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Preference Functions for Spatial Risk Analysis

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When outcomes are defined over a geographic region, measures of spatial risk regarding these outcomes can be more complex than traditional measures of risk. One of the main challenges is the need for a cardinal preference function that incorporates the spatial nature of the outcomes. We explore preference conditions that will yield the existence of spatial measurable value and utility functions, and discuss their application to spatial risk analysis. We also present a simple example on household freshwater usage across regions to demonstrate how such functions can be assessed and applied.

KEY WORDS: Measurable value; preferences; spatial risk analysis; utility

1. INTRODUCTION

In decision and risk analyses, geographically varying outcomes are fundamentally more complex than more traditional types of outcomes. To aid decisionmakers who use geographical data, more sophisticated functions must be developed to capture these decisionmakers' preferences accurately. It is usually not sufficient merely to list the positive or negative impacts in each region of different policies. Simple averages of the impacts across space will likely be too simplified, since decisionmakers will have preferences over the outcome levels to be taken into consideration. If relatively simple conditions are satisfied, it is possible to represent ordinal preferences over spatial outcomes using additive spatial multiattribute value functions.[1] However, in the context of risk analysis specifically, ordinal value functions may not be sufficient to capture the relevant preferences of decisionmakers, and measurable value functions or utility functions may be needed. This article develops the theory supporting measurable spatial value functions and spatial utility functions for risk analysis, and provides an illustrative example.

A spatial outcome can be expressed in terms of levels of one or more attributes over a set of geographic regions. The preference models developed in this article could be used in spatial decision support systems as presented by Ferretti and Montibeller.[2] Malczewski[3] reviews many instances of relevant spatial decision problems. These preference models can also be used to support spatial risk analyses, for instance, in the context of epidemics presented by Zagmutt *et al.*,[4] or incorporating vulnerability to natural disasters, as explored by Zhou *et al.*[5]

We will first develop the theory supporting several spatial preference models in Section 2, and then present an example of how one of these models (an additive measurable spatial value function) can be assessed and implemented in Section 3. Section 4 concludes the article.

2. MODELS

Consider a geographic space that is divided into m discrete regions indexed by $i = 1, \ldots, m$. An outcome will consist of a vector of levels of one or more attributes in each of the m regions. The remainder of this section is divided into four subsections. The first two subsections establish the cases in

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which measurable value functions are needed, and the last two establish the cases in which utility functions are needed. For both measurable value functions and utility functions, one subsection considers only a single attribute, while the other considers multiple attributes. The treatments of spatial value and spatial utility functions in this article are not completely parallel, as previous supporting work differs between the two.

Some of the preference conditions described in this article may seem restrictive. However, it is important to note that the use of expected values or other common summary metrics for decision making asserts conditions (implicitly or explicitly) that are *much* stronger; some such metrics are special cases of the models in this article. Simon *et al.*[1] discuss this issue in more depth.

2.1. Single-Attribute Measurable Spatial Value

Consider a decision under certainty with a single attribute Z, where z_i denotes the level of Z in region i. Let $\mathbf{Z}$ denote the set of attribute-level vectors $\mathbf{z} = (z_1, \ldots, z_m)$; that is, $\mathbf{Z}$ is the state space of $\mathbf{z}$. The preference relation over $\mathbf{Z}$ will be denoted as $\succsim$, with corresponding strict relation $\succ$ and indifference relation $\sim$. Then, relatively simple preference conditions will establish the existence of an additive ordinal spatial value function of the form:

$$V(z_1, \ldots, z_m) = \sum_{i=1}^{m} a_i v(z_i), \tag{1}$$

as shown by Simon *et al.*,[1] based on ordinal preference theory developed by Debreu,[6–8] Gorman,[9] Fishburn,[10] and Krantz *et al.*[11] In Equation (1), the a_i represent weights assigned to each region, and v is a single-region value function over the levels of Z. There are two notable preference conditions required to establish Equation (1): *pairwise spatial preferential independence* and *homogeneity*. Intuitively, pairwise spatial preferential independence asserts that value tradeoffs between attribute levels in two given regions do not depend on common attribute levels in the other regions, and homogeneity asserts that relative preferences for different levels of the attribute do not depend on the region. Simon *et al.* provide detailed specifications of these two conditions.

The function in Equation (1), however, is ordinal, and therefore does not provide any information about the relative preference differences between outcomes. Thus, while it is helpful in a decision analysis context to determine the rank order of alternatives, decisionmakers may wish to know how close alternatives are to one another. To show the existence of a spatial preference function incorporating information about preference differences, we adapt approaches used by Krantz *et al.*[11] and Dyer and Sarin.[12,13] First, let the binary relation $\succsim^*$ denote a preference difference relation over the set of pairs of outcomes, with corresponding strict and indifference relations defined in the standard manner. Consider four outcomes $\mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^3, \mathbf{z}^4 \in \mathbf{Z}$. Then, $\mathbf{z}^1\mathbf{z}^2 \succsim^* \mathbf{z}^3\mathbf{z}^4$ means that the strength of preference for $\mathbf{z}^1$ over $\mathbf{z}^2$ is at least as great as the strength of preference for $\mathbf{z}^3$ over $\mathbf{z}^4$. Two noteworthy conditions on $\succsim^*$ are needed. First, assume that the conditions on $\succsim$ needed for Equation (1) are met, and let $\bar{\mathbf{z}}_i \in \bar{\mathbf{Z}}_i$ denote a vector of attribute levels in all regions except for i. Then:

Definition 1. $\succsim^*$ *satisfies difference consistency if:*

1. For any region i, for all attribute levels z_i^1 and z_i^2, $(z_i^1, \bar{\mathbf{z}}_i) \succsim (z_i^2, \bar{\mathbf{z}}_i)$ if and only if $(z_i^1, \bar{\mathbf{z}}_i)(z_i^c, \bar{\mathbf{z}}_i) \succsim^* (z_i^2, \bar{\mathbf{z}}_i)(z_i^c, \bar{\mathbf{z}}_i)$ for some z_i^c and $\bar{\mathbf{z}}_i$.

2. For any $\mathbf{z}^1, \mathbf{z}^2 \in \mathbf{Z}$, if $\mathbf{z}^1 \sim \mathbf{z}^2$, then $\mathbf{z}^1\mathbf{z}^3 \sim^* \mathbf{z}^2\mathbf{z}^3$ for any $\mathbf{z}^3 \in \mathbf{Z}$.

Intuitively, difference consistency asserts that $\succsim^*$ is consistent with $\succsim$.

Definition 2. $\succsim^*$ *satisfies difference independence if for any region i, for all attribute levels z_i^1 and z_i^2 such that $(z_i^1, \bar{\mathbf{z}}_i) \succsim (z_i^2, \bar{\mathbf{z}}_i)$ for some $\bar{\mathbf{z}}_i \in \bar{\mathbf{Z}}_i$, $(z_i^1, \bar{\mathbf{z}}_i)(z_i^2, \bar{\mathbf{z}}_i) \sim^* (z_i^1, \bar{\mathbf{z}}_i')(z_i^2, \bar{\mathbf{z}}_i')$ for all $\bar{\mathbf{z}}_i' \in \bar{\mathbf{Z}}_i$.*

Difference independence asserts that preference differences between spatial outcomes are not affected by attribute levels in regions for which the attribute levels do not differ between outcomes. See Fig. 1 for an illustration of how to verify difference independence. In Fig. 1, the relevant attribute-level differences occur in the North region ($i = 1$); each region color and shading pattern represents an arbitrary attribute level such that the indifference relation holds. A similar process could verify difference consistency.

Difference independence precludes more complex types of preference interactions between regions. For example, if the attribute being analyzed is groundwater level, and groundwater can be transported easily between regions once extracted, then the value of an improvement in one region might depend on the quantities of groundwater in other regions. Such a relationship can be handled by as-

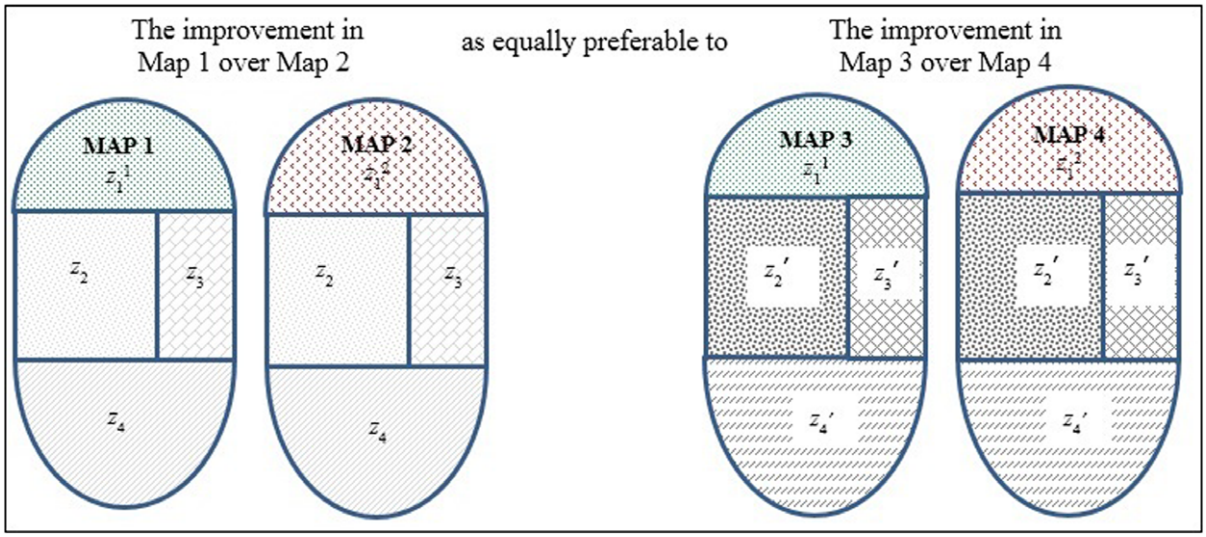

Fig. 1. Difference independence holds when a decisionmaker judges . . .

sessing more complicated preference functions than those used in this article, or in some simpler cases by redefining the attribute or aggregating regions.

Theorem 1. *For $\succsim$ defined over $\mathbf{Z}$ and $\succsim^*$ defined over $\mathbf{Z} \times \mathbf{Z}$, there exists a measurable spatial value function of the same form as Equation* (1) *if and only if pairwise spatial preferential independence, homogeneity, difference consistency, difference independence, and technical conditions specified in the Appendix are satisfied, where V is unique up to positive linear transformations, and for all $\mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^3, \mathbf{z}^4 \in \mathbf{Z}$, $V(\mathbf{z}^1) - V(\mathbf{z}^2) \geq V(\mathbf{z}^3) - V(\mathbf{z}^4)$ if and only if $\mathbf{z}^1\mathbf{z}^2 \succsim^* \mathbf{z}^3\mathbf{z}^4$.*

The proof of Theorem 1 is a straightforward combination of Theorem 1 from Simon *et al.* and the theorem from Dyer and Sarin,[12] and is described in the Appendix.

For risk analyses with different effects across a spatial area, such a measurable value function would allow a relative comparison of how good different policies are, rather than just a rank order of the policies.

2.2. Multiattribute Measurable Spatial Value

The previous subsection discusses the use of measurable value functions to represent preferences for levels of one attribute over a geographic space. It is also possible that a decisionmaker is concerned about the levels of multiple attributes, such as a watershed's quality of wildlife habitats (fish, invertebrates, amphibians, mammals, and birds) and human habitats (residential value and commercial value).[14] In that case, a multiattribute spatial value function is needed. Let n represent the number of attributes, indexed by j. Then, let z_{ij} denote the level of attribute j in region i, and modify $\mathbf{z}$ to represent a vector of levels of each attribute in each region, and $\mathbf{Z}$ to denote the set of $\mathbf{z}$.

Theorem 2. *For $\succsim$ defined over $\mathbf{Z}$ and $\succsim^*$ defined over $\mathbf{Z} \times \mathbf{Z}$, there exists a multiattribute measurable spatial value function of the form:*

$$V(\mathbf{z}) = \sum_{i=1}^{m} a_i \sum_{j=1}^{n} b_j v_j(z_{ij}), \tag{2}$$

if and only if preferential independence, homogeneity, difference consistency, difference independence, and technical conditions specified in the Appendix are satisfied, where V is unique up to positive linear transformations, and for all $\mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^3, \mathbf{z}^4 \in \mathbf{Z}$, $V(\mathbf{z}^1) - V(\mathbf{z}^2) \geq V(\mathbf{z}^3) - V(\mathbf{z}^4)$ if and only if $\mathbf{z}^1\mathbf{z}^2 \succsim^* \mathbf{z}^3\mathbf{z}^4$.

The approach to proving Theorem 2 is similar to that of Theorem 1, and can be found in the Appendix. An important distinction in the conditions from the single-attribute case is that preferential independence and difference independence now refer to region-attribute combinations rather than regions only.

2.3. Single-Attribute Spatial Utility

While magnitudes of relative preferences between outcomes can be helpful for risk analyses, further insight into levels of risk can be obtained via the use of utility functions that incorporate uncertainty regarding the possible outcomes. Simon *et al.*[1] discuss two approaches for spatial utility in an online appendix. They consider additive spatial utility functions, which require a condition called *additive independence*, asserting that only the marginal distributions of the levels of the attribute in each region are relevant. (That is, the combinations of which levels occur together do not affect preferences.) Because this is a strong condition, they also consider the possibility of constructing a utility function over the spatial value function, based on the approaches used by Dyer and Sarin[15] and Matheson and Abbas.[16] Constructing a spatial utility function over value requires no assertions about preferences regarding uncertainty, and imposes no particular form on the utility function. One could consider these two approaches to spatial utility as two extremes: the former approach requires a very strong preference condition leading to a particular functional form, while the latter allows for an extremely wide range of possible functional forms. In this article, we do not assume additive independence; we will consider the conditions that are more likely to hold in practice, but still lead to functional forms for the utility function (albeit more general ones). We present two such single-attribute spatial utility functions.

In this subsection, we use the single-attribute interpretations of $\mathbf{z}$ and $\mathbf{Z}$ describing a vector of levels of one attribute in each of the m regions. Let P denote a joint probability distribution over $\mathbf{Z}$, $p(z_1, \ldots, z_m)$ denote the probability of a specific outcome $(z_1, \ldots, z_m)$, and $p_i(z_i)$ denote the marginal probability of a specific attribute level in region i. Modify $\succsim$ to denote a preference relation over $\mathbf{P}$: the set of joint distributions over $\mathbf{Z}$. That is, an element $P \in \mathbf{P}$ represents a particular gamble.

Instead of additive independence, we use weaker conditions based on the utility independence conditions presented by Keeney and Raiffa.[17]

Definition 3. *$\succsim$ satisfies single spatial utility independence if preferences between gambles over the attribute levels in any individual region do not depend on the fixed levels of the attribute in the other $m-1$ regions.*

Definition 4. *$\succsim$ satisfies mutual spatial utility independence if preferences between gambles over the attribute levels in any subset S of the m regions do not depend on the fixed attribute levels in its complement $\bar{S}$.*

Note that mutual spatial utility independence is a stronger condition, and implies single spatial utility independence. Like difference independence, spatial utility independence precludes some complex preference interactions. However, as we will see shortly, it does not impose an additive relationship between regions, and does allow for some degree of interaction. Single spatial utility independence is an easier condition to verify; see Fig. 2 for an illustration of the procedure. In the top half of Fig. 2, Maps 1–4 have fixed attribute levels in all regions except region 1. In the bottom half of the figure, Maps 1′–4′ also have fixed attribute levels in all regions except region 1, but these attribute levels differ from those of the corresponding regions in Maps 1–4. If Gamble_{1-2} is preferred to Gamble_{3-4}, single spatial utility independence requires $\text{Gamble}_{1'-2'}$ to be preferred to $\text{Gamble}_{3'-4'}$, since the levels in region 1 follow the same pattern in both comparisons.

The visualization of uncertainty in spatial decision problems is an ongoing challenge, and not one that we will attempt to resolve here. See MacEachren *et al.*[18] for a discussion of the difficulties involved and the efforts being taken to overcome them.

A simpler definition of homogeneity can be used in this case to replace the definition used in the previous subsections:

Definition 5. *Consider two distinct regions i and i', and gambles $P^1, P^2, P^3, P^4 \in \mathbf{P}$ such that: for P^1 and P^2, attribute levels are fixed and common to both gambles for all regions except i; for P^3 and P^4, attribute levels are fixed and common to both gambles for all regions except i'; $p_i^1(z_i) = p_{i'}^3(z_{i'})$ for all possible attribute levels; and $p_i^2(z_i) = p_{i'}^4(z_{i'})$ for all possible attribute levels. Then, $\succsim$ satisfies homogeneity if $P^1 \succsim P^2$ implies $P^3 \succsim P^4$ for any choice of i, i', P^1, P^2, P^3, and P^4.*

Intuitively, this definition of homogeneity states that preferences over gambles on a single region do not depend on the region. The preceding definitions allow us to state the following two theorems:

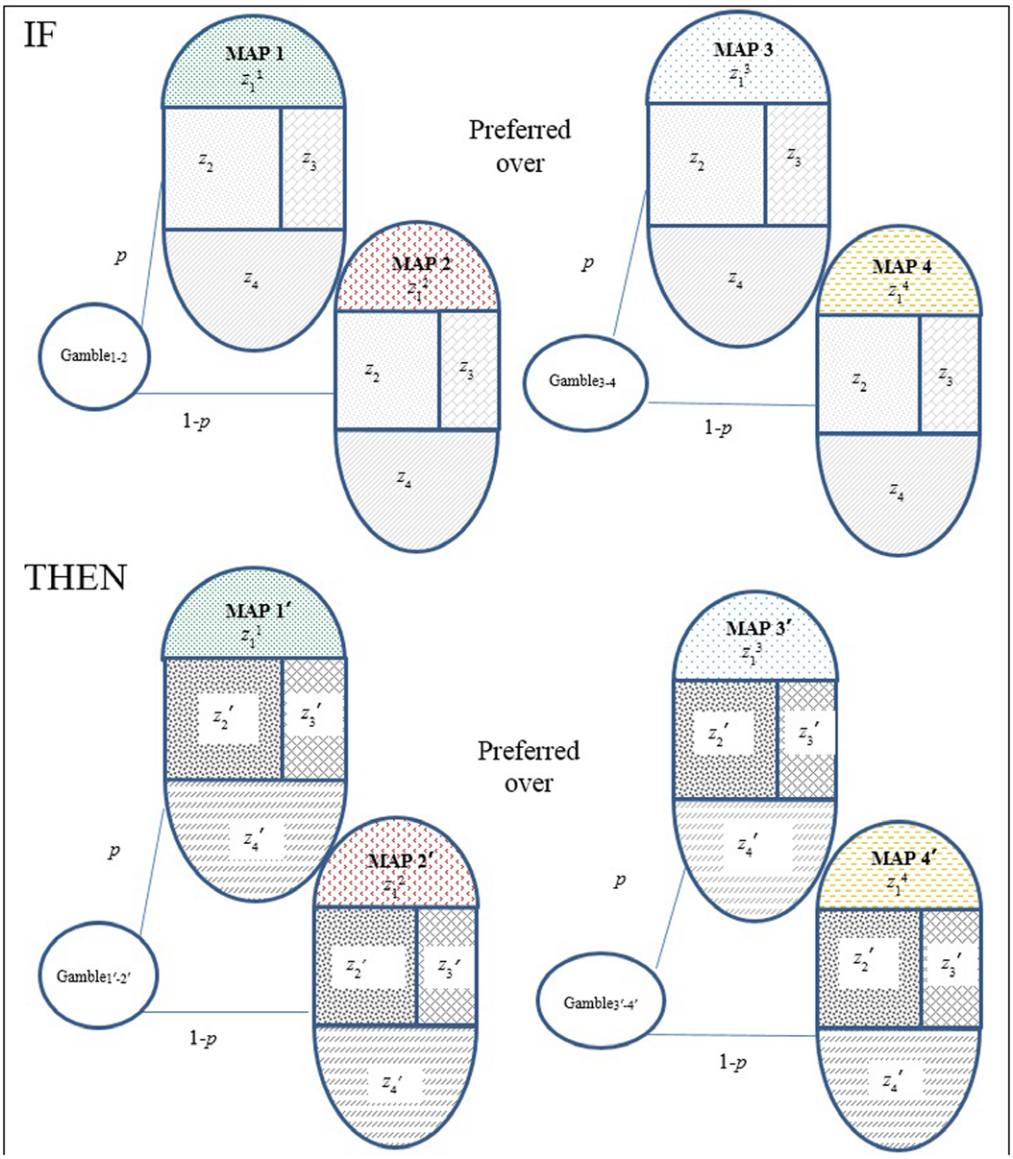

Fig. 2. Single spatial utility independence holds when . . .

Theorem 3. *For $\succsim$ defined over* **P**, *there exists a multilinear spatial utility function of the form:*

$$U(z_1, \ldots, z_m) = \sum_{i=1}^{m} a_i u(z_i) + \sum_{i=1}^{m} \sum_{i'>i} a_{ii'} u(z_i) u(z_{i'}) + \cdots + a_{123,\ldots,m} u(z_1) \ldots u(z_m), \quad (3)$$

*if $\succsim$ satisfies single spatial utility independence, homogeneity, and technical conditions given in the Appendix, where U is unique up to positive linear transformations, such that for all $P^1, P^2 \in$ **P**, $E[U]$ under P^1 is greater than or equal to $E[U]$ under P^2 if and only if $P^1 \succsim P^2$, where the "a" terms are the weights for individual regions and interaction terms, and u is a (common) single-attribute utility function.*

Theorem 4. *For $\succsim$ defined over* **P**, *there exists a multiplicative spatial utility function of the form:*

$$U(z_1, \ldots, z_m) = \sum_{i=1}^{m} a_i u(z_i) + a \sum_{i=1}^{m} \sum_{i'>i} a_i a_{i'} u(z_i) u(z_{i'}) + \cdots + a^{m-1} a_1 \ldots a_m u(z_1) \ldots u(z_m), \quad (4)$$

if $\succsim$ satisfies mutual spatial utility independence, homogeneity, and technical conditions given in the Appendix, where U is unique up to positive linear

transformations, such that for all $P^1, P^2 \in \mathbf{P}$, $E[U]$ under P^1 is greater than or equal to $E[U]$ under P^2 if and only if $P^1 \succsim P^2$, where a_i is the weight for region i, "a" is a scaling constant, and u is a (common) single-attribute utility function.

The proofs of these two Theorems can be found in the Appendix. They are based on Theorems 6.3 and 6.1, respectively, from Keeney and Raiffa[17] for multilinear and multiplicative utility models with no spatial component. Readers interested in further generalizations can refer to Fishburn[19] and Farquhar.[20] In both utility functions, U and u are normalized such that their ranges are $[0, 1]$. The approach for determining the weights and scaling terms is discussed in the Appendix, but it should be noted here that the process is substantially easier for multiplicative spatial utility functions, as the coefficients on the interaction terms do not require any additional assessment.

If the decisionmaker perceives some possible interactions across regions, it may be possible to reduce the decision model to an additive one by redefining region divisions or merging interacting regions.

The multilinear and multiplicative utility functions allow the possibility that, in addition to the contribution to utility from each region considered alone, part of the utility can come from each possible dual region interaction, triple region interaction, etc., up to an m-region interaction. The spatial utility independence conditions permit such interactions between regions, and thus are not generally considered to be restrictive.

In practice, the majority of the interaction terms will likely be insignificant, and can be removed in the multilinear form without any detriment to decision quality. However, there may be specific region pairs with nontrivial interactions; for instance, if two neighboring regions contain parks, their single-region utility functions regarding air pollution might not be able to capture fully the decisionmaker's preferences regarding pairs of air pollution levels in the two parks.

2.4. Multiattribute Utility

Just as Section 2.2 extends Section 2.1 to incorporate both multiple regions and multiple attributes, this section extends Section 2.3 to incorporate multiple attributes as well. We again let n represent the number of attributes, indexed by j, z_{ij} denote the level of attribute j in region i, and modify $\mathbf{z}$ to represent a vector of levels of each attribute in each region, and $\mathbf{Z}$ to denote the set of $\mathbf{z}$. P still denotes a joint distribution over $\mathbf{Z}$, $p(\mathbf{z}^k)$ denotes the probability of a specific outcome $\mathbf{z}^k$, and $p_{ij}(z_{ij})$ denotes the marginal probability of the level of the jth attribute in region i.

We modify the utility independence conditions to account for multiple attributes:

Definition 6. *$\succsim$ satisfies single utility independence if preferences between gambles over the levels of any one attribute in any one region do not depend on the fixed levels of that attribute in the other $m-1$ regions, nor on the fixed levels of the other $n-1$ attributes in any region.*

Definition 7. *$\succsim$ satisfies spatial utility independence if preferences between gambles over the attribute levels in any region i do not depend on any of the fixed attribute levels in the other $m-1$ regions.*

Definition 8. *$\succsim$ satisfies attribute utility independence if preferences between gambles over the levels of any individual attribute j across the m regions do not depend on the fixed levels of the other $n-1$ attributes.*

Theorem 5. *For $\succsim$ defined over $\mathbf{P}$, if $\succsim$ satisfies single utility independence, homogeneity, and technical conditions specified in the Appendix, then there exists a spatial multilinear utility function of the form:*

$$U(\mathbf{z}) = \sum_{i=1}^{m} a_i \sum_{j=1}^{n} b_j u_j(z_{ij}) + \sum_{i=1}^{m}\sum_{j=1}^{n} \sum_{i',j' \in S_{ij}} k_{iji'j'} u_j(z_{ij}) u_{j'}(z_{i'j'}) + \cdots + k_{11,12,\ldots,mn} \prod_{i=1}^{m}\prod_{j=1}^{n} u_j(z_{ij}), \tag{5}$$

such that for all $P_1, P_2 \in \mathbf{P}$, $E[U]$ under P_1 is greater than or equal to $E[U]$ under P_2 if and only if $P_1 \succsim P_2$, where the a_i represents region weights, the b_j represents attribute weights, u_j represents the single-attribute utility function for attribute j (common to all regions), the set S_{ij} consists of all i', j' such that either $i' > i$, or $i' = i$ and $j' > j$, the k terms are the coefficients on interactions, and U is unique up to positive linear transformations.

See the Appendix for the proof of Theorem 5. One portion of the proof also relates closely to an assessment method for a_i and b_j. As in the single-attribute multilinear case, the coefficients on

the interaction terms require additional assessment; they cannot easily be decomposed into attribute and region weights in the way that the coefficients of terms with only one single-attribute utility are. In practice, most of these interaction terms will likely be insignificant, but they can be used as previously to capture specific interactions that are of importance to the decisionmaker.

As in the single-attribute case, if stronger versions of utility independence hold, we can specify the utility function further.

Theorem *For $\succsim$ defined over* $\mathbf{P}$*, if $\succsim$ satisfies spatial utility independence, attribute utility independence, homogeneity, and technical conditions specified in the Appendix, then there exists a spatial multiplicative utility function of the form:*

$$U(\mathbf{z}) = \sum_{i=1}^{m} a_i \sum_{j=1}^{n} b_j u_j(z_{ij}) + k \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{i',j' \in S_{ij}} a_i b_j u_j(z_{ij}) a_{i'} b_{j'} u_{j'}(z_{i'j'}) + \cdots + k^{mn-1} \prod_{i=1}^{m} \prod_{j=1}^{n} a_i b_j u_j(z_{ij}), \quad (6)$$

such that for all $P_1, P_2 \in \mathbf{P}$, $E[U]$ under P_1 is greater than or equal to $E[U]$ under P_2 if and only if $P_1 \succsim P_2$, where the a_i represents region weights, the b_j represents attribute weights, u_j represents the single-attribute utility function for attribute j (common to all regions), the set S_{ij} consists of all i', j' such that either $i' > i$, or $i' = i$ and $j' > j$, k is a scaling constant, and U is unique up to positive linear transformations.

The proof of Theorem 6 can be found in the Appendix. Note that, as in the single-attribute case, the coefficients on the interaction terms do not require any additional assessment; they are determined by a_i and b_j.

3. EXAMPLE

In this section, we present an example of a decision problem that could be evaluated using one of the preference models from Section 2. For illustrative purposes, we present the analysis from start to finish, beginning with a basic explanation of the problem, through the eventual calculation of the overall value of each alternative. It would be straightforward to construct analogous decision problems corresponding to the other models in Section 2. Note also that, for the sake of clarity, our example is deliberately very small. Real applications will likely involve a much larger number of regions; however, the underlying preference concepts are identical.

The single attribute under consideration in this stylized example is the average annual freshwater consumption per household in each of four geographical regions, measured in acre-feet (af) (1 acre-foot = 325,851 gallons). Lower consumption is preferred, assuming health needs are met.

The example is based on the set of four regions used in Figs. 1 and 2. The status quo is Map 1. Currently, the cooler regions use less freshwater. The North region is the coolest, so households currently use an average of 1 af annually. The West region is cooled by the ocean, the East region contains a desert, and the South region is the hottest, so they have average annual household consumptions of 1.2, 1.8, and 2.0 af, respectively.

Two water policies are being considered (Map 2′ or Map 3′). Specific levels for each map are shown in Table II. Policy 1, leading to Map 2′, involves installing a separate pipe system for recycled water in the hot South region for household landscaping, with some of the costs offset by stopping inspections checking compliance with water conservation rules for landscaping in all regions. This would lead to less freshwater usage in the South, and slightly more freshwater usage in the other regions. Policy 2, leading to Map 3′, would also install the recycled water pipe system in the South, stop inspections of water conservation compliance in the East and the West, and would save added money by removing any water conservation restrictions or inspections in the cooler North. The outcomes would be the same as Policy 1, except in the North, where the water usage would be a bit higher.

3.1. Assessing a Single-Attribute Measurable Value Function

After verifying that difference consistency and difference independence (Fig. 1) hold, the water district decisionmaker specifies the measurable value of different average annual levels of acre-feet of freshwater usage per household, which will apply to all four regions. Suppose the decisionmaker divides the range from the worst level of 2 af to the best level of 1 af into four segments representing equal preference differences:

- {Changing from 2 af to 1.5 af} is the same improvement as {Changing from 1.5 af to 1.29 af}

Table I. Assessment of Swing Weight a_i of Each Region i

Region i	Benchmark Worst Map: All Regions at Worst Level of 2 Acre-Feet	**Map S**	**Map N**	**Map W**	**Map E**	Raw Swing Weight on Region i	Normalized Swing Weight a_i
North ($i = 1$)	**0**	**0** $= v(2\text{ af})$	**1**	**0**	**0**	10	**0.05** = 10/200
West ($i = 2$)	**0**	**0**	**0**	**1**	**0**	10	**0.05** = 10/200
East ($i = 3$)	**0**	**0**	**0**	**0**	**1**	80	**0.40** = 80/200
South ($i = 4$)	**0**	**1** $= v(1\text{ af})$	**0**	**0**	**0**	100	**0.50** = 100/200
	First step. Value arbitrarily set to 0.	Second step. Map S determined as most preferable. Value set at arbitrary best level of 100; record 100 as raw swing weight for South.	Third step. Value judged as 10 on 0–100 scale; record 10 as raw swing weight for North.	Fourth step. Value judged as 10 on 0–100 scale; record 10 as raw swing weight for West.	Fifth step. Value judged as 80 on 0–100 scale; record 80 as raw swing weight for East.	Sixth step. Sum = 200.	

Table II. Calculation of Overall Spatial Value of Annual Freshwater Usage per Household in Each Region for Status Quo Map and Two Policy Alternative Maps

Alternative Maps Resulting from Policies		Status Quo Map 1			Policy 1 Map 2′ Recycled Water System for South, Stop Inspecting Landscaping Water Use in All Regions			Policy 2 Map 3′ Recycled Water System in South, Rescind All Landscaping Water Restrictions in North, No Inspection of Landscaping Water in East and West		
Region i	Region Swing Weight a_i	Map 1	Acre-ft./ household	$v(z_i)$	Map 2′	Acre-ft./ household	$v(z_i)$	Map 3′	Acre-ft./ household	$v(z_i)$
North ($i = 1$)	0.05	z_1^1	1.0 af	1.00	z_1^2	1.1 af	0.81	z_1^3	1.5 af	0.25
West ($i = 2$)	0.05	z_2	1.2 af	0.64	z_2'	1.3 af	0.49	z_2'	1.3 af	0.49
East ($i = 3$)	0.40	z_3	1.8 af	0.04	z_3'	1.9 af	0.01	z_3'	1.9 af	0.01
South ($i = 4$)	0.50	z_4	2.0 af	0.00	z_4'	1.2 af	0.64	z_4'	1.2 af	0.64
Total overall value V of a map $= \sum_{i=1}^{4} a_i v(z_i)$				**0.10**			**0.39**			**0.36**

- {Changing from 1.5 af to 1.29 af} is the same improvement as {Changing from 1.29 af to 1.134 af}
- {Changing from 1.29 af to 1.134 af} is the same improvement as {Changing from 1.134 af to 1 af}

Arbitrarily setting $v(2\text{ af}) = 0$ and $v(1\text{ af}) = 1$, we get:

- $v(1.000\text{ af}) = 1.00$.
- $v(1.134\text{ af}) = 0.75$.
- $v(1.290\text{ af}) = 0.50$.
- $v(1.500\text{ af}) = 0.25$.
- $v(2.000\text{ af}) = 0.00$.

These points fit the function $v(z_i) = (2 - z_i)^2$, shown in Fig. 3.

The specific values in each region for each of the maps can be read off of the graph. In decisions involving multiple attributes, each single-attribute value function can be assessed without needing to know the levels of other attributes, based on the theoretical results in this article.

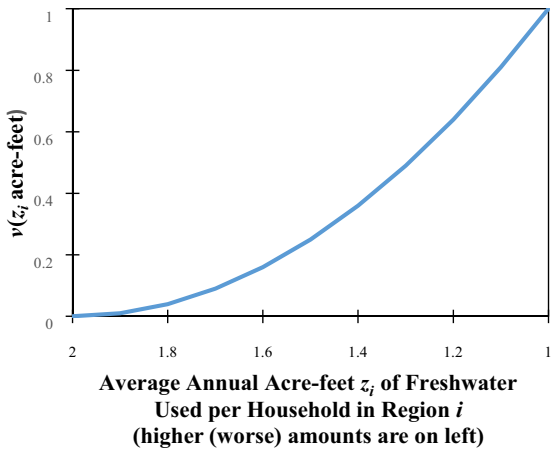

Fig. 3. Value of z_i acre-feet: $v(z_i) = (2 - z_i)^2$.

3.2. Assessing Swing Weights on Regions

The swing weight method[21] for determining the weights on the four regions is illustrated in Table I. Four hypothetical maps are created, each of which is at the best possible water usage, with a value of 1, in only one region. The other regions are at the worst possible water usage, with a value of 0.

In step 1, suppose the water manager starts with the benchmark worst map, in which all regions are at the worst level of 2 af of freshwater per household. Set the value of this benchmark map arbitrarily to 0. The decisionmaker provides a judgment of the value of the four hypothetical maps in steps 2–5, then the weight a_i for each region i can be derived.

In step 2, when asked which map the manager would choose to move to first (or "swing to" from the benchmark worst map) if only one improvement were allowed, suppose the manager chooses to move to Map S, in which the South region has the best level of water usage of 1 af per household. The manager might support this judgment saying: "I'd improve the South, due to its hotter weather and population size, it is the highest priority for our department." Set the value of Map S arbitrarily to 100. This now sets the scale of value for the hypothetical maps to range from 0 to 100.

In steps 3–5, using this 0–100 scale, the manager would directly rate the measurable value of the other three hypothetical maps, for the North, West, and East. Suppose the manager says the East map is close in value to the South map, partly because it is relatively hot in the desert area of the East, so water usage is a relatively high concern, with a value of 80, and the other two maps are at lower values of 10 each, partly because the West is cooled by the ocean and the North is the coolest region. The value of each of the four maps can be obtained using an additive measurable spatial value function. Since each map has one region obtaining a value of 1 (while the other regions obtain zero value), the raw swing weight on the region with a value of 1 is equal to the value assigned to that map. Thus, the raw swing weights on the South, East, West, and North regions, respectively, are: 100, 80, 10, 10.

In step 6, by convention, we normalize the raw weights to sum to 1. This results in normalized swing weights on the regions of 0.50 for the South, 0.40 for the East, and 0.05 each for the West and the North.

In decisions involving multiple attributes, the b_j weight on each attribute j could be assessed following a similar process.

3.3. Calculating the Overall Value of Each Map

The way to calculate the overall spatial value of each map is shown in Table II. An advantage of the theoretical results in this article is that the resulting numbers for the overall value of alternatives carry a strength of preference interpretation, so we can see how far apart possible future maps are in terms of preference differences. In this example, $V(\text{Status Quo Map 1}) = 0.10$ on a 0–1 scale. The two policy alternatives would lead to similarly valued improvements, with $V(\text{Policy 1's Map 2}') = 0.39$ and $V(\text{Policy 2's Map 3}') = 0.36$ is just slightly worse. The choice between the two new policies could consider the differences in the net costs of the two policies after the cost savings from stopping inspections and/or restrictions have been used up.

In decisions involving multiple attributes, each map outcome v_{ij} for region i and attribute j would be weighted by both the region weight a_i and the attribute weight b_j, then the products would be summed for all regions and all attributes.

4. CONCLUSION

We have presented theory to support the construction of measurable spatial value functions for decisions under certainty and spatial utility functions for decisions under risk when data vary across geographical regions. Such preference functions can be used in spatial risk analyses and spatial decision support systems. Most challenging is when there are multiple attributes as well as multiple regions, requiring

the theorist and analyst to keep track of variations along both dimensions. A stylized example is provided of a measurable spatial value function to evaluate different policies affecting the average annual freshwater usage per household in each region.

APPENDIX

Proof. Theorem 1. Pairwise spatial preferential independence, homogeneity, difference consistency, and difference independence are stated in the body of the article. The remaining conditions assert that $\succsim$ satisfies continuity, transitivity, completeness, and unrestricted solvability, and that it depends on every region.

Simon *et al.*[1] show that, given the stated conditions on $\succsim$, there exists an ordinal spatial value function of the form of Equation (1). Let $a_1 v(z_1) + \cdots + a_m v(z_m)$ be such a representation of $\succsim$. For $\succsim$ satisfying the stated conditions, the representation $a_1 v(z_1) + \cdots + a_m v(z_m)$, and conditions equivalent to the stated conditions on $\succsim^*$, Dyer and Sarin[12] show that, using the notation of this article, for given $\mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^3, \mathbf{z}^4 \in \mathbf{Z}$, $\mathbf{z}^1\mathbf{z}^2 \succsim^* \mathbf{z}^3\mathbf{z}^4$ iff:

$$\sum_{i=1}^{m} a_i v(z_i^1) - \sum_{i=1}^{m} a_i v(z_i^2) \geq \sum_{i=1}^{m} a_i v(z_i^3) - \sum_{i=1}^{m} a_i v(z_i^4), \quad \text{(A1)}$$

which establishes Theorem 1. There are two additional points of note relevant to this proof that arise in Dyer and Sarin's paper. First, they use the two-attribute case in the statement of their theorem and its proof; however, they state that it is straightforward to extend the result to more than two attributes. Second, they note that it "may be possible to substitute restricted solvability for unrestricted solvability, at the cost of a more tedious proof" (p. 272). While restricted solvability is a substantially weaker condition, such a proof would be beyond the scope of this article. □

Proof. Theorem 2. In addition to the conditions stated in the body of the article, $\succsim$ satisfies continuity, transitivity, completeness, and unrestricted solvability, and depends on every region-attribute combination. Difference independence can be adapted in a straightforward manner to assert that the preference difference between two levels of a specific region-attribute combination does not depend on any of the other region-attribute combination levels. Preferential independence between region-attribute combinations is a somewhat less intuitive condition. However, it can be replaced by two separate conditions asserting preferential independence between regions and preferential independence between attributes; Simon *et al.* show that preferential independence between any two region-attribute combinations is implied by these more intuitive versions of preferential independence.

From theorem 2 of Simon *et al.*, there exists an ordinal function of the form of Equation (2) representing $\succsim$. Let $a_1 b_1 v_1(z_{11}) + \cdots + a_m b_n v_n(z_{mn})$ be such a representation of $\succsim$. Then, as in the single-attribute case, for given $\mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^3, \mathbf{z}^4 \in \mathbf{Z}$, the method from Dyer and Sarin can be used to show that $\mathbf{z}^1\mathbf{z}^2 \succsim^* \mathbf{z}^3\mathbf{z}^4$ iff:

$$\sum_{i=1}^{m} a_i \sum_{j=1}^{n} v_j(z_{ij}^1) - \sum_{i=1}^{m} a_i \sum_{j=1}^{n} v_j(z_{ij}^2) \geq \sum_{i=1}^{m} a_i \sum_{j=1}^{n} v_j(z_{ij}^3) - \sum_{i=1}^{m} a_i \sum_{j=1}^{n} v_j(z_{ij}^4), \quad \text{(A2)}$$

which establishes Theorem 2. □

Proof. Theorem 3. In addition to the conditions stated in the body of the article, $\succsim$ satisfies continuity, transitivity, and completeness, and the domain of Z is a closed interval with least and most preferred levels.

From theorem 6.3 of Keeney and Raiffa, we obtain the utility function:

$$U(z_1, \ldots, z_m) = \sum_{i=1}^{m} a_i u_i(z_i) + \sum_{i=1}^{m} \sum_{i'>i} a_{ii'} u_i(z_i) u_{i'}(z_{i'}) + \cdots + a_{123,\ldots,m} u_1(z_1) \ldots u_m(z_m), \quad \text{(A3)}$$

where all u_i are normalized to a [0, 1] range (as is U). To establish Theorem 3, we must show that the m different single-attribute utility functions can be replaced by a common function u. We do so by contradiction:

Assume there exists an attribute level c such that $u_i(c) \neq u_{i'}(c)$ for two regions i and i'. Denote the vector of attribute levels in all regions except i as $\bar{\mathbf{z}}_i$, the least desirable level of Z as z^0, and the most desirable level as z^*. Since the utility functions have a [0, 1] range, it must be true that $u_i(z^0) = u_{i'}(z^0) = 0$ and $u_i(z^*) = u_{i'}(z^*) = 1$. Let $\bar{\mathbf{z}}_i^0$ denote the vector of the least desirable attribute levels in all regions except i. Then, consider outcomes $\mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^3, \mathbf{z}^4, \mathbf{z}^5, \mathbf{z}^6$ defined as follows:

$$\mathbf{z}^1 = (z^0, \bar{\mathbf{z}}_i^0),$$
$$\mathbf{z}^2 = (z^*, \bar{\mathbf{z}}_i^0),$$
$$\mathbf{z}^3 = (c, \bar{\mathbf{z}}_i^0),$$

$$\mathbf{z}^4 = (z^0, \bar{\mathbf{z}}^0_{i'}),$$
$$\mathbf{z}^5 = (z^*, \bar{\mathbf{z}}^0_{i'}),$$
$$\mathbf{z}^6 = (c, \bar{\mathbf{z}}^0_{i'}).$$

That is, the first three outcomes differ only in region i, the last three differ only in region i', and all fixed attribute levels are at the worst possible level.

Arbitrarily, let $u_i(c) < u_{i'}(c)$, and select a number d such that $u_i(c) < d < u_{i'}(c)$. Then, consider two gambles: the first consists of $\mathbf{z}^2$ with probability d and $\mathbf{z}^1$ with probability $1 - d$, and the second consists of $\mathbf{z}^5$ with probability d and $\mathbf{z}^4$ with probability $1 - d$. Homogeneity implies that the first gamble is preferred to $\mathbf{z}^3$ if and only if the second gamble is preferred to $\mathbf{z}^6$. However, it is straightforward to use Equation (A3), which represents $\succsim$, to show that the first gamble is preferred to $\mathbf{z}^3$, but the second gamble is not preferred to $\mathbf{z}^6$. (For each of the six outcomes, all terms in the right-hand side of the utility expression are zero except for $a_i u_i(z_i)$ for the first three outcomes, and $a_{i'} u_{i'}(z_{i'})$ for the last three outcomes.) This contradiction implies that there cannot exist such a c, and hence the single-region utility functions for i and i' are equal. Since the choices of i and i' were arbitrary, there must be a common utility function u for each region, which establishes Theorem 3.

The region weights a_i are determined by assessing the utility of $(z^*, \bar{\mathbf{z}}^0_i)$ from the decisionmaker (recalling that U is normalized to have a [0, 1] range), and setting a_i equal to that utility. The weight on an interaction term between each set of regions S requires an additional assessment from the decisionmaker, and is given by:

$$a_{\{s:s\in S\}} = U(\mathbf{z}^S) - \sum_{\{R:\emptyset\subsetneq R\subsetneq S\}} a_{\{r:r\in R\}}, \qquad \text{(A4)}$$

where $\mathbf{z}^S$ is an outcome with attribute level z^* in regions in S and z^0 in all other regions. For instance, $a_{123} = U(z^*, z^*, z^*, z^0, \ldots, z^0) - a_1 - a_2 - a_3 - a_{12} - a_{13} - a_{23}$. □

Proof. Theorem 4. In addition to the conditions stated in the body of the article, $\succsim$ satisfies continuity, transitivity, and completeness, and the domain of Z is a closed interval with least and most preferred levels.

From theorem 6.1 of Keeney and Raiffa, we obtain the utility function:

$$U(z_1, \ldots, z_m) = \sum_{i=1}^{m} a_i u_i(z_i) + \sum_{i=1}^{m}\sum_{i'>i} a a_i a_{i'} u_i(z_i) u_{i'}(z_{i'}) + \cdots + a^{m-1} a_1 \ldots a_m u_1(z_1) \ldots u_m(z_m), \qquad \text{(A5)}$$

where all u_i and U are normalized to a [0, 1] range. The remaining step is to show that the m different single-attribute utility functions can be replaced by a common function u. Since mutual spatial utility independence implies single spatial utility independence, this can be done using the same approach from the proof of Theorem 3, which establishes Theorem 4.

The region weights a_i can be determined using the same approach as for the multilinear form. The constant a is set to normalize U to have a [0, 1] range, and does not require any assessment. It is 0 if the a_i terms sum to 1 (which results in an additive utility function); otherwise, it is specified as the value that solves the equation:

$$1 + a = \prod_{i=1}^{m} (1 + a a_i). \qquad \text{(A6)}$$

The values for a and the a_i terms fully determine the coefficients on all interaction terms without the need for further assessment. □

Proof. Theorem 5. In addition to the conditions stated in the body of the article, $\succsim$ satisfies continuity, transitivity, and completeness, and the domain of Z is a closed interval with least and most preferred levels.

From theorem 6.3 of Keeney and Raiffa, we obtain the utility function:

$$U(\mathbf{z}) = \sum_{i=1}^{m}\sum_{j=1}^{n} k_{ij} u_{ij}(z_{ij}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{i',j'\in S_{ij}} k_{iji'j'} u_{ij}(z_{ij}) u_{i'j'}(z_{i'j'}) + \cdots + k_{11,12,\ldots,mn} \prod_{i=1}^{m}\prod_{j=1}^{n} u_{ij}(z_{ij}), \qquad \text{(A7)}$$

where all u_{ij} are normalized to a [0, 1] range. To establish Theorem 5, we must show that the m different single-attribute utility functions for each of the n attributes can be replaced by common functions u_j, and that the k terms are products of the weights on the attribute and the region.

The approach to establishing common single-attribute utility functions u_j across regions is identical to that used in the proof of Theorem 3, except that it is done separately for each attribute, and in each case, the other $n - 1$ attributes are fixed at their worst possible levels in all regions.

To show that the k terms are products of attribute and region weights, we adapt a similar technique used by Simon *et al.*, which also gives rise

to a straightforward assessment method for these weights. First, assume without loss of generality that k_{11} is the largest of the coefficients on the terms with only one single-attribute utility function. We then consider two outcomes: the first outcome has all attribute levels at their worst possible levels except for attribute $j \neq 1$ in region 1, which is at its best level (meaning that $u_j(z_{1j}) = 1$), and the second outcome has all attribute levels at their worst possible levels except for attribute 1 in region 1, which is set to a level z_{11}^j such that two outcomes are equally preferable. We can use Equation (A7) (with common u_j) to equate the utilities of the two outcomes, resulting in: $k_{1j} = k_{11}u_1(z_{11}^j)$. Homogeneity implies that the same attribute level z_{11}^j must make the analogous equation true for any region i. Thus, $k_{ij} = k_{i1}u_1(z_{11}^j)$ for any i. We can then define a_i as being equal to k_{i1}, and b_j as being equal to $u_1(z_{11}^j)$, and substitute into the preceding equation to obtain $k_{ij} = a_i b_j$. □

Proof. Theorem 6. In addition to the conditions stated in the body of the article, $\succsim$ satisfies continuity, transitivity, and completeness, and the domain of Z is a closed interval with least and most preferred levels.

We first show that the set of mn region-attribute combinations satisfies mutual utility independence. Two of the conditions of this theorem are spatial utility independence and attribute utility independence. Thus, any set $\{Z_{i1}, \ldots, Z_{in}\}$ is utility independent of its complement, as is any set $\{Z_{1j}, \ldots, Z_{mj}\}$. We now state a portion of theorem 6.2 from Keeney and Raiffa, temporarily using only a single subscript to convey their meaning clearly. That theorem asserts that the following two statements are equivalent:

1. Attributes $Z_1, \ldots, Z_n$ are mutually utility-independent.
2. $\{Z_i, Z_{i+1}\}$ are utility-independent, for $i = 1, 2, \ldots, n-1$.

The second statement implies that a "chain" of utility independent pairs covering the entire set of attributes (in our case, region-attribute combinations) is equivalent to mutual utility independence. We can then apply a portion of theorem 6.7 from Keeney and Raiffa, stating that unions and intersections of (overlapping) utility independent sets are also utility independent, as follows:

Consider a pair of region-attribute combinations $\{Z_{ij}, Z_{i'j'}\}$, where $i \neq i'$ and $j \neq j'$. Spatial utility independence implies that both $\{Z_{i1}, \ldots, Z_{in}\}$ and $\{Z_{i'1}, \ldots, Z_{i'n}\}$ are utility independent, and attribute utility independence implies that both $\{Z_{1j}, \ldots, Z_{mj}\}$ and $\{Z_{1j'}, \ldots, Z_{mj'}\}$ are utility independent. The union of $\{Z_{i1}, \ldots, Z_{in}\}$ and $\{Z_{1j'}, \ldots, Z_{mj'}\}$ is utility independent, as is the union of $\{Z_{i'1}, \ldots, Z_{i'n}\}$ and $\{Z_{1j}, \ldots, Z_{mj}\}$. The intersection of these two unions, which is $\{Z_{ij}, Z_{i'j'}\}$, is then also utility independent.

We can also show that a pair of region-attribute combinations $\{Z_{ij}, Z_{i'j}\}$ involving attribute j, where $i \neq i'$, is utility independent by applying another portion of theorem 6.7 from Keeney and Raiffa. Specifically, we apply the fact that the symmetric difference of two utility-independent subsets, i.e., elements that are in one but not the other, is utility independent. Consider a third region i'' (we assume $m \geq 3$) and a second attribute j'. We know that each of the pairs $\{Z_{ij}, Z_{i''j'}\}$ and $\{Z_{i'j}, Z_{i''j'}\}$ is utility independent, as each pair has neither a region nor an attribute in common. The symmetric difference of these two pairs is $\{Z_{ij}, Z_{i'j}\}$; hence, this combination is also utility independent.

A chain of such utility-independent pairs covering all region-attribute combinations can now be constructed in a straightforward manner by ordering the region-attribute combinations lexicographically based on the attribute first ($Z_{11}, Z_{21}, \ldots, Z_{m1}, Z_{12}, \ldots, Z_{m2}, \ldots, Z_{mn}$), and noting that each pair of adjacent combinations is utility independent. This establishes mutual utility independence of the region-attribute combinations.

From theorem 6.1 of Keeney and Raiffa, we obtain the utility function:

$$\begin{aligned} U(\mathbf{z}) = & \sum_{i=1}^{m}\sum_{j=1}^{n} k_{ij}u_{ij}(z_{ij}) \\ & + k\sum_{i=1}^{m}\sum_{j=1}^{n} \sum_{i',j' \in S_{ij}} k_{ij}u_{ij}(z_{ij})k_{i'j'}u_{i'j'}(z_{i'j'}) \\ & + \cdots + k^{mn-1}\prod_{i=1}^{m}\prod_{j=1}^{n} k_{ij}u_{ij}(z_{ij}), \end{aligned} \tag{A8}$$

where all u_{ij} are normalized to a $[0, 1]$ range. To establish Theorem 6, all that remains is to show that the m different single-attribute utility functions for each of the n attributes can be replaced by common functions u_j, and that $k_{ij} = a_i b_j$. Both can be accomplished in an identical manner to the proof of Theorem 5. □

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REFERENCES

1. Simon J, Kirkwood CW, Keller LR. Decision analysis with geographically varying outcomes: Preference models and illustrative applications. Operations Research, 2014; 62(1):182–194.
2. Ferretti V, Montibeller G. Key challenges and meta-choices in designing and applying multi-criteria spatial decision support systems. Decision Support Systems, 2016; 84:41–52.
3. Malczewski J. GIS-based multicriteria decision analysis: A survey of the literature. International Journal of Geographical Information Science, 2006; 20(7):703–726.
4. Zagmutt FJ, Schoenbaum MA, Hill AE. The impact of population, contact, and spatial heterogeneity on epidemic model predictions. Risk Analysis, 2016; 36(5):939–953.
5. Zhou Y, Li N, Wu W, Wu J, Shi P. Local spatial and temporal factors influencing population and societal vulnerability to natural disasters. Risk Analysis, 2014; 34(4):614–639.
6. Debreu G. Representation of a preference ordering by a numerical function. Pp. 159–165 in Thrall RM, Davis RL, Coombs CH (eds.) Decision Processes. New York: Wiley, 1954.
7. Debreu G. Topological methods in cardinal utility theory. Pp. 16–26 in Arrow KS, Suppes P, Karlin S (eds.) Mathematical Methods in the Social Sciences. Stanford, CA: Stanford University Press, 1960.
8. Debreu G. Continuity properties of Paretian utility. International Economic Review, 1964; 5(3):285–293.
9. Gorman WM. The structure of utility functions. Review of Economic Studies, 1968; 35(4):367–390.
10. Fishburn PC. Utility Theory for Decision Making. New York: John Wiley and Sons, 1970.
11. Krantz DH, Luce RD, Suppes P, Tversky A. Foundations of Measurement, Volume I: Additive and Polynomial Representations. New York and London: Academic Press, 1971.
12. Dyer JS, Sarin RK. On the relationship between additive conjoint and difference measurement. Journal of Mathematical Psychology, 1978; 18:270–272.
13. Dyer JS, Sarin RK. Measurable multiattribute value functions. Operations Research, 1979; 27(4):810–822.
14. Merrick JRW, Parnell GS, Barnett J, Garcia M. A multiple-objective decision analysis of stakeholder values to identify watershed improvement needs. Decision Analysis, 2005; 2(1): 44–57.
15. Dyer JS, Sarin RK. Relative risk aversion. Management Science, 1982; 28(8):875–886.
16. Matheson JE, Abbas AE. Utility transversality: A value-based approach. Journal of Multi-Criteria Decision Analysis, 2005; 13(5–6):229–238.
17. Keeney RL, Raiffa H. Decisions with Multiple Objectives: Preferences and Value Tradeoffs. New York: John Wiley and Sons, 1976.
18. MacEachren AM, Robinson A, Hopper S, Gardner S, Murray R, Gahegan M, Hetzler E. Visualizing geospatial information uncertainty: What we know and what we need to know. Cartography and Geographic Information Science, 2005; 32(3):139–160.
19. Fishburn PC. Bernoullian utilities for multiple-factor situations. Pp. 47–61 in Cochrane JL, Zeleny M (eds). Multiple Criteria Decision Making. Columbia, SC.: University of South Carolina Press, 1973.
20. Farquhar PH. A fractional hypercube decomposition theorem for multiattribute utility functions. Operations Research, 1975; 23:941–967.
21. Clemen RT, Reilly T. Making Hard Decisions with Decision Tools Suite, 3rd ed. Mason, OH: South-Western's Cengage Learning, 2014.