *ex post* equity. Under this axiom, u(Person 1 lives, Person 2 dies) + u(Person 1 dies, Person 2 lives) = $a_1 + a_2 = 2$.

***Ex Post* Equity Measure θ.** As one example of an *ex post* equity measure, Keller and Sarin [9] suggest the following equity measure, θ, which depends on the distribution of the number of fatalities among the m groups. Suppose there are n_i fatalities in group i, $i = 1$ to m. Then

$$\theta = \sum_{n_m=0}^{N_m} \sum_{n_{m-1}=0}^{N_{m-1}} \cdots \sum_{n_1=0}^{N_1} \theta(n_1, n_2, \ldots, n_m)\, \pi(n_1, n_2, \ldots, n_m),$$

where π is the joint probability mass function that can be computed from the estimates of risks to members in each group. N_i is the maximum possible number of fatalities in group i, $i = 1$ to m.

If $\bar{n}$ is the expected number of fatalities per group and each group experiences exactly $\bar{n}$ fatalities, there is no *ex post* inequity, from the perspective of a comparison across groups. So, any deviation from $\bar{n}$ is an indication of *ex post* inequity, especially when the size of the population in each group is approximately the same or is substantially larger than the number of fatalities actually experienced. In this case, Keller and Sarin [9] define the *ex post* equity of a specified number of fatalities n_i in each group i as

$$\theta(n_1, n_2, \ldots, n_m) = -\sum_{i=1}^{m} (n_i - \bar{n})^2.$$

For the simple *ex post* equity scenario above at the beginning of the section titled "*Ex Post* Equity," there are two groups ($i = 1$ and 2), each having one person in it. The data to compute θ are organized in the following table for a generic policy alternative.

In each row in the table, multiply the probability from column 1 times $\theta(n_1, n_2)$ in the last column, then sum up the four products to get the overall *ex post* equity measure θ.

In the *ex post* equity scenario,

Alternative C has $p = 0$, $q = 0$, $r = 0.5$ and $\# = 0.5$, so $\theta = 0.5(0) + 0.5(0) = 0$.

Alternative A has $p = 0.5$, $q = 0.5$, $r = 0$ and $\# = 0$, so $\theta = 0.5(-0.5) + 0.5(-0.5) = -0.5$.

Higher numbers indicate better performance in terms of *ex post* equity, so Alternative C has the best *ex post* equity, with $\theta = 0$. One can interpret this as zero inequity. More negative numbers show greater *ex post* inequity. See the section below on fair-risk models for a societal utility function which includes θ.

Catastrophe Avoidance

Merely calculating expected number of lives lost and choosing protective actions that minimize the number of expected lives lost can lead to choices that might not be agreed upon by members of society. By observation of the nightly TV news we see that the public often reacts more to disasters when 100 people are lost all at once rather than 100 less newsworthy smaller events when one person at a time dies. It is as if the disutility of 100 deaths at once is greater than 100 times the disutility of 1 death.

Probability of Specific Number of Deaths in Group 1 and Group 2 $\pi(n_1, n_2)$	Group 1 Fate of Person 1	Number of Fatalities in Group 1 n_1	Group 2 Fate of Person 2	Number of Fatalities in Group 2 n_2	Average Number of Fatalities $\bar{n}$	$\theta(n_1, n_2)$
p chance	Person 1 lives	0	Person 2 dies	1	0.5	$-((0-0.5)^2 + (1-0.5)^2) = -0.5$
q chance	Person 1 dies	1	Person 2 lives	0	0.5	$-((1-0.5)^2 + (0-0.5)^2) = -0.5$
r chance	Person 1 lives	0	Person 2 lives	0	0	$-((0-0)^2 + (0-0)^2) = 0$
$1-(p+q+r) = \#$ chance	Person 1 dies	1	Person 2 dies	1	1	$-((1-1)^2 + (1-1)^2) = 0$

According to Keeney's [5] catastrophe-avoidance condition and Fishburn's [7] *catastrophe-avoidance preference axiom E6*,

$$u(p, q, 1-p-q, 0) > u(0, 0, 1-(p+q)/2, \#).$$

In this axiom, it is better for at least one person to survive (just Person 1 in consequence 1 or just Person 2 in consequence 2) than to have both die when the expected number of deaths is equal. Under this axiom, u(Person 1 lives, Person 2 dies) $= a_1 > 1$ and u(Person 1 dies, Person 2 lives) $= a_2 > 1$.

Consider the scenario below given by Keller and Sarin [9] to Americans and by Bian and Keller [10] to Chinese. In this scenario, both alternatives have an expected loss of one life, so in terms of expected number of fatalities, they are equal.

Serum-Producing Scenario.

One hundred islanders were born highly susceptible to contracting a fatal disease. Recently, it was discovered that the presence of a naturally occurring noxious gas led to this condition and the gas has been eradicated. However, there is still some chance of the islanders contracting the disease and thus dying. You could decide to give an injection to all 100 islanders. This injection will prevent everyone from contracting the disease. However, the serum for the injection can only be obtained from the blood of a person who has artificially been made to contract the fatal disease. The serum cannot be obtained from a person who has naturally contracted the disease, so you cannot just wait to see if one person contracts the disease and then make the serum from the sick person's blood. If one islander is sacrificed by being made to contract the disease, enough serum will be obtained to eliminate the risk of death to the remaining 99 islanders. If nothing is done, there is a 1% chance of an epidemic breaking out in which all 100 islanders will contract the disease and thus die. There is a 99% chance that no epidemic will break out, so all 100 islanders will live.

The two options are summarized below. Circle your choice/the option that is fairer.

Alternative D: Do nothing, and thus take a 1% chance of all 100 islanders dying (*Considered fairer by most Americans and Chinese, and chosen by most Americans*).

Alternative E: Sacrifice one islander (*Chosen by most Chinese*).

Alternative D was seen as fairer by most Chinese and Americans. It was chosen as an action by most Americans. It has the chance of the catastrophe of all 100 dying, yet each person has an equal 1% chance of dying. Thus, there is *ex ante* equity and *ex post* equity. There is also common-fate equity, since in one outcome all 100 live and in the other outcome all 100 die.[1]

Alternative E avoids any risk of catastrophe, but involves one specific person dying for sure. Most Chinese chose Alternative E, so they preferred *catastrophe avoidance*. It may be that the Chinese cultural value of collectivism led Chinese to protect the 100 person group from all dying, even if one individual had to be sacrificed. In contrast, more Americans may have been led by the American cultural value of individualism to choose Alternative D.

Keeney [4,5] shows that a preference for more equitable distributions of risk, like in the "Do nothing" Alternative D, implies a risk-prone attitude and that catastrophe avoidance, like in the "Sacrifice 1 islander" Alternative E, reveals a risk-averse attitude. Table 1 provides examples of utility functions over number of fatalities which are risk prone, risk neutral, or risk averse. The scale is from 0 to 1, but can be rescaled from −1 to 0, if decision makers prefer to use negative numbers for outcomes of deaths. (See the section titled "Utility Function Demonstrating a Preference for *Ex Post* Equity" for another risk-prone utility function.)

Envy-Free Allocation

Imagine you give some cookies to two children. When one child whines "That's not fair," what is meant by this complaint? It could mean that she *envies* the cookie allocation given to the other child, in comparison to her own cookie allocation.

The concept of an *envy-free allocation* means that neither child wants to switch his allocation with the other child.

An envy-free cookie allocation might be attained by following a process where the first child divides the cookies and the second child chooses which of two piles of cookies to take, leaving the rejected pile for the child who divided them.

Envy-Free Axiom.

> The chosen allocation should be envy free. An allocation (of risks and/or benefits) between two people is envy free if neither person would want to switch his/her allocation with the other person.

Keller and Sarin [9] discuss the idea of an envy-free allocation of risks and benefits as a way to resolve controversial facility siting problems. See the section titled "Envy-Free Allocation Model."

DECISION MODELS CONTAINING EQUITY MEASURES

One purpose of identifying equity axioms is to examine what equity principles are satisfied in different existing decision models. A related purpose is to build a decision model based upon the desired set of equity axioms. This section provides a few examples of decision models incorporating a variety of equity preferences and applications. For more models see papers cited earlier and Refs 15–29 for additional references on societal equity.

Utility Function Demonstrating a Preference for *Ex Post* Equity

If societal decision makers prefer more equitable distributions of *ex post* risk, then they may choose to evaluate each individual at risk identically to all others [2,4]. A utility function over the number of fatalities can sufficiently summarize preference for this concept of *ex post* equity of public risk. One specific utility function form examined in

[1]This scenario's context is a modification of one presented in Hammerton *et al.* [13]. Fischhoff [14] used the same probabilities and outcomes to illustrate framing effects and only 29% chose the sure-loss option E. Thus, subjects generally preferred the more equitable option D in a civil defense context also.

Table 1. Examples of Utility Functions Over Number of Fatalities

Utility in Serum-Producing Scenario	Risk Prone: Prefer D (Do Nothing) over E (Sacrifice One Islander)	Risk Neutral: Indifferent between D (Do Nothing) and E (Sacrifice One Islander)	Risk Averse: Prefer E (Sacrifice One Islander) Over D (Do Nothing)
Utility of 0 deaths	1.000	1.000	1.000
Utility of 1 death	0.950	0.990	0.995
Utility of 100 deaths	0.000	0.000	0.000

Keeney [4] is a constantly risk-prone utility function $u(y)$, over the number of fatalities y:

$$u(y) = (1/d)[(1-d)^y - 1], 0 < d < 1,$$
$$y = 0, 1, 2, \ldots, N,$$

where u(0 fatalities) $= 0$ and u(1 fatality) $= -1$ to set the origin and scale. Figure 1 below shows the utility over number of fatalities y when the curvature parameter $d = 0.01$. In this convex utility curve with $d = 0.01$, $u(1) = -1$, u(10 fatalities) $= -9.56$, u(20 fatalities) $= -18.21$, u(100 fatalities) $= -64.40$, and u(200 fatalities) $= -86.60$. For smaller values of d, the curve is more linear. For $d = 0.0000001$, the utility curve is nearly linear. Also since $\lim_{y\to+\infty} u(y) = -1/d$, the negative reciprocal of d can be interpreted as the lower bound on the utility due to the societal impact of a tragedy where y is large. Alternatively, d can also be interpreted as the ratio of the societal utility of the first involuntary risk fatality to that of a large tragedy.

To include preference over fatalities, the utility function should be monotonically decreasing since fewer fatalities are always preferred; To include preference for *ex post* equity, the utility function should be risk prone (convex) since a lottery with an expected number of fatalities should be preferred to a sure consequence with the same fatalities which has less spread over society members. The utility function above is both monotonically decreasing and convex. Thus the utility function above addresses both attitudes toward fatalities and *ex post* equity without separating them into individual components.

Fair-Risk Model, Incorporating *Ex Post* and *Ex Ante* Equity Preferences

Keller and Sarin [9] present a simple weighted additive model that includes the number of fatalities, *ex ante* equity of the distribution of the risks, and the *ex post* equity of final consequences. (Keeney and Winkler [2] propose a similar weighted additive structure, in their Equation (3).) Suppose the N_i members in each of m groups, indexed by i, have an independent probability p_i of becoming a fatality. The *fair-risk model* is

$$K_Y U_Y(y) + K_\theta \theta + K_\Phi \Phi,$$

where U_Y is the utility function defined over number of fatalities y, θ is the *ex post* equity measure defined in the respective section, Φ is the *ex ante* equity measure defined in its respective section, and K_Y, K_θ, and K_Φ are scaling constants that reflect trade-offs among the three attributes. A simple utility function U_Y for total number of fatalities is the negative of the expected number of fatalities.

Examination of Antarctica Operations in Terms of Equity to Different Groups

Broder and Keller [17] provide examples of equity measures applied to Antarctica operations. They created two stylized scenarios, based on interviews with US Antarctica operations managers at the US National Science Foundation and reviews of planning documents. In scenario 1, the *status quo* situation was represented, with different fatality risks from a tourist airplane crash or a fatal emergency in Antarctica operations for 3000 tourists, 50 rescue workers, and 950

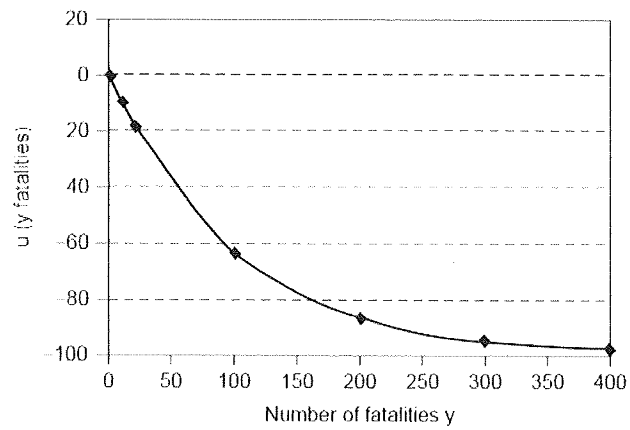

Figure 1. Utility function over number of fatalities when $d = 0.01$.

other Antarctica workers. In scenario 2, a streamlining plan then under consideration was represented in a stylized example where the other Antarctic workers were decreased to 900, and the probabilities and numbers of deaths in groups changed a bit. They then numerically calculated the following equity measures, based on formulae proposed in Fishburn and Sarin [28]:

- individual risk (*ex ante*) inequity;
- group risk (*ex post*) inequity;
  - intergroup inequity (nonuniformity of expected fatality rates across groups);
  - within-group risk inequity;
  - total group risk inequity, that is, combining intergroup and within-group inequity measures,
  - social outcome inequity, assuming common-fate preferences;
- dispersive inequity, for example, risks to scientists versus support staff compared with risks to governmental versus nongovernmental workers, when there are overlapping memberships in these groups (since there are governmental scientists, nongovernmental scientists, governmental support staff, and nongovernmental support staff).

Envy-Free Allocation Model

Keller and Sarin [9] presented the following envy-free risk benefit model, which incorporates the concept of an envy-free allocation, as described in the corresponding section. This model also includes information on both the risks and the benefits received by each party. For scenarios involving risks and benefits, see Keller and Sarin's [9] scenario 6 and the scenarios in Keller and Sarin [15].

The model argues that an allocation of benefits b_i and risks r_i to each group i expressed as $(b_1, r_1; b_2, r_2; \ldots; b_m, r_m)$ is fair if $u_i(b_i, r_i) \geq u_i(b_j, r_j)$ for $i = 1$ to m, $j = 1$ to m. Then if we assume that a decision maker has a utility function u_D, which is defined over the utility functions u_i of each of the $i = 1$ to m groups, the following mathematical program will give you the fair allocation:

Envy-Free Risk–Benefit Model

Maximize $u_D(u_1, u_2, \ldots, u_m)$

Subject to

$$u_i(b_i, r_i) \geq u_i(b_j, r_j), i = 1 \text{ to } m, \quad j = 1 \text{ to } m$$

(envy-free allocations)

$$\sum_{i=1}^{m} b_i = 1 \quad \text{(all benefits allocated)}$$

$$\sum_{i=1}^{m} r_i = 1 \quad \text{(all risks allocated)}$$

$$b_i, r_i \geq 0, i = 1 \text{ to } m$$

$$\text{(nonnegative variables).}$$

In the above model u_D can be regarded as a social welfare function. u_D increases in group i's utility u_i. Group i's utility u_i increases in the benefits b_i and decreases in the risks r_i.

SUMMARY

This article has provided an introduction to a number of *ex ante* and *ex post* equity concepts. When guiding societal policy making, such concepts will often be part of the discussion. They can be dealt with qualitatively in policy discussions, or be formally incorporated into preference models for guiding decision making. Some examples of societal utility function models are discussed in the section titled "Decision Models Containing Equity Measures." See Refs 15–29 for additional references on societal equity, and the article titled ***Fairness and Equity in Societal Decision Analysis***.

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