Geometry and Spacetime
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Model Solutions for Problems

Problem 2.1.1 Prove that for all vectors u in V, there is a *unique* vector v in V such that $u + v = \mathbf{0}$.

Proof Assume that for some vector u in V, there are vectors v_1 and v_2 in V such that $u + v_1 = \mathbf{0}$ and $u + v_2 = \mathbf{0}$. Then

$$\begin{aligned}
v_1 &= v_1 + \mathbf{0} && \text{(by VS3)}\\
&= v_1 + (u + v_2) && \text{(by our assumption that } u + v_2 = \mathbf{0})\\
&= (v_1 + u) + v_2 && \text{(by VS2)}\\
&= (u + v_1) + v_2 && \text{(by VS1)}\\
&= \mathbf{0} + v_2 && \text{(by our assumption that } u + v_1 = \mathbf{0})\\
&= v_2 + \mathbf{0} && \text{(by VS1)}\\
&= v_2. \quad \square && \text{(by VS3)}
\end{aligned}$$

Problem 2.1.2 Prove that for all vectors u in V, if $u + u = u$, then $u = \mathbf{0}$.

Proof Let u be a vector in V such that $u + u = u$. Then

$$\begin{aligned}
u &= u + \mathbf{0} && \text{(by VS3)}\\
&= u + (u + (-u))\\
&= (u + u) + (-u) && \text{(by VS2)}\\
&= u + (-u) && \text{(by our assumption that } u + u = u)\\
&= \mathbf{0}. \quad \square
\end{aligned}$$

Problem 2.1.3 Prove that for all vectors u in V, and all real numbers a,

(i) $0 \cdot u = \mathbf{0}$
(ii) $-u = (-1) \cdot u$
(iii) $a \cdot \mathbf{0} = \mathbf{0}$.

Proof Let u be a vector in V and let a be a real number. (i) By VS 6,

$$0 \cdot u = (0 + 0) \cdot u = 0 \cdot u + 0 \cdot u.$$

So, by problem 2.1.2, $0 \cdot u = \mathbf{0}$.

(ii) By VS 8 and VS 6,

$$u + (-1) \cdot u = 1 \cdot u + (-1) \cdot u = (1 - 1) \cdot u = 0 \cdot u.$$

But, by part (i), $0 \cdot u = \mathbf{0}$. So $u + (-1) \cdot u = \mathbf{0}$. Thus, $(-1) \cdot u$ is the additive inverse of u, i.e., $(-1) \cdot u = -u$.

(iii) By VS 3 and VS 5,

$$a \cdot \mathbf{0} = a \cdot (\mathbf{0} + \mathbf{0}) = a \cdot \mathbf{0} + a \cdot \mathbf{0}.$$

So, by problem 2.1.2 again, $a \cdot \mathbf{0} = \mathbf{0}$. $\square$

Problem 2.1.4 Prove that the intersection of any non-empty set of subspaces of V is a subspace of V.

Proof Let X be a non-empty set of subspaces of V. We show that the intersection set $\cap X$ is also a subspace of V, i.e., we show that (i) $\cap X$ is non-empty, and (ii) $\cap X$ is closed under vector addition and scalar multiplication.

(i) The zero vector $\mathbf{0}$ belongs to every subspace of V. So, in particular, it belongs to every subspace of V that is in X. So $\mathbf{0} \in \cap X$.

(ii) Let u and v be vectors in $\cap X$, and let a be a real number. Then, for all W in X, u and v belong to W. It follows – since each individual W in X is a subspace – that $u+v$ and $a \cdot u$ belong to all W in X. So $u+v$ and $a \cdot u$ belong to the intersection set $\cap X$. Thus, as required, $\cap X$ is closed under vector addition and scalar multiplication. $\square$

Problem 2.1.5 Let S be a subset of V. Show that $L(S) = S$ iff S is a subspace of V.

Proof Let X be the set of all subspaces of V that contain S as a subset. So $L(S) = \cap X$. Since S is a subset of each element of X, it is certainly a subset of the intersection of all those elements, i.e., $S \subseteq \cap X$. So $S \subseteq L(S)$. This much holds for any subset S of V. But if S is itself a subspace of V, i.e., $S \in X$, then it is also true that $\cap X \subseteq S$, and so $L(S) \subseteq S$. Thus, if S is a subspace of V, it follows that L(S) = S. Conversely, suppose that $L(S) = S$. Then S is certainly a subspace of V, for in this case $S = \cap X$ and we know from problem 2.1.4 that $\cap X$ is a subspace of V. $\square$

Problem 2.1.6 Let S be a subset of V. Show that S is linearly dependent iff there is a vector u in S that belongs to the linear span of $S - \{u\}$.

Proof Let us first dispose of one special case. If S is the empty set, then S is *not* linearly dependent. And in this case, there does *not* exist a vector u in S that belongs to the linear span of $S - \{u\}$. So the stated equivalence holds. Thus we may assume that S is non-empty.

Suppose that S is linearly dependent. Then, for some $k \geq 1$, there exist (distinct) vectors $u_1, ..., u_k$ in S and real numbers $a_1, ..., a_k$, not all 0, such that $a_1 \cdot u_1 + ... + a_k \cdot u_k = \mathbf{0}$. Without loss of generality – because we can always renumber the vectors – we may assume that $a_k \neq 0$. Now consider two cases: (i) a_k is the only non-zero coefficient in the indicated sum, or (ii) otherwise. Assume first that (i) obtains. Then $a_k \cdot u_k = \mathbf{0}$ and, hence, by VS8 and VS7,

$$u_k = 1 \cdot u_k = ((1/a_k)\, a_k) \cdot u_k = (1/a_k) \cdot (a_k \cdot u_k) = (1/a_k) \cdot \mathbf{0}.$$

But $(1/a_k) \cdot \mathbf{0} = \mathbf{0}$ by problem 2.1.3. Therefore, $u_k = \mathbf{0}$. It follows that there is a vector u in S, namely $\mathbf{0}$, that belongs to the linear span of $S - \{u\}$. (Why? The the linear span of $S - \{u\}$ is a subspace of V, and $\mathbf{0}$ belongs to *every* subspace of V.)

Next assume that case (ii) obtains. Then, by problem 2.1.3 again, we have

$$a_k \cdot u_k = -(a_1 \cdot u_1 + ... + a_{k-1} \cdot u_{k-1}) = (-1) \cdot (a_1 \cdot u_1 + ... + a_{k-1} \cdot u_{k-1}),$$

and the indicated sum has at least one term. It follows by VS8 and VS7, once again, that

$$u_k = (-1/a_k) \cdot (a_1 \cdot u_1 + ... + a_{k-1} \cdot u_{k-1}).$$

So, in this case too, we see that there is a vector u in S, namely u_k, that belongs to the linear span of $S - \{u\}$.

Conversely, assume that there is a vector u in S that belongs to the linear span of $S - \{u\}$. Again, we consider two cases: (i) $S - \{u\}$ is the empty set, and (ii) $S - \{u\}$ is non-empty. In case (i), the linear span of $S - \{u\}$ is $\{\mathbf{0}\}$. So $u = \mathbf{0}$. Therefore $\mathbf{0}$ belongs to S and, hence, the latter is linearly dependent. In case (ii), the linear span of $S - \{u\}$ is the set of all linear combinations of elements in $S - \{u\}$. So, since u is in that linear span, there is a $k \geq 1$, vectors $u_1, ..., u_k$ in S, and real numbers $a_1, ..., a_k$, such that

$$u = a_1 \cdot u_1 + ... + a_k \cdot u_k.$$

Hence, by problem 2.1.3,

$$\begin{aligned} \mathbf{0} &= u + (-u) = (a_1 \cdot u_1 + ... + a_k \cdot u_k) + (-u) \\ &= (a_1 \cdot u_1 + ... + a_k \cdot u_k) + (-1) \cdot u. \end{aligned}$$

Since the vectors $u_1, ..., u_k$ and u all belong to S, and since at last one of the coefficients in the final sum is non-zero, namely the final coefficient (-1), we see that S is linearly dependent. $\square$

Problem 2.1.7 Show that two finite dimensional vector spaces are isomorphic iff they have the same dimension.

Proof Let $\mathbf{V} = (V, +, \mathbf{0}, \cdot)$ and $\mathbf{V}' = (V', +', \mathbf{0}', \cdot')$ be finite dimensional vector spaces. Assume first that there exists an isomorphism $\Phi : V \to V'$. Let $n = \dim(\mathbf{V})$. If $n = 0$, then $V = \{\mathbf{0}\}$ and $V' = \{\Phi(\mathbf{0})\} = \{\mathbf{0}'\}$. So, $\dim(\mathbf{V}') = 0 = \dim(\mathbf{V})$. Thus we may assume $n \geq 1$. Let $S = \{u_1, ..., u_n\}$ be a basis for $\mathbf{V}$. We claim that $S' = \{\Phi(u_1), ..., \Phi(u_n)\}$ is a basis for $\mathbf{V}'$ and, therefore, in this case too, $\dim(\mathbf{V}') = \dim(\mathbf{V})$.

First we verify that S' is linearly independent. Assume to the contrary that there exist coefficients $a_1, ..., a_n$, not all 0, such that

$$a_1 \cdot' \Phi(u_1) +' ... +' a_n \cdot' \Phi(u_n) = \mathbf{0}'.$$

Since Φ is linear it follows that

$$\Phi(a_1 \cdot u_1 + ... + a_n \cdot u_n) = a_1 \cdot' \Phi(u_1) +' ... +' a_n \cdot' \Phi(u_n) = \mathbf{0}'.$$

Hence, since $\ker(\Phi) = \{\mathbf{0}\}$, $a_1 \cdot u_1 + ... + a_n \cdot u_n = \mathbf{0}$. But this is impossible since S is a basis (and, therefore, linearly independent). So S' is linearly independent, as claimed.

Next we verify that $L(S') = V'$. Let u' be a vector in V'. Since Φ maps V onto V', there is a vector u in V such that $\Phi(u) = u'$. Since S is a basis for $\mathbf{V}$, there exist coefficients $a_1, ..., a_n$ such that $u = a_1 \cdot u_1 + ... + a_n \cdot u_n$. Hence, by the linearity of Φ again,

$$u' = \Phi(u) = \Phi(a_1 \cdot u_1 + ... + a_n \cdot u_n) = a_1 \cdot' \Phi(u_1) +' ... +' a_n \cdot' \Phi(u_n).$$

Thus u' is in $L(S')$. Since u' was an arbitrary vector in V', $L(S') = V'$. Thus S' is a basis for $\mathbf{V}'$, as claimed.

Conversely, assume that $\mathbf{V}$ and $\mathbf{V}'$ both have dimension n. If $n = 0$, then $V = \{\mathbf{0}\}$, $V' = \{\mathbf{0}'\}$, and the trivial map Φ that take $\mathbf{0}$ to $\mathbf{0}'$ qualifies as an isomorphism between the vector spaces. So we may assume that $n \geq 1$. Let $S = \{u_1, ...u_n\}$ be a basis for $\mathbf{V}$, and let $S' = \{u'_1, ..., u'_n\}$ be a basis for $\mathbf{V}'$. We define a map $\Phi : V \to V'$ as follows. Given any vector u in V, it can be expressed uniquely in the form $u = a_1 \cdot u_1 + ... + a_n \cdot u_n$. We take $\Phi(u)$ to be $a_1 \cdot' u'_1 +' ... +' a_n \cdot' u'_n$. We claim that Φ, so defined, qualifies as an isomorphism between $\mathbf{V}$ and $\mathbf{V}'$.

First, it is injective, i.e., $\ker(\Phi) = \{\mathbf{0}\}$. For suppose $\Phi(u) = \mathbf{0}'$ for some vector $u = a_1 \cdot u_1 + ... + a_n \cdot u_n$. Then $\mathbf{0}' = \Phi(u) = a_1 \cdot' u'_1 +' ... +' a_n \cdot' u'_n$. And therefore, since S' is linearly independent, all the coefficents a_i must be 0, i.e., $u = \mathbf{0}$. So $\ker(\Phi) = \{\mathbf{0}\}$, as claimed.

Next, Φ maps V onto V'. For let u' be any vector in V'. It can be expressed as $u' = a_1 \cdot' u'_1 +' ... +' a_n \cdot' u'_n$. Hence, if $u = a_1 \cdot u_1 + ... + a_n \cdot u_n$, $\Phi(u) = u'$. So $\Phi[V] = V'$, as claimed.

Finally, Φ is linear. For given any vectors $u = a_1 \cdot u_1 + ... + a_n \cdot u_n$ and $v = b_1 \cdot u_1 + ... + b_n \cdot u_n$ in V, and any real number a, it follows (by VS 1, VS 2, and VS 6) that

$$\begin{aligned}
\Phi(u+v) &= \Phi((a_1+b_1)\cdot u_1 + ... + (a_n+b_n)\cdot u_n) \\
&= (a_1+b_1)\cdot' u'_1 +' ... +' (a_n+b_n)\cdot' u'_n \\
&= (a_1 \cdot' u'_1 +' ... +' a_n \cdot' u'_n) +' (b_1 \cdot' u'_1 +' ... +' b_n \cdot' u'_n) \\
&= \Phi(u) +' \Phi(v),
\end{aligned}$$

and (by VS 5 and VS 7) that

$$\begin{aligned}
\Phi(a \cdot u) &= \Phi(a \cdot (a_1 \cdot u_1 + ... + a_n \cdot u_n)) \\
&= \Phi((aa_1)\cdot u_1 + ... + (aa_n)\cdot u_n) \\
&= (aa_1)\cdot' u'_1 +' ... +' (aa_n)\cdot' u'_n \\
&= a \cdot' (a_1 \cdot' u'_1 +' ... +' a_n \cdot' u'_n) \\
&= a \cdot' \Phi(u).
\end{aligned}$$

So we are done. □

Note: At this stage, we allow ourselves to perform simple computations with vectors (e.g., rearranging terms in a sum) without justifying every step with a direct appeal to clauses VS 1 - VS 8 in the definition of a vector space.

Problem 2.2.1 Show that for all points p and q in A, and all subspaces W of V, the following conditions are equivalent.

(i) q belongs to p+W

(ii) p belongs to q+W

(iii) $\overrightarrow{pq} \in W$

(iv) p+W and q+W coincide (i.e., contain the same points)

(v) p+W and q+W intersect (i.e., have at least one point in common)

Proof Let p and q be points in A, and let W be a subspace of V.

(i) ⇒ (ii) Assume that q belongs to $p + W$. Then there is a vector u in W such that $q = p + u$. It follows that $p = q + (-u)$. Since u is in W (and since W is a subspace of V), $(-u)$ is in W as well. So p belong to $q + W$.

(ii) $\Rightarrow$ (iii) Assume that p belongs to $q + W$. Then there is a vector v in W such that $p = q + v$. So $\overrightarrow{qp} = v \in W$. But W is a subspace of V. So, since $\overrightarrow{qp}$ belongs to W, $-\overrightarrow{qp}$ belongs to W as well. It follows that $\overrightarrow{pq} = -\overrightarrow{qp} \in W$.

(iii) $\Rightarrow$ (iv) Assume that $\overrightarrow{pq}$ belongs to W. We show that $(p+W) \subseteq (q+W)$. (A similar argument shows that $(q+W) \subseteq (p+W)$.) Let r be a point in $p+W$. Then there is a vector u in W such that $r = p + u$. It follows that

$$r = (q + \overrightarrow{qp}) + u = q + (\overrightarrow{qp} + u) \in q + W$$

(since both $\overrightarrow{qp}$ and u belong to W and W is a subspace of V). So r is in $q+W$. Thus, $(p + W) \subseteq (q + W)$, as claimed.

(iv) $\Rightarrow$ (v) This one is trivial.

(v) $\Rightarrow$ (i) Assume there is a point r that belongs to both $p+W$ and $q+W$. Then there exist vectors u and v in W such that $r = p + u$ and $r = q + v$. It follows that

$$q = r + (-v) = (p + u) + (-v) = p + (u - v).$$

Since u and v are both in W, and since W is a subspace of V, $(u - v)$ is in W. So q belongs to $p + W$. □

Problem 2.2.2 Let p_1+W_1 and p_2+W_2 be lines, and let u_1 and u_2 be non-zero vectors, respectively, in W_1 and W_2. Show that the lines intersect iff $\overrightarrow{p_1p_2}$ is a linear combination of u_1 and u_2.

Proof Assume first that the lines intersect. Then there is a point q in A, and real numbers a_1 and a_2, such that $q = p_1 + a_1u_1$ and $q = p_2 + a_2u_2$. So, $p_1 + a_1u_1 = p_2 + a_2u_2$. It follows that $p_2 = p_1 + a_1u_1 + (-a_2u_2)$ and, hence, that $\overrightarrow{p_1p_2} = a_1u_1 - a_2u_2$. So $\overrightarrow{p_1p_2}$ is a linear combination of u_1 and u_2.

Conversely, assume that $\overrightarrow{p_1p_2}$ is a linear combination of u_1 and u_2, i.e., assume there exist real numbers a_1 and a_2 such that $\overrightarrow{p_1p_2} = a_1u_1 + a_2u_2$. Then $p_2 = p_1 + (a_1u_1 + a_2u_2)$ and, therefore, $p_2 + (-a_2u_2) = p_1 + a_1u_1$. It follows that there exists a point – namely this point $p_1 + a_1u_1$ – that belongs to both $p_1 + W_1$ and $p_2 + W_2$. □

Problem 2.2.3 Let p, q, r, s be any four distinct points in A. Show that the following conditions are equivalent.

(i) $\overrightarrow{pr} = \overrightarrow{sq}$

(ii) $\overrightarrow{sp} = \overrightarrow{qr}$

(iii) The midpoints of the line segments $LS(p, q)$ and $LS(s, r)$ coincide, i.e.,

$$p + \frac{1}{2}\overrightarrow{pq} \;=\; s + \frac{1}{2}\overrightarrow{sr}.$$

Proof
(i) ⇒ (ii) Assume $\overrightarrow{pr} = \overrightarrow{sq}$. Then

$$\overrightarrow{sp} \;=\; \overrightarrow{sq} + \overrightarrow{qr} + \overrightarrow{rp} \;=\; \overrightarrow{pr} + \overrightarrow{qr} + \overrightarrow{rp} \;=\; \overrightarrow{qr} + (\overrightarrow{pr} + \overrightarrow{rp}) \;=\; \overrightarrow{qr} + \mathbf{0} \;=\; \overrightarrow{qr}.$$

So we have (ii).

(ii) ⇒ (iii) Assume $\overrightarrow{sp} \;=\; \overrightarrow{qr}$. Then

$$\begin{aligned}
p + \frac{1}{2}\,\overrightarrow{pq} &= (s + \overrightarrow{sp}) + \frac{1}{2}\,(\overrightarrow{pr} + \overrightarrow{rq}) \\
&= s + \frac{1}{2}\,(\overrightarrow{sp} + \overrightarrow{sp} + \overrightarrow{pr} + \overrightarrow{rq}) \\
&= s + \frac{1}{2}\,(\overrightarrow{sp} + \overrightarrow{qr} + \overrightarrow{pr} + \overrightarrow{rq}) \\
&= s + \frac{1}{2}\,((\overrightarrow{sp} + \overrightarrow{pr}) + (\overrightarrow{qr} + \overrightarrow{rq})) \\
&= s + \frac{1}{2}\,(\overrightarrow{sr} + \mathbf{0}) = s + \frac{1}{2}\,\overrightarrow{sr}.
\end{aligned}$$

So we have (iii).
(ii) ⇒ (i) Assume $p + \frac{1}{2}\,\overrightarrow{pq} \;=\; s + \frac{1}{2}\,\overrightarrow{sr}$. Then

$$\begin{aligned}
p &= s + \frac{1}{2}\,\overrightarrow{sr} + \left(-\frac{1}{2}\,\overrightarrow{pq}\right) \\
&= s + \frac{1}{2}\,(\overrightarrow{sr} - \overrightarrow{pq}) \\
&= s + \frac{1}{2}\,((\overrightarrow{sp} + \overrightarrow{pr}) - (\overrightarrow{ps} + \overrightarrow{sq})) \\
&= s + \frac{1}{2}\,((\overrightarrow{sp} - \overrightarrow{ps}) + (\overrightarrow{pr} - \overrightarrow{sq})) \\
&= s + \frac{1}{2}\,(2\,\overrightarrow{sp} + (\overrightarrow{pr} - \overrightarrow{sq})) \\
&= s + \left(\overrightarrow{sp} + \frac{1}{2}\,(\overrightarrow{pr} - \overrightarrow{sq})\right).
\end{aligned}$$

So $\overrightarrow{sp} \;=\; \overrightarrow{sp} + \frac{1}{2}\,(\overrightarrow{pr} - \overrightarrow{sq})$. It follows that $(\overrightarrow{pr} - \overrightarrow{sq}) \;=\; \mathbf{0}$ and, hence, that $\overrightarrow{pr} \;=\; \overrightarrow{sq}$. So we have (i). □

Problem 2.2.4 Let $p_1, ..., p_n$ $(n \geq 1)$ be distinct points in A. Show that there is a point o in A such that $\overrightarrow{op}_1 + ... + \overrightarrow{op}_n = \mathbf{0}$. (If particles are present at the points $p_1, ..., p_n$, and all have the same mass, then o is the "center of mass" of the n particle system.)

Proof Let q be any point at all, and let o be defined by

$$o = q + \frac{1}{n}(\overrightarrow{qp}_1 + ... + \overrightarrow{qp}_n).$$

Clearly, $n\,\overrightarrow{qo} = (\overrightarrow{qp}_1 + ... + \overrightarrow{qp}_n)$. It follows that

$$\begin{aligned}(\overrightarrow{op}_1 + ... + \overrightarrow{op}_n) &= (\overrightarrow{oq} + \overrightarrow{qp}_1) + ... + (\overrightarrow{oq} + \overrightarrow{qp}_n) \\ &= n\,\overrightarrow{oq} + (\overrightarrow{qp}_1 + ... + \overrightarrow{qp}_n) \\ &= n\,\overrightarrow{oq} + n\,\overrightarrow{qo} \;=\; n\,(\overrightarrow{oq} - \overrightarrow{oq}) \;=\; \mathbf{0}. \quad \square\end{aligned}$$

Problem 2.2.5 Let $(V, \mathbf{A}, +)$ be a two-dimensional affine space. Let $\{p_1, q_1, r_1\}$ and $\{p_2, q_2, r_2\}$ be two sets of non-collinear points in A. Show that there is a unique affine space isomorphism $\varphi: A \to A$ such that $\varphi(p_1) = p_2$, $\varphi(q_1) = q_2$, and $\varphi(r_1) = r_2$.

Proof
Let $\{p_1, q_1, r_1\}$ and $\{p_2, q_2, r_2\}$ be two sets of non-collinear points in A. Then the vectors $\overrightarrow{p_1q_1}$ and $\overrightarrow{p_1r_1}$ are linearly independent and, so, form a basis for V. Similarly, $\overrightarrow{p_2q_2}$ and $\overrightarrow{p_2r_2}$ form a basis for V. It follows that there is a unique isomorphism $\Phi: V \to V$ such that

$$\begin{aligned}\Phi(\overrightarrow{p_1q_1}) &= \overrightarrow{p_2q_2} \\ \Phi(\overrightarrow{p_1r_1}) &= \overrightarrow{p_2r_2}.\end{aligned}$$

Now consider the map $\varphi: A \to A$ defined by

$$\varphi(s) \;=\; p_2 \;+\; \Phi(\overrightarrow{p_1s}). \tag{1}$$

It follows from proposition 2.2.6 that $\varphi(p_1) = p_2$, that φ is a bijection, and that

$$\varphi(s) \;=\; \varphi(t) \;+\; \Phi(\overrightarrow{ts}). \tag{2}$$

for all s and t in A. Thus φ qualifies as an affine space isomorphism. And it further follows from (1) that

$$\begin{aligned}\varphi(q_1) &= p_2 \;+\; \Phi(\overrightarrow{p_1q_1}) \;=\; p_2 \;+\; \overrightarrow{p_2q_2} \;=\; q_2 \\ \varphi(r_1) &= p_2 \;+\; \Phi(\overrightarrow{p_1r_1}) \;=\; p_2 \;+\; \overrightarrow{p_2r_2} \;=\; r_2,\end{aligned}$$

as required.

To establish uniqueness, suppose that $\varphi' : A \to A$ is an affine space isomorphism such that $\varphi'(p_1) = p_2$, $\varphi'(q_1) = q_2$, and $\varphi'(r_1) = r_2$. Suppose that $\Phi': V \to V$ is the corresponding vector space isomorphism. So we have

$$\varphi'(s) \;=\; \varphi'(t) \;+\; \Phi'(\overrightarrow{ts}). \tag{3}$$

for all s and t in A. It now follows by (3) and (1) that

$$\Phi'(\overrightarrow{p_1 q_1}) = \overrightarrow{\varphi'(p_1)\varphi'(q_1)} = \overrightarrow{p_2\, q_2} = \overrightarrow{\varphi(p_1)\varphi(q_1)} = \Phi(\overrightarrow{p_1 q_1}).$$

Similarly, we have

$$\Phi'(\overrightarrow{p_1 r_1}) \quad = \quad \Phi(\overrightarrow{p_1 r_1}).$$

So the isomorphisms Φ and Φ' agree in their action on the elements of a basis for V. It follows that they are agree in their action on all vectors in V, i.e., $\Phi' = \Phi$. From this, in turn, it follows that φ and φ' must be equal. For by (3) and (1) again, we have

$$\begin{aligned} \phi'(s) &= \phi'(p_1) + \Phi'(\overrightarrow{p_1 s}) \\ &= p_2 + \Phi'(\overrightarrow{p_1 s}) \\ &= \phi(p_1) + \Phi(\overrightarrow{p_1 s}) \\ &= \phi(s) \end{aligned}$$

for all s in A. $\square$

Problem 2.3.1 Prove that for all vectors v and w in V,

$$\langle v, w\rangle = \tfrac{1}{2}(\langle v, v\rangle + \langle w, w\rangle - \langle v - w, v - w\rangle).$$

Proof This follows immediately from the fact that (by IP1, IP2, IP3, and problem 2.1.3),

$$\begin{aligned} \langle v - w, v - w\rangle &= \langle v + (-1)w, v + (-1)w\rangle \\ &= \langle v, v\rangle + \langle v, (-1)w\rangle + \langle (-1)w, v\rangle + \langle (-1)w, (-1)w\rangle \\ &= \langle v, v\rangle - 2\langle v, w\rangle + \langle w, w\rangle. \ \square \end{aligned}$$

Problem 2.3.2 Let W be a subspace of V. Show that the following conditions are equivalent.

(i) W is definite.

(ii) There does not exist a non-zero vector w in W with $\langle w, w\rangle = 0$.

Proof One direction ((i) $\Rightarrow$ (ii)) is immediate. If W is definite, then either $\langle w, w\rangle > 0$ for all non-zero w in W, or $\langle w, w\rangle < 0$ for all non-zero w in W. Either way, there cannot be a non-zero vector w in W such that $\langle w, w\rangle = 0$.

For the converse, suppose that (ii) holds, but (i) does not. Then there exist non-zero vectors u and v in W such that $\langle u, u\rangle < 0$ and $\langle v, v\rangle > 0$. (It follows alone from the fact that W is not definite that there exist non-zero vectors u and v in W such that $\langle u,u\rangle \leq 0$ and $\langle v,v\rangle \geq 0$. And by (ii), the inequalities must be strict.) Consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x) = \langle x\,u + (1-x)v,\ x\,u + (1-x)v\rangle.$$

Clearly, it can be expressed in the form

$$f(x) = ax^2 + bx + c,$$

where $a = \langle u-v, u-v\rangle$, $b = 2\langle u-v, v\rangle$, and $c = \langle v,v\rangle$. Now f must have a real root, i.e., there must be a real number x_0 such that $f(x_0) = 0$. We will verify this shortly, but let us assume it for now. Then

$$w = x_0\,u + (1-x_0)\,v$$

is in W (since u and v are) and $\langle w,w\rangle = 0$ (since $\langle w,w\rangle = f(x_0)$). Moreover, w is not the zero vector. Why? If it were, it would follow that $x_0\,u = -(1-x_0)\,v$ and, hence, that $x_0^2\,\langle u,\,u\rangle = (1-x_0)^2\,\langle v,\,v\rangle$. And this is impossible, since $\langle u,\,u\rangle$ is negative and $\langle v,\,v\rangle$ is positive. So w is a non-zero vector in W satisfying $\langle w,\,w\rangle = 0$. But this contradicts (ii). So it must be the case that if (ii) holds, then (i) holds as well.

It only remains to verify that there is a real number x_0 such that $f(x_0) = 0$. There are various ways to see this. First, f is certainly continuous. (All polynomials are.) And $f(0) = \langle v,\,v\rangle > 0$, while $f(1) = \langle u,\,u\rangle < 0$. So (by the "intermediate value theorem"), there must be an "intermediate point" x_0, between 0 and 1, where f switches from positive to negative values.

Second, it follows from simple algebraic considerations. Any polynomial of form $f(x) = ax^2 + bx + c$ has a real root iff $(b^2 - 4ac) \geq 0$. (Recall the formula that gives the roots, real or not, for the quadratic equation $ax^2 + bx + c = 0$.) So it suffices to verify the latter inequality in the case at hand. What we have to work with are the two conditions

$$\begin{aligned} c \ &= \ \langle v,\,v\rangle \ > \ 0 \\ a+b+c \ = \ f(1) \ &= \ \langle u,\,u\rangle \ < \ 0. \end{aligned}$$

So suppose the inequality does not hold, i.e., suppose that $b^2 < 4ac$. Then it must be the case that $a > 0$ and $0 < a + c < -b$. But this leads to a contradiction:

$$(a+c)^2 \ < \ b^2 \ < \ 4ac \ = \ (a+c)^2 - (a-c)^2 \ \leq \ (a+c)^2.$$

So, $(b^2 - 4ac) \geq 0$. $\square$

Problem 2.4.1 Prove that for all vectors u and v in V,

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Proof Let u and v be vectors in V. Then

$$\begin{aligned} \|u+v\|^2 + \|u-v\|^2 &= \; < u+v,\, u+v > \; + \; < u-v,\, u-v > \\ &= \; 2 < u,\, u > \; + \; 2 < v,\, v > \\ &= \; 2\,(\|u\|^2 + \|v\|^2). \qquad \square \end{aligned}$$

Problem 2.4.2 Give a second proof of proposition 2.4.1.

Proof Let u and v be vectors in V. We may assume that $u \neq \mathbf{0}$, since otherwise the proposition is trivial. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \langle xu+v,\, xu+v\rangle = ax^2 + bx + c,$$

where $a = \langle u, u\rangle$, $b = 2\langle u, v\rangle$, and $c = \langle v, v\rangle$. Since the inner product $\langle\, ,\, \rangle$ is positive-definite, we have

(i) $a > 0$;

(ii) $f(x) \geq 0$ for all x in $\mathbb{R}$;

(iii) $f(x) = 0$ iff $xu + v = \mathbf{0}$.

Now *any* function $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax^2+bx+c$, with $a > 0$, assumes a minimal value $\dfrac{(-b^2 + 4ac)}{4a}$ at $x = \dfrac{-b}{2a}$. (Here we invoke basic principles of algebra or calculus.) In the case at hand, by (ii), that minimal value must be greater than or equal to 0. So, $(-b^2 + 4ac) \geq 0$. Substituting for a, b, and c in this inequality yields

$$\langle u,\, v\rangle^2 \;\leq\; \|u\|^2\, \|u\|^2.$$

This gives us the first clause of proposition 2.4.1. For the second clause, notice that (working backwards), $\langle u,\, v\rangle^2 = \|u\|^2\, \|u\|^2$, i.e., $(-b^2 + 4ac) = 0$ iff the minimal value of f is 0 iff $f(-b/2a) = 0$. But, by (iii), $f(-b/2a) = 0$ iff $v = (b/2a)u$. $\square$

Problem 2.4.3 (The measure of a straight angle is π.) Let p, q, r be (distinct) collinear points, and suppose that q is between p and r (i.e., $\overrightarrow{pq} = a\,\overrightarrow{pr}$ with $0 < a < 1$). Show that $\measuredangle(p, q, r) = \pi$.

Proof Since $\overrightarrow{pq} = a\,\overrightarrow{pr}$, we have $\overrightarrow{qp} = -a\,\overrightarrow{pr}$ and $\overrightarrow{qr} = (1-a)\,\overrightarrow{pr}$. Hence,

$$< \overrightarrow{qp},\; \overrightarrow{qr} > \; = \; < -a\,\overrightarrow{pr},\, (1-a)\,\overrightarrow{pr} > \; = \; -a\,(1-a)\,\|\overrightarrow{pr}\|^2$$

and, since $0 < a < 1$,

$$\|\overrightarrow{qp}\|\,\|\overrightarrow{qr}\| \;=\; \|-a\,\overrightarrow{pr}\|\,\|(1-a)\,\overrightarrow{pr}\| \;=\; a\,(1-a)\,\|\overrightarrow{pr}\|^2.$$

It follows that

$$cos(\measuredangle(p,q,r)) \;=\; \frac{<\overrightarrow{qp},\; \overrightarrow{qr}>}{\|\overrightarrow{qp}\|\,\|\overrightarrow{qr}\|} \;=\; -1.$$

The only number between 0 and π whose cosine is -1 is π. So, $\measuredangle(p,q,r) = \pi$.
□

Problem 2.4.4 (Law of Cosines) Let p, q, r be points, with q distinct from p and r. Show that

$$\|\overrightarrow{pr}\|^2 = \|\overrightarrow{qp}\|^2 + \|\overrightarrow{qr}\|^2 - 2\|\overrightarrow{qp}\|\,\|\overrightarrow{qr}\|\cos\measuredangle(p,q,r).$$

Proof By the polarization identity (problem 2.3.1), with $v = \overrightarrow{qr}$ and $w = \overrightarrow{qp}$, we have

$$2\,\langle\overrightarrow{qr},\, \overrightarrow{qp}\rangle \;=\; \langle\overrightarrow{qr},\, \overrightarrow{qr}\rangle + \langle\overrightarrow{qp},\, \overrightarrow{qp}\rangle - \langle\overrightarrow{qr} - \overrightarrow{qp},\, \overrightarrow{qr} - \overrightarrow{qp}\rangle.$$

But, $\overrightarrow{qr} - \overrightarrow{qp} = \overrightarrow{pr}$. So

$$2\,\langle\overrightarrow{qr},\, \overrightarrow{qp}\rangle \;=\; \|\overrightarrow{qr}\|^2 + \|\overrightarrow{qp}\|^2 - \|\overrightarrow{pr}\|^2.$$

Furthermore,

$$\langle\overrightarrow{qr},\, \overrightarrow{qp}\rangle \;=\; \cos\measuredangle(p,q,r)\,\|\overrightarrow{qp}\|\,\|\overrightarrow{qr}\|.$$

So,

$$\|\overrightarrow{pr}\|^2 = \|\overrightarrow{qp}\|^2 + \|\overrightarrow{qr}\|^2 - 2\|\overrightarrow{qp}\|\,\|\overrightarrow{qr}\|\cos\measuredangle(p,q,r). \quad \square$$

Problem 2.4.5 (Right Angle in a Semicircle Theorem) Let p, q, r, o be (distinct) points such that (i) p, o, r are collinear, and (ii) $\|\overrightarrow{op}\| = \|\overrightarrow{oq}\| = \|\overrightarrow{or}\|$. (So q lies on a semicircle with diameter $LS(p,r)$ and center o.) Show that $\overrightarrow{qp} \perp \overrightarrow{qr}$, and so $\measuredangle(p,q,r) = \dfrac{\pi}{2}$.

Proof By (i), we have $\overrightarrow{op} = a\,\overrightarrow{or}$ for some a. Hence, $\|\overrightarrow{op}\| = |a|\,\|\overrightarrow{or}\|$ and, therefore, by (ii), $|a| = 1$. Now a cannot be 1. For if $\overrightarrow{op} = \overrightarrow{or}$, then

$$\overrightarrow{op} \;=\; \overrightarrow{or} + \overrightarrow{rp} \;=\; \overrightarrow{op} + \overrightarrow{rp}.$$

And so it would follow that $\overrightarrow{rp} = \mathbf{0}$, which is impossible since p and r are distinct. So $a = -1$ and $\overrightarrow{op} = -\overrightarrow{or}$. This implies that

$$\overrightarrow{qr} \;=\; \overrightarrow{qo} + \overrightarrow{or} \;=\; -\overrightarrow{oq} - \overrightarrow{op}.$$

We also clearly have

$$\overrightarrow{qp} \;=\; \overrightarrow{qo} + \overrightarrow{op} \;=\; -\overrightarrow{oq} + \overrightarrow{op}.$$

Hence, by (ii) again,

$$\begin{aligned} < \overrightarrow{qp}, \overrightarrow{qr} > \;&=\; < -\overrightarrow{oq} + \overrightarrow{op}, -\overrightarrow{oq} - \overrightarrow{op} > = < \overrightarrow{oq}, \overrightarrow{oq} > - < op, op > \\ &=\; \|\overrightarrow{oq}\|^2 - \|\overrightarrow{op}\|^2 \;=\; 0. \end{aligned}$$

Thus, $\overrightarrow{qp} \perp \overrightarrow{qr}$ and

$$cos(\measuredangle(p,q,r)) \;=\; \frac{< \overrightarrow{qp},\; \overrightarrow{qr} >}{\|\overrightarrow{qp}\|\,\|\overrightarrow{qr}\|} \;=\; 0.$$

The only number between 0 and π whose cosine is 0 is $\frac{\pi}{2}$. So, $\measuredangle(p,q,r) = \frac{\pi}{2}$.
□

Problem 2.4.6 (Stewart's Theorem) Let p, q, r, s be points (not necessarily distinct), with s between q and r (i.e., $\overrightarrow{qs} = a\,\overrightarrow{qr}$ with $0 \leq a \leq 1$). Show that

$$\|\overrightarrow{pq}\|^2\|\overrightarrow{sr}\| + \|\overrightarrow{pr}\|^2\|\overrightarrow{qs}\| - \|\overrightarrow{ps}\|^2\|\overrightarrow{qr}\| = \|\overrightarrow{qr}\|\|\overrightarrow{qs}\|\|\overrightarrow{sr}\|.$$

Proof We are given that $\overrightarrow{qs} = a\,\overrightarrow{qr}$ and, hence, $\overrightarrow{sr} = (1-a)\,\overrightarrow{qr}$ for some a with $0 \leq a \leq 1$. It follows that

$$\|\overrightarrow{qr}\|\|\overrightarrow{qs}\|\|\overrightarrow{sr}\| \;=\; a\,(1-a)\,\|\overrightarrow{qr}\|^3. \tag{4}$$

We also have

$$\begin{aligned} \|\overrightarrow{pq}\|^2\,\|\overrightarrow{sr}\| \;&=\; \langle \overrightarrow{ps} + \overrightarrow{sq},\, \overrightarrow{ps} + \overrightarrow{sq} \rangle\,(1-a)\,\|\overrightarrow{qr}\| \\ &=\; \langle \overrightarrow{ps} - a\,\overrightarrow{qr},\, \overrightarrow{ps} - a\,\overrightarrow{qr} \rangle\,(1-a)\,\|\overrightarrow{qr}\| \\ &=\; \left[\|\overrightarrow{ps}\|^2 + a^2\|\overrightarrow{qr}\|^2 - 2\,a\,\langle \overrightarrow{ps},\, \overrightarrow{qr} \rangle\right](1-a)\,\|\overrightarrow{qr}\| \end{aligned} \tag{5}$$

and, similarly,

$$\|\overrightarrow{pr}\|^2\,\|\overrightarrow{qs}\| = \left[\|\overrightarrow{ps}\|^2 + (1-a)^2\|\overrightarrow{qr}\|^2 + 2\,(1-a)\,\langle \overrightarrow{ps},\, \overrightarrow{qr} \rangle\right] a\,\|\overrightarrow{qr}\|. \tag{6}$$

Adding (5) and (6) yields

$$\|\overrightarrow{pq}\|^2\,\|\overrightarrow{sr}\| + \|\overrightarrow{pr}\|^2\,\|\overrightarrow{qs}\| - \|\overrightarrow{ps}\|^2\,\|\overrightarrow{qr}\| \;=\; a\,(1-a)\,\|\overrightarrow{qr}\|^3. \tag{7}$$

Comparing (4) and (7) yields the desired conclusion. □

Problem 3.1.1 Show that there are no subspaces of dimension higher than 1 all of whose vectors are causal.

Proof Assume there are non-zero, linearly independent vectors u and v such that, for all real numbers a and b, the vector $au + bv$ is causal, i.e.,

$$a^2\langle u,\, u\rangle + 2ab\langle u,\, v\rangle + b^2\langle v,\, v\rangle \geq 0. \tag{8}$$

Of course (taking $a = 1, b = 0$ and $a = 0, b = 1$) u and v must be causal themselves. There are two cases to consider. Either one of the two is timelike, or both are null. Assume first that one of the two, say u, is timelike. Then, by proposition 3.1.1, we can express v in the form $v = au + w$, with w in $u^{\perp}$. w is in the space spanned by u and v. So it must be causal. But since w is in $u^{\perp}$, it must be spacelike or the zero vector (by proposition 3.1.1). So, $w = \mathbf{0}$. This contradicts our assumption that u and v are linearly independent. So we may assume next that u and v are null. (This is our second case.) Then $\langle u,\, u\rangle = 0 = \langle v,\, v\rangle$ and so, by (8), $ab\langle u,\, v\rangle \geq 0$ for all a and b. But this is only possible if $\langle u,\, v\rangle = 0$. So, by proposition 3.1.2, u and v must proportional. This contradicts, once again, our assumption that u and v are linearly independent. So we may conclude that there are no subspaces of dimension higher than 1 all of whose vectors are causal. □

Problem 3.1.2 One might be tempted to formulate the extended definition this way: two causal vectors are "co-oriented" if $\langle u, v\rangle \geq 0$. But this will not work. Explain why.

There are (at least) two related problems with the proposal. First, it allows two null vectors to qualify as "co-oriented" when, intuitively, they have opposite orientations. (Consider any non-zero null vector v and its negation $(-v)$.) Second, the proposed relation is not transitive on the set of causal vectors. To see this, let u and v be, respectively, timelike and null vectors such that $\langle u, v\rangle > 0$. Then the pairs $\{u, v\}$ and $\{v, -v\}$ qualify as "co-oriented" under the proposal, but the pair $\{u, -v\}$ does not.

Problem 3.1.3 Let o, p, q be three points in A such that p is spacelike related to o, and q is timelike related to o. Show that any two of the following conditions imply the third.

(i) $\overrightarrow{pq}$ is null.

(ii) $\overrightarrow{op} \perp \overrightarrow{oq}$

(iii) $\|\overrightarrow{op}\| = \|\overrightarrow{oq}\|$

Proof Since $\overrightarrow{pq} = -\overrightarrow{op} + \overrightarrow{oq}$,

$$\langle \overrightarrow{pq}, \overrightarrow{pq} \rangle = \langle \overrightarrow{op}, \overrightarrow{op} \rangle - 2\langle \overrightarrow{op}, \overrightarrow{oq} \rangle + \langle \overrightarrow{oq}, \overrightarrow{oq} \rangle. \tag{9}$$

All three implications follow easily from (9). For example, if (i) and (ii) hold, then $\langle \overrightarrow{pq}, \overrightarrow{pq} \rangle = 0 = \langle \overrightarrow{op}, \overrightarrow{oq} \rangle$. So (9) yields $-\langle \overrightarrow{op}, \overrightarrow{op} \rangle = \langle \overrightarrow{oq}, \overrightarrow{oq} \rangle$. This is equivalent to $\|\overrightarrow{op}\|^2 = \|\overrightarrow{oq}\|^2$, since $\overrightarrow{op}$ is spacelike and $\overrightarrow{oq}$ is timelike. Therefore (iii) holds. (The other two implications are handled the same way.) $\square$

Problem 3.1.4 Let p, q, r, s be distinct points in A such that

(i) r, q, s lie on a timelike line with q between r and s;

(ii) $\overrightarrow{rp}$ and $\overrightarrow{ps}$ are null.

Show that $\overrightarrow{qp}$ is spacelike, and $\|\overrightarrow{qp}\|^2 = \|\overrightarrow{rq}\| \, \|\overrightarrow{qs}\|$.

Proof We know from (i) that

$$\overrightarrow{rq} = a\,\overrightarrow{rs} \tag{10}$$

$$\overrightarrow{qs} = (1-a)\,\overrightarrow{rs} \tag{11}$$

for some real number a where $0 < a < 1$. Hence,

$$a\,\overrightarrow{rs} + \overrightarrow{qp} = \overrightarrow{rp} \tag{12}$$

$$(1-a)\,\overrightarrow{rs} - \overrightarrow{qp} = \overrightarrow{ps}. \tag{13}$$

But we know from (ii) that that $\langle \overrightarrow{rp}, \overrightarrow{rp} \rangle = 0 = \langle \overrightarrow{ps}, \overrightarrow{ps} \rangle$. So, by (12) and (13),

$$a^2 \langle \overrightarrow{rs}, \overrightarrow{rs} \rangle + 2a \langle \overrightarrow{rs}, \overrightarrow{qp} \rangle + \langle \overrightarrow{qp}, \overrightarrow{qp} \rangle = 0 \tag{14}$$

$$(1-a)^2 \langle \overrightarrow{rs}, \overrightarrow{rs} \rangle - 2(1-a) \langle \overrightarrow{rs}, \overrightarrow{qp} \rangle + \langle \overrightarrow{qp}, \overrightarrow{qp} \rangle = 0. \tag{15}$$

If we multiply (14) by $(1-a)$, multiply (15) by a, and then add, we arrive at

$$a(1-a)\langle \overrightarrow{rs}, \overrightarrow{rs} \rangle + \langle \overrightarrow{qp}, \overrightarrow{qp} \rangle = 0. \tag{16}$$

Since $\overrightarrow{rs}$ is timelike, and since $0 < a < 1$, it follows that $\langle \overrightarrow{qp}, \overrightarrow{qp} \rangle < 0$, i.e., $\overrightarrow{qp}$ is spacelike. In addition, it follows from (10) and (11) that

$$\|\overrightarrow{rq}\| \, \|\overrightarrow{qs}\| = a(1-a)\, \|\overrightarrow{rs}\|^2 = \|\overrightarrow{qp}\|^2. \quad \square$$

Problem 3.1.5 Let L be a timelike line, and let p be any point in A. Show the following.

(i) There is a unique point q on L such that $\overrightarrow{pq} \perp L$.

(ii) If $p \notin L$, there are exactly two points on L that are null related to p. (If $p \in L$, there is exactly one such point, namely p itself.)

Proof Let o and r be distinct points on L with $\|\overrightarrow{or}\| = 1$. Every point q on L can be uniquely expressed in the form $q = o + x\,\overrightarrow{or}$, where x is a real number. For every such point,

$$\overrightarrow{pq} = \overrightarrow{po} + \overrightarrow{oq} = -\overrightarrow{op} + x\,\overrightarrow{or}. \tag{17}$$

Hence, it suffices for us to show

(i) there is a unique real number x such that $(-\overrightarrow{op} + x\,\overrightarrow{or}) \perp \overrightarrow{or}$;

(ii) if $p \notin L$, there are exactly two real numbers x such that $\|-\overrightarrow{op} + x\,\overrightarrow{or}\| = 0$.

The first claim is immediate. Since

$$\langle -\overrightarrow{op} + x\,\overrightarrow{or},\, \overrightarrow{or}\rangle = -\langle \overrightarrow{op},\, \overrightarrow{or}\rangle + x\,\langle \overrightarrow{or},\, \overrightarrow{or}\rangle = -\langle \overrightarrow{op},\, \overrightarrow{or}\rangle + x,$$

the orthogonality condition in (i) will be satisfied iff $x = \langle \overrightarrow{op},\, \overrightarrow{or}\rangle$. To verify (ii), we need to do just a bit more work. Since

$$\begin{aligned}\langle -\overrightarrow{op} + x\,\overrightarrow{or},\, -\overrightarrow{op} + x\,\overrightarrow{or}\rangle &= \langle \overrightarrow{or},\, \overrightarrow{or}\rangle\, x^2 - 2\langle \overrightarrow{or},\, \overrightarrow{op}\rangle\, x + \langle \overrightarrow{op},\, \overrightarrow{op}\rangle \\ &= x^2 - 2\langle \overrightarrow{or},\, \overrightarrow{op}\rangle\, x + \langle \overrightarrow{op},\, \overrightarrow{op}\rangle,\end{aligned}$$

we need to consider the equation

$$a\,x^2 + b\,x + c = 0, \tag{18}$$

where $a = 1$, $b = -2\,\langle \overrightarrow{or},\, \overrightarrow{op}\rangle$, and $c = \langle \overrightarrow{op},\, \overrightarrow{op}\rangle$. Its solutions are given by

$$x = \frac{-b \pm \sqrt{D}}{2}$$

where $D = b^2 - 4ac$. So to establish (ii), it will suffice to verify that

(iii) $D \geq 0$;

(iv) $D = 0 \iff p \in L$.

To do so, we invoke proposition 3.1.1, and express $\overrightarrow{op}$ in the form $\overrightarrow{op} = k\,\overrightarrow{or} + w$, with $w \perp \overrightarrow{or}$. Then,

$$\begin{aligned} b &= -2\,\langle \overrightarrow{or},\, \overrightarrow{op}\rangle = -2k \\ c &= \langle k\,\overrightarrow{or} + w,\, k\,\overrightarrow{or} + w\rangle = k^2 + \langle w,\, w\rangle \end{aligned}$$

and, therefore,

$$D = 4\,k^2 - 4(k^2 + \langle w,\, w\rangle) = -4\,\langle w,\, w\rangle.$$

Since w is orthogonal to the timelike vector $\overrightarrow{or}$, it is either spacelike or the zero-vector (by proposition 3.1.1 again). Either way, $\langle w,\, w\rangle \leq 0$. So we have (iii). And precisely because w is either spacelike or the zero-vector, we also have

$$D = 0 \iff \langle w,\, w\rangle = 0 \iff w = \mathbf{0} \iff \overrightarrow{op} = k\,\overrightarrow{or} \iff p \in L. \quad \square$$

Problem 3.2.1 Let o, p, q, r, s be distinct points where

(i) o, p, q lie on a timelike line L with p between o and q;

(ii) o, r, s lie on a timelike line L' with r between o and s;

(iii) $\overrightarrow{pr}$ and $\overrightarrow{qs}$ are null;

(iv) $\overrightarrow{oq}$, $\overrightarrow{pr}$, and $\overrightarrow{qs}$ are co-oriented.

Show that

$$\frac{\|\overrightarrow{rs}\|}{\|\overrightarrow{pq}\|} = \left[\frac{1+v}{1-v}\right]^{\frac{1}{2}},$$

where v is the speed that the individual with worldline L attributes to the individual with worldline L' .

Proof It follows from (i) and (ii) that there exist numbers a and b, with $0 < a < 1$ and $0 < b < 1$, such that $\overrightarrow{pq} = a\,\overrightarrow{oq}$ and $\overrightarrow{rs} = b\,\overrightarrow{os}$. We claim that $a = b$. To see this, note first that

$$a\,\overrightarrow{oq} + \overrightarrow{qs} = \overrightarrow{pq} + \overrightarrow{qs} = \overrightarrow{pr} + \overrightarrow{rs} = \overrightarrow{pr} + b\,\overrightarrow{os} = \overrightarrow{pr} + b\,(\overrightarrow{oq} + \overrightarrow{qs})$$

and, therefore,

$$(a-b)\,\overrightarrow{oq} = \overrightarrow{pr} - (1-b)\,\overrightarrow{qs}. \tag{19}$$

It follows by (iii), and the fact that $\overrightarrow{oq}$ is timelike, that

$$-2\,(1-b)\,\langle\overrightarrow{pr},\, \overrightarrow{qs}\rangle = (a-b)^2\,\|\overrightarrow{oq}\|^2 \geq 0. \tag{20}$$

(Here we have just taken the inner product of each side of (19) with itself.) Hence, $\langle\overrightarrow{pr},\, \overrightarrow{qs}\rangle \leq 0$. But, by (iv), $\overrightarrow{pr}$ and $\overrightarrow{qs}$ are co-oriented. So $\langle\overrightarrow{pr},\, \overrightarrow{qs}\rangle = 0$ and, therefore (by (20)), a = b as claimed. Thus

$$\frac{\|\overrightarrow{rs}\|}{\|\overrightarrow{pq}\|} = \frac{\|\overrightarrow{os}\|}{\|\overrightarrow{oq}\|}. \tag{21}$$

Now we compute the right side of (21). To do so, we use the fact that $\overrightarrow{qs} = \overrightarrow{os} - \overrightarrow{oq}$. Taking inner products of each side (and using the fact that $\overrightarrow{qs}$ is null), we have

$$0 = \|\overrightarrow{os}\|^2 + \|\overrightarrow{oq}\|^2 - 2\,\langle\overrightarrow{os},\, \overrightarrow{oq}\rangle.$$

It follows that $\overrightarrow{os}$ and $\overrightarrow{oq}$ are co-oriented (i.e., $\langle\overrightarrow{os},\, \overrightarrow{oq}\rangle > 0$) and, therefore, that

$$\langle\overrightarrow{os},\, \overrightarrow{oq}\rangle = \|\overrightarrow{os}\|\,\|\overrightarrow{oq}\|\cosh\theta = \|\overrightarrow{os}\|\,\|\overrightarrow{oq}\|\,(1-v^2)^{-\frac{1}{2}},$$

where θ is the hyperbolic angle between $\overrightarrow{os}$ and $\overrightarrow{oq}$, and v is the relative velocity between the worldlines determined by the two vectors. (Here we are using equation (3.2.3) in the notes.) Thus, if we take X to be the ratio $\dfrac{\|\overrightarrow{os}\|}{\|\overrightarrow{oq}\|}$, we have

$$X^2 - 2\,X(1-v^2)^{-\frac{1}{2}} + 1 = 0. \tag{22}$$

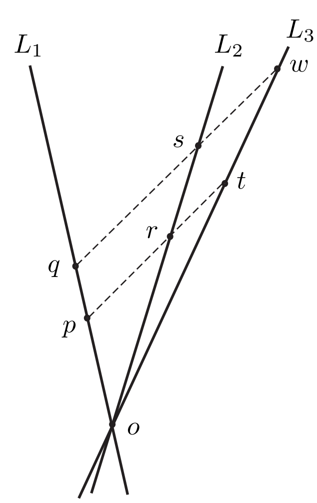
We also have a side constraint on X. Since $\overrightarrow{oq} + \overrightarrow{qs} = \overrightarrow{os}$,

$$\|\overrightarrow{os}\|^2 = \|\overrightarrow{oq}\|^2 + 2\,\langle\overrightarrow{oq},\, \overrightarrow{qs}\rangle > \|\overrightarrow{oq}\|^2.$$

(Here we use the fact that $\overrightarrow{qs}$ is null, and $\overrightarrow{oq}$ and $\overrightarrow{qs}$ are co-oriented.) So $X > 1$. It is a matter of simple algebra now to check that the quadratic equation (22) has exactly one solution satisfying the constraint, namely

$$X = \left[\frac{1+v}{1-v}\right]^{\frac{1}{2}}. \quad \square$$

Problem 3.2.2 Give a second derivation of the "relativistic addition of velocities formula" using the result of problem 3.2.1.



Proof Let the points p, q, r, s, t, w be as in the figure. (Here the dotted lines containing p, r, t and q, s, w, respectively, are understood to be null.) Then, by problem 2.3.1, we have:

$$\begin{aligned}
\frac{\|\overrightarrow{rs}\|}{\|\overrightarrow{pq}\|} &= \left[\frac{1+v_{12}}{1-v_{12}}\right]^{\frac{1}{2}} \\
\frac{\|\overrightarrow{tw}\|}{\|\overrightarrow{rs}\|} &= \left[\frac{1+v_{23}}{1-v_{23}}\right]^{\frac{1}{2}} \\
\frac{\|\overrightarrow{tw}\|}{\|\overrightarrow{pq}\|} &= \left[\frac{1+v_{13}}{1-v_{13}}\right]^{\frac{1}{2}}.
\end{aligned}$$

Mutiplying the first and second equations, and comparing with the third, yields:

$$\frac{1+v_{13}}{1-v_{13}} = \left[\frac{1+v_{12}}{1-v_{12}}\right]\left[\frac{1+v_{23}}{1-v_{23}}\right].$$

The rest is simple algebra. One need only solve for v_{13} in terms of v_{12} and v_{23}.
□

Problem 3.3.1 Formulate and prove a uniqueness result for Euclidean angular measure that corresponds to Proposition 3.3.1.

In what follows, let $(\mathbf{A}, \langle\, ,\, \rangle)$ be an n-dimensional Euclidean space, with $n \geq 2$. Our uniqueness result can be formulated as folows.

Proposition 1 Let o be a point in A, and let S_o be the set of all points p in A such that $\|\overrightarrow{op}\| = 1$. Further, let $f : S_o \times S_o \to \mathbb{R}$ be a continuous map satisfying the following two conditions.

(i) (Additivity): For all points p, q, r in S_o co-planar with o, if $\overrightarrow{oq}$ is between $\overrightarrow{op}$ and $\overrightarrow{or}$,

$$f(p, r) = f(p, q) + f(q, r).$$

(ii) (Invariance): If $\varphi : A \to A$ is an isometry of $(\mathbf{A}, \langle\ ,\ \rangle)$ that keeps o fixed, i.e., $\varphi(o) = o$, then, for all p and q in S_o,

$$f(\varphi(p), \varphi(q)) = f(p, q).$$

Then there is a constant K such that, for all p and q in S_o, $f(p, q) = K\,\measuredangle(p, o, q)$, where $\measuredangle(p, o, q)$ is understood to be defined by the requirement that $\langle\overrightarrow{op}, \overrightarrow{oq}\rangle = \cos\measuredangle(p, o, q)$.)

Note that we have the resources in hand for understanding the requirement that $f : S_o \times S_o \to \mathbb{R}$ be "continuous". This comes out as the condition that, for all p and q in S_o, and all sequences $\{p_i\}$ and $\{q_i\}$ in S_o, if $\{p_i\}$ converges to p and $\{q_i\}$ converges to q, then $f(p_i, q_i)$ converges to $f(p, q)$. (And the condition that $\{p_i\}$ converges to p can be understood to mean that the sequence $\{\|\overrightarrow{p_i\, p}\|\}$ converges to 0.)

Note also that the invariance condition is well formulated. For if $\varphi\colon A \to A$ is an isometry of $(\mathbf{A}, \langle\ ,\ \rangle)$ that keeps o fixed, then $\varphi(p)$ and $\varphi(q)$ are both points on S_o (and so $(\varphi(p), \varphi(q))$ is in the domain of f). $\varphi(p)$ belongs to S_o since

$$\|\overrightarrow{o\,\varphi(p)}\| = \|\overrightarrow{\varphi(o)\,\varphi(p)}\| = \|\Phi(\overrightarrow{op})\| = \|\overrightarrow{op}\| = 1.$$

And similarly for $\varphi(q)$. (Here Φ is the vector space isomorphism associated with ϕ.)

Proof Given any four points p_1, q_1, p_2, q_2 in S_o with $\langle \overrightarrow{op}_1, \overrightarrow{oq}_1\rangle = \langle \overrightarrow{op}_2, \overrightarrow{oq}_2\rangle$, there is an isometry $\varphi\colon A \to A$ such that $\varphi(o) = o$, $\varphi(p_1) = p_2$, and $\varphi(q_1) = q_2$. (We prove this after completing the main part of the argument.) It follows from the invariance condition that $f(p_1, q_1) = f(p_2, q_2)$. Thus we see that the number $f(p, q)$ depends only on the inner product $\langle \overrightarrow{op}, \overrightarrow{oq}\rangle$, i.e., there is a map $g\colon[-1, +1] \to \mathbb{R}$ such that

$$f(p, q) = g(\langle \overrightarrow{op}, \overrightarrow{oq}\rangle),$$

for all p and q in S_o. Since f is continuous, so must g be.

Next we use the fact that f satisfies the additivity condition to extract information about g. Let θ_1 and θ_2 be any two numbers in the interval $(0, \pi)$ such that $(\theta_1 + \theta_2)$ is in the interval as well. We claim that

$$g(\cos(\theta_1 + \theta_2)) = g(\cos\,\theta_1) + g(\cos\,\theta_2). \tag{23}$$

To see this, let p be any point in S_o, and let s be any point in A such that $\overrightarrow{os}$ is a unit vector orthogonal to $\overrightarrow{op}$. (Certainly such points exist. It suffices to start with any unit vector u in $\overrightarrow{op}^{\perp}$, and take $s = o + u$.) Further, let points q and r be defined by:

$$\overrightarrow{oq} = (\cos\,\theta_2)\,\overrightarrow{op} + (\sin\,\theta_2)\,\overrightarrow{os} \tag{24}$$

$$\overrightarrow{or} = \cos(\theta_1 + \theta_2)\,\overrightarrow{op} + \sin(\theta_1 + \theta_2)\,\overrightarrow{os}. \tag{25}$$

Clearly, q and r belong to S_o (since $\cos^2\theta + \sin^2\theta = 1$ for all θ). Multiplying the first of these equations by $\sin(\theta_1 + \theta_2)$, the second by $\sin\,\theta_2$, and then subtracting the second from the first, yields

$$\begin{aligned}
&\sin(\theta_1 + \theta_2)\overrightarrow{oq} - (\sin\,\theta_2)\,\overrightarrow{or} \\
&= [\sin(\theta_1 + \theta_2)\,\cos\,\theta_2 - \cos(\theta_1 + \theta_2)\,\sin\,\theta_2]\,\overrightarrow{op} \\
&= [\sin((\theta_1 + \theta_2) - \theta_2)]\,\overrightarrow{op} = (\sin\,\theta_1)\,\overrightarrow{op}.
\end{aligned}$$

So we can express $\overrightarrow{oq}$ in the form $\overrightarrow{oq} = a\,\overrightarrow{op} + b\,\overrightarrow{or}$, with positive coefficients

$$a = \frac{\sin\,\theta_1}{\sin(\theta_1 + \theta_2)}$$

$$b = \frac{\sin\,\theta_2}{\sin(\theta_1 + \theta_2)}.)$$

Thus $\overrightarrow{oq}$ is between $\overrightarrow{op}$ and $\overrightarrow{or}$. So, by the additivity assumption,

$$g(\langle\overrightarrow{op},\overrightarrow{or}\rangle) \;=\; f(p,r) \;= f(p,q) + f(q,r) \;=\; g(\langle\overrightarrow{op},\overrightarrow{oq}\rangle) + g(\langle\overrightarrow{oq},\overrightarrow{or}\rangle). \qquad (26)$$

But equations (24) and (25) (and the orthogonality of $\overrightarrow{op}$ and $\overrightarrow{os}$) imply that:

$$\begin{aligned}
\langle\overrightarrow{op},\overrightarrow{or}\rangle &= \cos(\theta_1+\theta_2)\\
\langle\overrightarrow{op},\overrightarrow{oq}\rangle &= \cos\,\theta_2\\
\langle\overrightarrow{oq},\overrightarrow{or}\rangle &= \cos(\theta_1+\theta_2)\,\cos\,\theta_2\;+\;\sin(\theta_1+\theta_2)\,\sin\,\theta_2\\
&= \cos((\theta_1+\theta_2)\;-\;\theta_2)\;=\;\cos\,\theta_1.
\end{aligned}$$

Substituting these values into (26) yields our claim (23).

Our argument to this point has established that the composite map

$$g\circ\cos\colon(0,\infty)\to\mathbb{R}$$

is additive. It follows by the continuity of g (and cos) that there is a number K such that $g(\cos(x)) = Kx$, for all x in $[0,\infty)$. Given any p and q in S_o, we need only substitute for x the number $\measuredangle(p,o,q)$ to reach the conclusion: $f(p,q) = g(\langle\overrightarrow{op},\overrightarrow{oq}\rangle) = g(\cos\measuredangle(p,o,q)) = K\measuredangle(p,o,q)$. $\square$

The lemma we need to complete the proof is the following.

Proposition 2 Let o and S_o be as in proposition 1. Given any four points p_1,q_1,p_2,q_2 in S_o with $\langle\overrightarrow{op}_1,\overrightarrow{oq}_1\rangle = \langle\overrightarrow{op}_2,\overrightarrow{oq}_2\rangle$, there is an isometry $\varphi\colon A\to A$ of $(\mathbf{A},\langle\;,\;\rangle)$ such that $\varphi(o)=o$, $\varphi(p_1)=p_2$, and $\varphi(q_1)=q_2$.

Proof It will suffice for us to show that there is a vector space isomorphism $\Phi\colon V\to V$ preserving the Euclidean inner product such that

$$\begin{aligned}
\Phi(\overrightarrow{op}_1) &= \overrightarrow{op}_2\\
\Phi(\overrightarrow{oq}_1) &= \overrightarrow{oq}_2.
\end{aligned}$$

For then the corresponding map $\varphi : A\to A$ defined by setting $\varphi(p) = o+\Phi(\overrightarrow{op})$ will be an isometry of $(\mathbf{A},\langle,\rangle)$ that makes the correct assignments to o, p_1, and q_1:

$$\begin{aligned}
\varphi(o) &= o+\Phi(\overrightarrow{oo}) &&= o+\Phi(\mathbf{0}) &&= o+\mathbf{0}=o\\
\varphi(p_1) &= o+\Phi(\overrightarrow{op}_1) &&= o+\overrightarrow{op}_2 &&= p_2\\
\varphi(q_1) &= o+\Phi(\overrightarrow{oq}_1) &&= o+\overrightarrow{oq}_2 &&= q_2.
\end{aligned}$$

We will realize Φ as a composition of two maps. The first will be a rotation $\Phi_1 : V\to V$ that takes $\overrightarrow{op}_1$ to $\overrightarrow{op}_2$. The second will be a rotation $\Phi_2 : V\to V$ that leaves $\overrightarrow{op}_2$ fixed, and takes $\Phi_1(\overrightarrow{oq}_1)$ to $\overrightarrow{oq}_2$. (Clearly, if these conditions are satisfied, then $(\Phi_2\circ\Phi_1)(\overrightarrow{op}_1) = \overrightarrow{op}_2$ and $(\Phi_2\circ\Phi_1)(\overrightarrow{oq}_1) = \overrightarrow{oq}_2$.) We consider Φ_1 and Φ_2 in turn.

If $p_1 = p_2$, we can take Φ_1 to be the identity map. Otherwise, the vectors $\overrightarrow{op}_1$ and $\overrightarrow{op}_2$ span a two-dimensional subspace W of V. In this case, we define Φ_1 by setting

$$\begin{aligned} \Phi_1(\overrightarrow{op}_1) &= \overrightarrow{op}_2 \\ \Phi_1(\overrightarrow{op}_2) &= -\overrightarrow{op}_1 + 2\,\langle \overrightarrow{op}_1, \overrightarrow{op}_2 \rangle\, \overrightarrow{op}_2 \\ \Phi_1(w) &= w \quad \text{for all } w \text{ in } W^{\perp}. \end{aligned}$$

(A linear map is uniquely determined by its action on the elements of a basis.) Thus, Φ_1 reduces to the identity on $W^{\perp}$, takes W to itself, and (within W) takes $\overrightarrow{op}_1$ to $\overrightarrow{op}_2$. Moreover, it preserves the inner product. (Notice, in particular, that

$$\begin{aligned} \langle \Phi_1(\overrightarrow{op}_1), \Phi_1(\overrightarrow{op}_2) \rangle &= \langle \overrightarrow{op}_2, -\overrightarrow{op}_1 + 2\,\langle \overrightarrow{op}_1, \overrightarrow{op}_2 \rangle\, \overrightarrow{op}_2 \rangle \\ &= -\langle \overrightarrow{op}_1, \overrightarrow{op}_2 \rangle + 2 \langle \overrightarrow{op}_1, \overrightarrow{op}_2 \rangle\, \langle \overrightarrow{op}_2, \overrightarrow{op}_2 \rangle = \langle \overrightarrow{op}_1, \overrightarrow{op}_2 \rangle, \end{aligned}$$

since $\langle \overrightarrow{op}_2, \overrightarrow{op}_2 \rangle = 1$.)

Next we turn to Φ_2. Since $\langle \overrightarrow{op}_2, \overrightarrow{op}_2 \rangle \neq 0$, it follow from proposition 2.3.1 that we can express $\Phi_1(\overrightarrow{oq}_1)$ and $\overrightarrow{oq}_2$ in the form

$$\Phi_1(\overrightarrow{oq}_1) = a\,\overrightarrow{op}_2 + u \tag{27}$$

$$\overrightarrow{oq}_2 = b\,\overrightarrow{op}_2 + v, \tag{28}$$

where u and v are orthogonal to $\overrightarrow{op}_2$. Now we must have $a = b$ since, by our initial assumption that $\langle \overrightarrow{op}_1, \overrightarrow{oq}_1 \rangle = \langle \overrightarrow{op}_2, \overrightarrow{oq}_2 \rangle$,

$$a = \langle \overrightarrow{op}_2,\, \Phi_1(\overrightarrow{oq}_1) \rangle = \langle \Phi_1(\overrightarrow{op}_1),\, \Phi_1(\overrightarrow{oq}_1) \rangle = \langle \overrightarrow{op}_1,\, \overrightarrow{oq}_1 \rangle = \langle \overrightarrow{op}_2,\, \overrightarrow{oq}_2 \rangle = b.$$

Moreover, since $\Phi_1(\overrightarrow{oq}_1)$ and $\overrightarrow{oq}_2$ are both unit vectors, it follows from (27) and (28) that

$$a^2 + \langle u,\, u \rangle = 1 = b^2 + \langle v,\, v \rangle.$$

So $\|u\| = \|v\|$.

Now $(\overrightarrow{op}_2)^{\perp}$, together with the induced inner product on it, is an $(n-1)$-dimensional Euclidean space. So we can certainly find a vector space isomorphism of $(\overrightarrow{op}_2)^{\perp}$ onto itself that preserves the inner product and takes u to v. We can extend this map to a vector space isomorphism $\Phi_2 : V \to V$ that preserves the inner product by simply adding the requirement that Φ_2 leave $\overrightarrow{op}_2$ fixed. This map serves our purposes because it takes $\Phi_1(\overrightarrow{oq}_1)$ to $\overrightarrow{oq}_2$, as required:

$$\Phi_2(\Phi_1(\overrightarrow{oq}_1)) = \Phi_2(a\,\overrightarrow{op}_2 + u) = a\,\Phi_2(\overrightarrow{op}_2) + \Phi_2(u) = b\,\overrightarrow{op}_2 + v = \overrightarrow{oq}_2. \quad \square$$

Problem 3.4.1 Prove the following result.

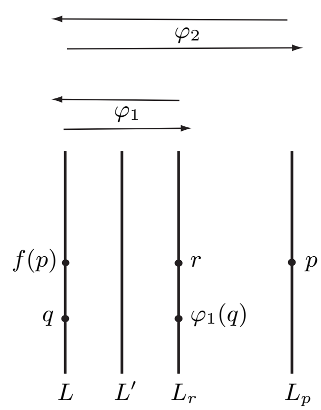
Proposition Let $(\mathbf{A}, \langle\, , \rangle)$ be an n-dimensional Minkowskian space, with $n \geq 2$. Let $\mathcal{L}$ be a frame, and let S be a two-place relation on A. Suppose S satisfies (S1) and, for some L in $\mathcal{L}$, satisfies (S2). Further, suppose S is is invariant under all $\mathcal{L}$-isometries of type (e). Then $S = Sim_{\mathcal{L}}$.

Proof For every point $p \in A$, let $f(p)$ be the unique point q on L such that $\overrightarrow{pq} \perp L$. It will suffice for us to show the following.

(iii) For all $p \in A$, $(p, f(p)) \in S$.

For then we can complete the proof *exactly* as in the case of proposition 3.4.1.

Let p be a point in A, and let r be the midpoint of the line segment connecting p and $f(p)$. (So $r = p + \frac{1}{2}\overrightarrow{p\,f(p)}$.) Further, let L_p and L_r be the (unique) lines in $\mathcal{L}$ that contain p and r respectively. Finally, let L' be the line in $\mathcal{L}$ that is midway between L and L_r. (So all four lines L, L', L_p, and L_r are subsets of a common two-dimensional subspace W. See the accompanying figure.)



By (S2), there is a unique point q on L such that

$$(r, q) \in S. \tag{29}$$

Now let $\varphi_1 : A \to A$ be a non-trivial $\mathcal{L}$-isometry of type (e) – either a reflection or rotation – that leaves L' intact and maps W onto itself. Then we have

$$\begin{aligned} \varphi_1(r) &= f(p) \\ \varphi_1(q) &= r + \overrightarrow{f(p)\,q}. \end{aligned}$$

Further, let $\varphi_2 : A \to A$ be a non-trivial $\mathcal{L}$-isometry of type (e) – either a reflection or rotation – that leaves L_r intact and maps W onto itself. Then

$$\begin{aligned} \varphi_2(f(p)) &= p \\ \varphi_2(\varphi_1(q)) &= \varphi_1(q). \end{aligned}$$

It now follows from (29) and our invariance assumption that

$$(f(p), \varphi_1(q)) = (\varphi_1(r),\, \varphi_1(q)) \in S$$

and

$$(p, \varphi_1(q)) = ((\varphi_2 \circ \varphi_1)(r),\, (\varphi_2 \circ \varphi_1)(q)) \in S.$$

So (by the symmetry and transitivity of S), we have $(p,\, f(p)) \in S$. $\square$

Problem 4.1.1 Exhibit a sentence ϕ_{par} in the language L that captures the "parallel postulate", the assertion that given a line L_1 and a point p not on L_1, there is a unique line L_2 that contains p and does not intersect L_1.

It will be convenient to introduce two abbreviations. We write

$$\begin{array}{rll} Coll(x,y,z) & \text{for} & (Bxyz \lor Bzxy \lor Byzx) \\ NoInt(x,y,u,v) & \text{for} & (x \neq y \;\&\; u \neq v) \;\&\; \neg(\exists w)(Coll(x,y,w) \;\&\; Coll(u,v,w)) \end{array}$$

Under the standard interpretation of our language, $Coll(x,y,z)$ holds if the three points x, y, z, are collinear; and $NoInt(x,y,u,v)$ holds if the line determined by x and y does not intersect the line determined by u and v.

We can take ϕ_{par} to be the sentence:

$$\begin{array}{l} (\forall x)(\forall y)(\forall z)(\ \neg Coll(x,y,z) \ \rightarrow \\ \qquad (\exists w)(NoInt(x,y,z,w) \;\&\; (\forall u)(NoInt(x,y,z,u) \ \rightarrow \ Coll(z,w,u))\ )\ ). \end{array}$$

Here is a paraphrase: Given any three points x, y, z that are not collinear, we can find a point w such that (i) the line determined by x and y does not intersect the one determined by z and w, and (ii) given any point u, if it is also true that the line determined by x and y does not intersect the one determined by z and u, then z, w, u must be collinear.

Problem 4.2.1 Verify that the map φ defined on the top of page 68 is, as claimed, a bijection between H_o^+ and D.

Recall how φ is defined. Given some point t in H_o^+, we have taken D to be the set of all points d such that $\overrightarrow{td} \perp \overrightarrow{ot}$ and $\|\overrightarrow{td}\| < 1$. And we have defined $\varphi\colon H_o^+ \rightarrow A$ by setting

$$\varphi(p) = o + \langle \overrightarrow{op}, \overrightarrow{ot} \rangle^{-1}\, \overrightarrow{op}$$

for all points p in H_o^+. We have three things to check.

(1) For all $p \in H_o^+$, $\varphi(p) \in D$, i.e., $\overrightarrow{t\,\varphi(p)} \perp \overrightarrow{ot}$ and $\|\overrightarrow{t\,\varphi(p)}\| < 1$.

(2) φ is injective.

(3) The image of H_o^+ under φ is all of D.

We take them in turn.

(1) Let p be a point in H_o^+. Then

$$\overrightarrow{t\,\varphi(p)} \;=\; -\overrightarrow{ot} + \overrightarrow{o\,\varphi(p)} \;=\; -\overrightarrow{ot} + \langle \overrightarrow{op}, \overrightarrow{ot}\rangle^{-1}\,\overrightarrow{op}.$$

It follows, since $\overrightarrow{ot}$ is a unit timelike vector, that

$$\langle \overrightarrow{t\,\varphi(p)}, \overrightarrow{ot}\rangle \;=\; -\langle \overrightarrow{ot},\, \overrightarrow{ot}\rangle \;+ \langle \overrightarrow{op}, \overrightarrow{ot}\rangle^{-1}\langle \overrightarrow{op},\, \overrightarrow{ot}\rangle \;=\; 0.$$

This gives us our first claim. Next, $\overrightarrow{t\,\varphi(p)}$ is either spacelike or equal to the zero vector, since it is orthogonal to the timelike vector $\overrightarrow{ot}$. And $\overrightarrow{op}$ is also a unit timelike vector. Hence

$$\begin{aligned}
\|\overrightarrow{t\,\varphi(p)}\|^2 \;&=\; -\langle \overrightarrow{t\,\varphi(p)},\, \overrightarrow{t\,\varphi(p)}\rangle \\
&=\; -[\,\langle \overrightarrow{ot},\, \overrightarrow{ot}\rangle \;-2\langle \overrightarrow{op}, \overrightarrow{ot}\rangle^{-1}\langle \overrightarrow{op},\, \overrightarrow{ot}\rangle \;+\; \langle \overrightarrow{op}, \overrightarrow{ot}\rangle^{-2}\langle \overrightarrow{op},\, \overrightarrow{op}\rangle\,] \\
&=\; -1 + 2 \;-\; \langle \overrightarrow{op}, \overrightarrow{ot}\rangle^{-2} \;<\; 1.
\end{aligned}$$

This gives us our second claim.

(2) Suppose p and q are in H_o^+, and $\varphi(p) = \varphi(q)$. It follows that

$$\langle \overrightarrow{op}, \overrightarrow{ot}\rangle^{-1}\,\overrightarrow{op} \;=\; \overrightarrow{o\,\varphi(p)} \;=\; \overrightarrow{o\,\varphi(q)} \;=\; \langle \overrightarrow{oq}, \overrightarrow{ot}\rangle^{-1}\,\overrightarrow{oq}.$$

But $\overrightarrow{oq}$ and $\overrightarrow{oq}$ are unit timelike vectors. So (taking the inner product of each side with itself),

$$\langle \overrightarrow{op}, \overrightarrow{ot}\rangle^{-2} \;=\; \langle \overrightarrow{oq}, \overrightarrow{ot}\rangle^{-2}.$$

And the three vectors $\overrightarrow{op}$, $\overrightarrow{oq}$, $\overrightarrow{ot}$ are co-oriented. So $\langle \overrightarrow{op}, \overrightarrow{ot}\rangle = \langle \overrightarrow{oq}, \overrightarrow{ot}\rangle$ and $\overrightarrow{op} = \overrightarrow{oq}$. Therefore, $p = o + \overrightarrow{op} = o + \overrightarrow{oq} = q$. Thus, φ is injective.

(3) Let d be any point in D. So $\overrightarrow{td} \perp \overrightarrow{ot}$ and $\|\overrightarrow{td}\| < 1$. We claim there is a point p in H_o^+ such that $\varphi(p) = d$. In fact, it suffices to take $p = o + k\,\overrightarrow{od}$ with $k = \langle \overrightarrow{od}, \overrightarrow{od}\rangle^{-\frac{1}{2}}$. (Note that $\overrightarrow{od}$ is timelike since

$$\langle \overrightarrow{od}, \overrightarrow{od}\rangle = \langle \overrightarrow{ot} + \overrightarrow{td}, \overrightarrow{ot} + \overrightarrow{td}\rangle = 1 \;+\; \langle \overrightarrow{td}, \overrightarrow{td}\rangle = 1 \;-\; \|\overrightarrow{td}\|^2 \;>\; 0.)$$

This point is certainly in H_o^+ since $\overrightarrow{op} = k\,\overrightarrow{od}$ and, therefore,

$$\langle \overrightarrow{op}, \overrightarrow{op}\rangle \;=\; k^2\langle \overrightarrow{od}, \overrightarrow{od}\rangle = 1.$$

Moreover, $\langle \overrightarrow{od}, \overrightarrow{ot}\rangle = \langle \overrightarrow{ot} + \overrightarrow{td}, \overrightarrow{ot}\rangle = \langle \overrightarrow{ot}, \overrightarrow{ot}\rangle = 1$ and, therefore,

$$\varphi(p) \;=\; o + \langle \overrightarrow{op}, \overrightarrow{ot}\rangle^{-1}\,\overrightarrow{op} \;=\; o + \langle k\,\overrightarrow{od}, \overrightarrow{ot}\rangle^{-1}\,k\,\overrightarrow{od} \;=\; o + \overrightarrow{od} \;=\; d.$$

So we are done. $\square$