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Wittgenstein

The parties to Wittgenstein's defining philosophical disputes often aren't philosophical opponents, but distinct aspects of his own thought, conflicting inclinations whose pull he finds hard to shake. As a result, his writings can be interpreted in various ways, each with passable textual support, so I don't pretend that the reading sketched here is the only one, or even necessarily the best. Still, it is the reading that's provided what have been, for me, the deepest insights, the most valuable lessons.¹

1. Representation and logic - early²

I take Wittgenstein's central contribution to be his sustained engagement with questions of logic and representation, beginning with

¹ The topics of §§1, 3, and 4 are treated more properly in chapters 3-7 of (2014a).

² On terminology, I use 'early' for the *Tractatus* and surrounding manuscripts up to 1922, 'transitional' for material from his return to philosophy in 1929 to roughly the mid-30s (*PR*, *BT*), 'mature' for the period of *PI* and *RFM*, Part I, 'post-*PI*' for 1946-1949 (*PPF*, *RPP*), and 'final' for 1949-1951 (*OC*, *LWPP*).

the *Tractatus*.³ There I take him to be asking a deceptively simple question - what must the world be like for us to represent it as we do? - and, crucially, to be presupposing what I call the 'priority of sense' - a proposition's sense, the conditions that would make it either true or false, must be in place before any consultation of the contingent state of the world.⁴ The priority of sense is what powers his case for necessarily existing simples named by simple names.⁵ To advance from an account of naming to a full account of representation, Wittgenstein adds the picture theory: arranging names a certain way represents the corresponding simple objects as arranged in the same way, even if they aren't. The final ingredient is the assumption that elementary propositions are logically independent. This soon revealed itself as the Achilles' heel of the edifice (see §2), but while in place, it yields a remarkable vision: the world is constituted by a necessarily existing store of simple objects with intrinsic possibilities of combination that actually combine in one particular way to form our contingent world; this world is represented by a particular selection of elementary propositions, essentially a truth-table row; the sense of a complex proposition is the way it divides the truth table into the rows that make it true and the rows that make it false.

³ My understanding of the book falls in a family of readings that David Stern (2003, pp. 125-132) classifies as 'metaphysical'.

⁴ This might have been more before the empirical science of linguistics came into its own in the second half of the 20th century.

⁵ This is spelled out on pp. 40-42 of (2014a).

As a bonus, this account of representation brings with it a philosophy of logic: logic is the 'scaffolding' of the world; it's what makes representation possible. Logical validities are senseless because they don't divvy up the truth table, but what they show about the world is profound: that it has this fundamental logical structure. This is what I find so attractive about the *Tractatus* - because I agree! I came to this position, not by reading Wittgenstein, but by meditating on the way Kant's logical forms of judgment generate the pure categories, which in turn characterize the world of any discursive intellect.⁶ In this way, logical forms like object-with-properties and ground-consequent structure the empirical world. Eschewing Kant's characteristic transcendental idealism (as a good naturalist⁷ should) and embracing Frege's deeper insights into the logical forms of judgment, I propose⁸ that large swaths of the world do in fact consist of objects-in-relations with ground-consequent dependencies (I call this KF-structure, after Kant and Frege), that these features validate a certain rudimentary logic,⁹ and that we humans come equipped, most likely in response to evolutionary

⁶ See (2007), §III.2, or (2014a), chapter 1.

⁷ The 'naturalism' I have in mind throughout is that practiced by the 'second philosopher' of (2007), (2014a), and (2014b), also the 'plain inquirer' of (2017).

⁸ See (2007), §III.3-8, (2014a), chapter 2, or (2014b).

⁹ I use this term for a simple logic of 'or', 'and', and 'not', with truth-value gaps and limited use of 'if-then'. Both the law of excluded middle and the material conditional fail, to be added later as beneficial idealizations. See (2007), §III.7, or (2014b) for more.

pressures, with basic cognitive mechanisms that detect these logical structures.

When I next returned to Wittgenstein after this foray of my own into the philosophy of logic, I was surprised to realize that a roughly similar point could be reached from the *Tractatus*, simply by eschewing Wittgenstein's characteristic priority of sense.¹⁰ Swap in ordinary physical things for the esoteric ontology of simple objects, retain the picture-theoretic idea that linguistic structures like subject-predicate and if-then often mirror real-worldly structures, and something like my naturalistic view reemerges. So I feel a certain kinship with the early Wittgenstein on these themes.

2. Representation - transitional

When he returned to philosophy in 1929, Wittgenstein was sorely troubled by the color exclusion problem. He could have stuck with *Tractatus* 6.3751, trusting that analysis of 'this is red' and 'this is blue' down to their basic constituents - to elementary propositions composed of simple names, representing atomic facts about simple objects - would reveal that the two propositions are logically inconsistent. Nothing on the surface makes this claim plausible, but the author of the *Tractatus* had supreme confidence in his Kant-like inference pattern: given that we do represent the world, such-and-such must be true. Now, instead, his faith in the logical

¹⁰ Strictly speaking, the assumption that elementary propositions are logically independent wouldn't vanish along with priority of sense, but the hidden ontology of simple objects is all that protects logical independence from immediate refutation (see below).

independence of elementary propositions falters, as in Waissman's notes from his conversations with Wittgenstein and Schlick in 1929:

Once I wrote, 'A proposition is laid against reality like a ruler. Only the end-points of the graduating lines actually touch the object that is to be measured'. Now I prefer to say that a *system of propositions* is laid against reality like a ruler. ... If I know that the object extends to graduating line 10, I also know immediately that it does not extend to graduating lines 11, 12, and so forth. (Waissman (1929-1931), pp. 63-64)

In *Philosophical Remarks*, the 1930 progress report on his Cambridge research grant, he realizes that 'this is red' also exists in a system of propositions ('my remark [*Tractatus* 2.0131] that a coloured body is in a colour space ... should have put me straight on this' (*PR* §83)).

Still, the *Tractarian* spirit hasn't entirely dissipated:

interconnections between color propositions are 'internal' (§80); the inferences between them are 'only formal' (§78).

If elementary propositions are now ordinary claims like 'this is red', then they're no longer the esoteric endpoint of analysis, entailing a further erosion of *Tractarian* doctrine. In *Big Typescript* of 1933, he writes:

My view [in the *Tractatus*] was wrong ... because I ... thought that logical analysis would have to bring hidden things to light (as do chemical and physical analysis). (*BT* §28)¹¹

He confesses that he'd been 'seduced by a false concept of such a reduction' (*BT* §28).

Without elementary propositions and logical analysis, what replaces the sense of a proposition as that which divides the truth

¹¹ Citations from *BT* stick to the 1933 typescript itself, avoiding the handwritten material written later (up to 1937, the editors report, into the mature period).

table in two? The *PR* offers a fairly blunt verificationism - 'to understand the sense of a proposition means to know how the issue of its truth or falsity is to be decided' (*PR* §43) - where the identity of those verification conditions isn't an empirical or otherwise contingent matter but holds 'no matter what may be the case' (*PR* §46).¹² In some parts of the *Big Typescript*, this softens into the potentially more elastic notion of grammar - 'grammatical rules determine the sense of a proposition' (*BT* §19, p. 63) - where these rules are entirely arbitrary, 'not accountable to any reality' (*BT* §56, p. 184). The rules can't be justified because justification is only possible within the game that's constituted by those very rules.

There's much more to be said about the ins and outs of the transitional period, but for present purposes, what matters is that the sense of a proposition - verification conditions or rules of grammar - must be in place before any inquiry into truth or falsity can begin. This is priority of sense, carried over from the *Tractatus*, reconstituted in different forms as the context shifts. (Also retained, by the way, is the idea that propositions are picture-like (*BT* §21).)

3. Representation - mature

Having elevated grammatical rules to such a central role, Wittgenstein turned his critical eye on the notion of rule itself and

¹² Cf. Carnap's inclusion of evidential rules in the linguistic framework, rendering them a priori.

began to doubt that it could bear the weight he'd placed on it.¹³

Preliminary versions of what would become the *Philosophical Investigations* §§1-188 were drafted as early as 1937, but the famous rule-following discussion didn't appear until sometime between 1939 and 1943, as a lead-in to material on the philosophy of mathematics. Late in 1944, Wittgenstein made a profound change: except for the cherry-picked passages on rule-following (*PI* §§192-197), all the material after §§1-188 was removed and relegated to what would eventually become Part I of *Remarks on the Foundations of Mathematics*, and the old material on the philosophy of mathematics was replaced by the now-familiar discussions of private language and the philosophy of psychology. The book was essentially complete by 1945.

In the early sections of *PI*, the gains of the transitional period are consolidated and developed: in place of ethereal propositions, we're to consider ordinary sentences; our attention is drawn to the complexity of the naming process, of ostensive definition; simples, elementary propositions, and analysis are thoroughly debunked; we're brought to recognize that words have uses beyond naming, that language has uses beyond pure description. In this context, where the meaning (sense) of a word or a sentence is often to be explained by looking to its use (*PI* §43), the pressing question is how we ever know it:

we *understand* the meaning of a word when we hear or say it; we grasp the meaning at a stroke, and what we grasp in this way is surely something different from the 'use' which is extended in time. (*PI* §138)

¹³ Seeds of this doubt were present in the transitional writings, e.g., *PR* §149, *BT* §110, p. 379.

If the use is to be encapsulated in rule, how do we grasp that rule and all its implications 'at a stroke'?

Enter the Wayward Student, the one who answers 1004 instead of 1002 when asked what follows 1000 in a series +2 (*PI* §185). The example comes from arithmetic, perhaps with the eye to the material on mathematics that was initially intended to follow, but of course this is inessential; the problem could just as well be raised by a student who apparently understands the word 'table' then surprises us by applying it to a house cat. Surely this is an incorrect use of the term 'table', but what makes it incorrect? There follows the familiar push and pull between efforts to identify what makes it incorrect - intentions or intuitions or mental images - and corresponding efforts to debunk each attempt in turn. These passages have been treated to a bewildering variety of interpretations, but the one I've found most helpful derives from the writings of David Stern.¹⁴ Roughly speaking, this discussion involves three separate voices: 'the voice of temptation', who offers various attempts to pin down the rule governing '+2' or 'table'; 'the voice of correctness', who rebuts each attempt, concluding that nothing predetermines the next step or the next proper use; and a third voice, presumably Wittgenstein himself, who sets out to dissolve the dispute with a string of ordinary observations.

This process begins with the observation that we don't come across the likes of the Wayward Student in real life:

¹⁴ See Stern (2004). He traces these ideas to Stanley Cavell.

What do I mean when I say 'the pupil's ability to learn *may* come to an end here'? Do I report this from my own experience? Of course not. (PI §144)

If we were actually confronted with a student who behaved in this way, we'd worry about his prospects, but we wouldn't be moved to doubt that 1000 plus 2 is 1002. Wittgenstein goes on to observe that our ordinary ability to follow a rule rests on everyday facts: we humans react similarly to training, find similar traits salient; we share a range of interests and motivations; and we live in a world with certain very general features. Nothing here is particularly mysterious.

So, if the purported problem of the Wayward Student isn't a problem in real life, if our ability to follow rules can be explained in pedestrian ways, why does it seem so problematic? For the third voice in the story, it isn't enough simply to describe the realities of rule following - the philosopher's discomfort won't be removed by these commonplaces alone - he must also identify the presupposition that keeps both the voice of temptation and the voice of correctness from feeling the force of those mundane observations. The philosopher's problem arises, the third voice suggests, when the case of the Wayward Student leads us to think we need an account of the correctness of '1002' as the result of adding 2 to 1000 or the incorrectness of applying 'table' to a cat that ties these judgments down no matter how unnatural a person's reactions, no matter how bizarre their practices, no matter how wild the world around them might be.

In other words, we trick ourselves into demanding what comes to the priority of sense from the *Tractatus*, now described as our 'tendency to sublimate the logic of our language' (*PI* §38). The voice of temptation thinks something or other determines correctness or incorrectness no matter what; the voice of correctness thinks no such thing exists and falls into skepticism; both assume this high standard is the one that sense must meet. Viewing the debate from a remove, as the third voice, Wittgenstein reminds us that 'the crystalline purity of logic' in his earlier book 'was, of course, not something I had discovered: it was a requirement' (*PI* §107) - purportedly, a requirement for the very possibility of representation. The mature Wittgenstein now simply observes that we manage to represent quite successfully - thank you very much - with decidedly more mundane equipment, and thus that the earlier requirement was excessive, inappropriate, wrong.

4. Logic - mature

What, then, about logical inference, say the inference from $(x)\phi(x)$ to $\phi(a)$? This story is found, not in *PI*, but in *RFM*, Part I, as prelude to the material on the philosophy of mathematics. The pattern of argument is familiar. On one side, we hear from the beleaguered voice of temptation -

But doesn't e.g. 'f(a)' have to follow from '(x)f(x)' if '(x)f(x)' is meant in the way we mean it? (*RFM* I, §10)

- and on the other, the voice of correctness -

However many rules you give me - I give a rule which justifies my employment of your rules. (*RFM* I, §113)

- which elicits the incredulous response:

Then according to you everybody could ... infer *anyhow!* (*RFM I*, §116)

The ordinary observations Wittgenstein calls on in response are also familiar: we find certain modes of arguing natural, a shared practice of inferring plays a vital role in our lives, and we live in a fairly well-behaved world. Here, as in the *Tractatus*, representation and logic are intertwined.

I noted earlier that my return to Wittgenstein after working out my own thinking on the philosophy of logic revealed both an unexpected affinity with the *Tractatus* and the priority of sense as the fundamental roadblock to further rapprochement. Moving up to the mature period, it's immediately obvious that Wittgenstein, too, is rejecting priority of sense, so - if my earlier analysis was correct - he should be drawn even closer to the naturalistic position I've been advocating. While it's admittedly counterintuitive to align Wittgenstein at any stage with a kind of naturalism as scientific as my own, I think this is in fact what's happened. We now both inhabit the everyday world of ordinary objects, with fairly stable properties, standing in relations, and so on, a world that validates laws like universal instantiation, disjunctive syllogism, and $1000+2=1002$.¹⁵ Humans in this world come equipped with shared natural reactions that make some elementary inferences so obvious as to pass without notice,

¹⁵ Wittgenstein takes elementary arithmetic identities to have the same standing as logical laws (see *RFM I*, §5). So do I (see (2007), pp. 318-319, (2014c), pp. 231-232).

and as communities, they engage in shared practices of inference that advance their understanding of themselves and their surroundings.

So, what is the flawed presupposition that leaves philosophers unmoved by these commonplaces? The third voice replies emphatically: the conviction that logic is sublime. It's one thing to forgo the priority of sense - it no longer seems so bold to claim that what we mean rests on contingencies of our psychology, behavior, and environment - but even today, many insist that the laws of logic are necessary truths, holding in all circumstances, no matter what, come what may, true in all possible worlds.¹⁶ Any position on the nature of logic that denies this, like Wittgenstein's or mine, will face considerable resistance in more traditional philosophical quarters.

If I'm right and the propensity to think in terms of KF-structures is part of our genetic and developmental endowment, then it will be hard for us to conceptualize a world in which our logic fails - which makes a direct defense of the contingency of the rudimentary laws a difficult undertaking. My approach has been to rely on a few weird thought experiments and especially on weird features of our own micro-world, where subatomic particles don't have determinate properties, and various statistical results undermine the assumption that objects are determinately individuated. If the validity of rudimentary logic rests on the presence of KF-structure, as I've claimed, then it should fail in the micro-world, and indeed it does (for example, the distributive law is violated). KF-structure isn't

¹⁶ Running in a tight circle, possible worlds are often very like the models of set-theoretic model theory, which, of course, have classic logic built in.

present in all possible worlds; it isn't even present in all sectors of our own world!

Not one to rely on natural science (see §6), Wittgenstein's case for the contingency of logic sticks with thought experiments. This is a difficult challenge for him, as well, since our logic is so intertwined with our form of life.¹⁷ In *PI*, he imagines what kinds of reactions and practices might be natural in worlds whose 'very general facts' are different from ours - like the world of wildly growing and shrinking lumps of cheese at *PI* §142 - but the direct confrontation with logic comes, as we should expect, in *RFM*.

For our purposes, two examples will do. First, consider the rubber rulers:

How should we get into conflict with the truth, if our footrules were made of very soft rubber instead of wood or steel? ... A shopkeeper might use [such a ruler] to treat different customers differently. (*RFM* I, §5)

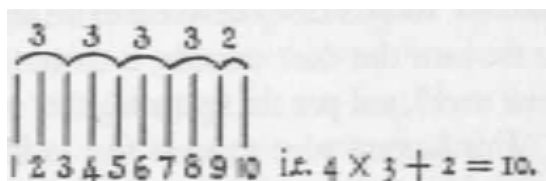
Here the physical world is the same as ours, but the shopkeeper's community has different interests that make it natural to conduct its measuring-like practice in these ways. This example doesn't concern logic, but we could just as easily (or with equal difficulty) imagine the shopkeeper's counting procedures vary from one customer to another, for similar reasons.¹⁸

The other example is the odd nut-sharer:

¹⁷ Recall the classic Stroud (1965).

¹⁸ Recall footnote 15.

Imagine someone so bewitched that he calculated



Now he is to apply this calculation. He takes 3 nuts four times over, and then 2 more, and he divides them among 10 people and each gets *one* nut; for he shares of them out in a way corresponding to the loops of the calculation, and as often as he gives someone a second nut it disappears. (*RFM I*, §137)

Here the goals of the practice are familiar - sharing nuts fairly - but the physical world and consequently the sharer's natural inclinations are not. If he were to count and calculate as we do, but in his world, with his natural inclinations, he'd come to think he needs only 10 nuts to give one to each of 10 people and he'd 'come into conflict with the truth', that is, distributing the nuts as he finds it natural to do (a second, disappearing nut to persons 3, 5, and 7), he'd run out before the end.

The moral is that our ways of inferring work for us, but they wouldn't work for people whose interests or natural reactions or worldly surroundings differ significantly from ours - the validity of our logic is contingent on these three pillars. On this much, Wittgenstein and my naturalist agree. Perhaps we also agree that the practice in which our logic is embedded is aimed at finding out how the world is, though I'm not sure about that. Where we definitely diverge is when I appeal to physical science to confirm the existence of KF-structure where it's present and to explore its absence where it's not, and to experimental psychology and evolutionary theory to explain our natural reactions. Wittgenstein's hostility to natural

science is well-known, but this is independent of his line of thought on logic and can be excised without loss - or so I've argued!¹⁹ (See §6 for more.)

5. Mathematics

As mentioned above, I take Wittgenstein's account of logic, as it developed from early to mature, to be his highest achievement and objectively one of philosophy's finest efforts in this arena. (I'm biased, of course, because I think we've reached roughly the same position from very different directions: working down from the sublime vs. working up within natural science.) Alas, I can't say the same for the broad sweep of his philosophy of mathematics.²⁰ What makes this especially poignant is that he has all the means he needs for an analysis as insightful as his treatment of logic, but his blind spots block the path.

He gets off to a bad start in the *Tractatus* with wild claims like 'The propositions of mathematics are equations' (6.2). Commentators struggle to reconstruct his view of quantification and how it might enable a viable account of mathematical induction, but there's no hope here for a treatment of mathematics as a whole. Matters don't improve much in the verificationist setting of the *PR*. If we think of the sense of an ordinary proposition as a full description of its

¹⁹ See (2014a), chapter 7.

²⁰ There are local highlights, e.g., his analyses of the significance of Russellian calculations in *RFM* III and of the force of Cantor's argument in *RFM* II (discussed below).

verification conditions, then the sense of a mathematical proposition is a full description of its proof (e.g., *PR* §§162, 166). If I'm given a full description of the verification conditions of an ordinary proposition, the truth or falsity of the proposition depends on whether or not those conditions are satisfied; but if I'm given a full description of the proof of a mathematical proposition, the question of its truth value has already been answered.²¹

Wittgenstein struggles mightily with the unacceptable consequences of this position: how can I set out to prove a mathematical proposition if it has no sense until it's been proved?; how can there be two different proofs of the same theorem?; how can there be an open question or a conjecture? Even Wittgenstein seems to realize that nothing like mathematics as we know it could be conducted under these strictures, and none of his tentative patches are remotely satisfactory, even to him.²²

In some parts of *BT*, the sense of a mathematical proposition is identified with a 'method of checking' its truth or falsity (*BT* §120, p. 423), which allows him to claim that a statement like 'there's a number between 5 and 100 such that so-and-so' - where there's a method of checking so-and-so for each number in that range - has sense even before the check has been executed. But that's not much. The general tone of the book is encapsulated in his famous revisionist rallying cry:

²¹ This revives the old problem the *Tractatus* was designed to fix: how can a proposition be both meaningful and false?

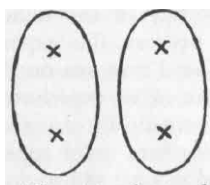
²² See, e.g., *PR*, chapter XIII.

What will distinguish the mathematicians of the future from those of today will really be a greater sensitivity; and *that* will - as it were - prune mathematics ... Philosophical clarity will have the same effect on the growth of mathematics as sunlight has on the growth of potato shoots. (In a dark cellar they grow several metres long.) (*BT* §122, pp. 432-433)

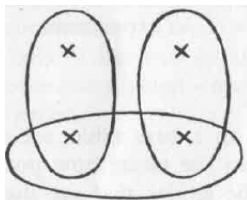
In both *PR* and *BT*, an apparent target for this pruning is mathematical uses of the infinite, the theory of sets.²³

Obviously, this clashes directly with the anti-revisionism of the mature period: 'Philosophy ... leaves mathematics as it is' (*PI* §124). By this time, Wittgenstein has renounced the demand for a unified 'theory of sense' (observing instead that words and sentences have varied uses), which largely frees him from the uncomfortable consequences of quasi-verification that bedevil the transitional writings. His discussions of arithmetic mirror the treatment of logic from *PI*, as would be expected. There is, for example, this appeal to our natural reactions:

'You only need to look at the figure



to see that $2+2$ are 4.' - Then I only need to look at the figure



to see that $2+2+2$ are 4. (*RFM* I, §38)

²³ See, e.g., *PR*, chapter XII, *BT*, chapter 137.

Anyone convinced by the second figure rather than the first would obviously be Wayward! As for the role of the world:

Put two apples on a bare table, see that no one comes near them and nothing shakes the table; now put another two apples on the table; now count the apples that are there ... The result of the counting is probably four. (We should present the result like this: when, under such-and-such circumstances, one puts first 2 apples and then another 2 on a table, mostly none disappear and none get added.) And analogous experiments can be carried out, with the same result, with all kinds of solid bodies. - This is how our children learn sums; for one makes them put down three beans and then another three beans and then count what is there. If the result at one time were 5, at another 7 ... then the first thing we said would be that beans are no good for teaching sums. But if the same thing happened with sticks, fingers, lines and most other things, that would be the end of all sums. (*RFM I*, §37)

There could hardly be a more eloquent description of KF-structure.

This is all familiar, but then there's more to mathematics than arithmetic: despite his general commitment to non-revisionism, Wittgenstein can't seem to reconcile himself to higher mathematics. For example, his criticism of Dedekind's account of the reals begins:

That every rational number can be called a principle of division of the rational numbers is perfectly clear. Now we discover something else that we can call a principle of division, e.g. what corresponds to $\sqrt{2}$. Then other similar ones - and now we are already quite familiar with the possibility of such divisions. (*RFM V*, §34)

So far, so good: each division is determined by one of these principles; this is what Wittgenstein calls an 'intensional' approach. The trouble comes nearly unnoticed when we

see them [the divisions] under the aspect of a cut made somewhere along the straight line, *hence extensionally*.

This misunderstands the role of the principle of division:

if a *principle* of division is a cut, it surely is so only because it is possible to say of any arbitrary rational number that it is on one side or the other of the cut.

Dedekind forgets this:

Dedekind's idea is that the division into an upper and lower class ... is the real number. The cut is an extensional *image*. (RFM V, §34)

The idea of a 'cut' is [a] dangerous illustration. (RFM V, §29)
 In departing from the intensional principle of division, falling into the extensional, Dedekind's account 'becomes stupid' (RFM V, §29).

Commentators differ on how to interpret Wittgenstein's alternative intensional account of the real numbers, but it's widely understood to be revisionist.²⁴ The obvious interpretive turn would be to regard this as the voice of correctness, with the corresponding voice of temptation defending the extensional account with remarks like this:

God knows *all* irrational numbers ... They are already all there, even though we only know certain of them. (RFM VII, §41)

What's missing, unfortunately, is the third voice, the one that dissolves the conflict by unmasking the flawed presupposition that pushes the combatant voices into their respective corners - at least in the selections the editors have passed on to us as the *Remarks*, no such voice materializes. The critique lodged by the voice of correctness against the metaphysical excesses of the voice of temptation is fierce, which leaves the impression that the author's sympathies lie with the intensional position. I suspect that this impression is correct, but it obviously conflicts with his avowed anti-revisionism.

²⁴ Bangu and Schatz (202?) raise questions about this assessment.

The irony is that Wittgenstein has the means available for a third voice in this conflict and others like it. The voice of temptation thinks classical analysis is justified because it describes a pre-theoretic abstract realm where the extensional cuts are all arrayed independently of us - thus validating Dedekind's move from intensional to extensional. In contrast, the voice of correctness thinks no such metaphysical justification is to be had, so we must retreat to cases where we can formulate and implement an explicit rule. A third voice could easily have pointed out that this restriction imposed by the voice of correctness is motivated by an assumption they share with the voice of temptation, namely, that the move from an intensional notion of division by a rule to an extensional notion of cut requires an extra-mathematical, presumably metaphysical, justification. Temptation thinks this extra-mathematical justification can be provided, correctness thinks it can't, but they're both missing the ordinary justification,²⁵ that is, the justification from within mathematics: the undeniable power and reach of classical analysis, especially with Dedekind's clarification of its foundations.

In Part I of *RFM*, the most definitive portion of the book, Wittgenstein comes tantalizingly close to what could have been this third voice. He's considering the notion that mathematics is a system of rules, and the question arises:

What, then - does it just twist and turn about within these rules?

²⁵ The 'the grey rags and dust' - see §7.

No, comes the reply:

it forms ever new rules: is always building new roads for traffic; by extending the network of old ones. (*RFM I*, §166)

This is what Dedekind has done, opening the new road of extensional cuts.

But then doesn't it need a sanction for this? Can it extend the network *arbitrarily*? (*RFM I*, §167)

This is the question both the voice of temptation and the voice of correctness have asked, and their shared assumption is that some kind of extra-mathematical justification is required. But here's the response in Wittgenstein's text

Well, I could say: a mathematician is always inventing new forms of description. Some, stimulated by practical needs, others, from aesthetic needs, - and yet others in a variety of ways. (*RFM I*, §167)

These passages end with the famous remark: 'The mathematician is an inventor, not a discoverer' (*RFM by*, §168). Understood in this way, Dedekind has invented something, the extensional cut, stimulated by mathematical needs - founding the calculus - and it turns out this was just the beginning of the fruitfulness of this idea. This is a perfectly Wittgensteinian dissolution, perfectly open to Wittgenstein himself. He just doesn't take it.

The much-discussed passages on Cantor's theorem (in *RFM II*) provide a related case study. Wittgenstein registers considerable disapproval:

Our suspicion is always to be aroused when a proof proves more than its means allow it. Something of this sort might be called 'a puffed-up proof'. (*RFM II*, §21)

A clever man [presumably Cantor] got caught in this net of language! (*RFM II*, §15)

I believe, and hope, that a future generation will laugh at this hocus pocus. (*RFM* II, §22)

What lies behind this alarming rhetoric, it seems to me, is the astute observation that nothing in the diagonal argument forces us to conclude that the cardinal number of the reals is greater than that of the natural numbers:

'These considerations may lead us to say that $2^{\aleph_0} > \aleph_0$.' That is to say: we may *make* the considerations lead us to that. Or: we can say *this* and give *this* as our reason. (*RFM* II, §35)

The proof is 'puffed-up' in the sense that 'it makes the determination of the concept - concept formation - look like a fact of nature' (*RFM* II, §19). What Cantor has done is place the bare diagonal argument in the context of his bold, new characterization of 'cardinal number' in terms of one-to-one correlation. It's only in that context that the proof proves what we take it to prove.

But that isn't enough, this astute line of thought continues.

Suppose we say $2^{\aleph_0} > \aleph_0$ and give the diagonal argument as our reason.

But if we do say it - what are we to do next? In what practice is this proposition anchored? It is for the time being a piece of mathematical architecture which hangs in the air, and looks as if it were, let us say, an architrave, but not supported by anything and supporting nothing. (*RFM* II, § 35)

An employment is not: yet to be discovered, but: still to be *invented*. (*RFM* II, §38)

Does this proposition have an employment? Today this seems a strange question to ask: the foundations of the calculus, which enable higher analysis, rest on the set-theoretic continuum; the theory of the transfinite is a thriving branch of pure mathematics, with its own methods and theorems; much of contemporary pure mathematics is stated in set-theoretic terms. The depth of set theory and set-theoretic

mathematics has become increasingly obvious over the many decades since Dedekind and Cantor.

These mathematical merits should have been obvious to Wittgenstein even in the 1930s, but if they were, he didn't let on; he never acknowledged set theory as a legitimate mathematical discourse with its own practices and goals. His treatment of Cantor's theorem runs parallel to that of Dedekind's cuts: the voice of temptation thinks the proof 'introduces us to the mysteries of the mathematical world' (*RFM* II, §40), shrugging off concerns; the voice of correctness thinks '*This* is the aspect against which I want to give a warning' (*RFM* II, §40), presumably making the astute observation noted above, that the diagonal argument doesn't force us to a conclusion about infinite numbers. As we've seen, Wittgenstein was similarly clear-eyed in Dedekind's case: the move to extensional cuts wasn't automatic; it was an important innovation, an 'invention'. What's missing in both cases is the dissolution of the conflict, the further observation - entirely available to him - that what justifies Dedekind's and Cantor's moves isn't something external to mathematics, as the two voices implicitly assume, but the internal developments they enable. For whatever reason, Wittgenstein can't see his way to these resolutions. His revisionist tendencies block him from the anti-revisionism to which he aspires, and in the end, he's caught vacillating between the two, without remedy.

6. Science and philosophy - final

Let me conclude this historical sweep with a brief comment on one thread in the writings of Wittgenstein's final two years (1949-1951),²⁶ one that culminates in final pages of the *Last Writings on the Philosophy of Psychology*, volume 2.²⁷ As noted at the end of §4, the greatest gulf between Wittgenstein's account of logic and my own effort is my willingness to call on physical science to investigate where the world exhibits KF-structure and where it doesn't, and on developmental and evolutionary psychology to account for our natural reactions and some of our basic interests. Not only does the mature Wittgenstein insist that philosophy offers no explanations (e.g., *PI* §§109, 126), only descriptions (e.g., *PI* §124), he also goes so far as to declare the complete irrelevance of science: 'our considerations must not be scientific ones' (*PI* §109). In the post-*PI* period, he takes explicit aim at experimental psychology:

Above all, don't wonder: 'What might be going on in the eyes or brain here?' (*PPF* §243)

So, there's no avoiding the question: why is science irrelevant?

One possible answer²⁸ traces to his discussion of sensations like pain, starting in the mature period: 'An "inner process" stands in need of outward criteria' (*PI* §580). Returning to the point later, in *Philosophy of Psychology - a Fragment* (once *PI*, Part II, the most polished of the post-*PI* writings), he asks us to 'imagine a

²⁶ In ((2022a)), I offer a reading of *On Certainty* (OC) that sees it as treating the problem of skepticism in terms of the three voices (with hinge epistemology in the voice of correctness), but I won't repeat this here.

²⁷ I'm grateful to Klagge (2017) for calling this to my attention.

²⁸ I consider others in (2014a), chapter 7.

physiological explanation of the experience' (PPF §236).²⁹ For example, suppose we've found a neural correlate for pain, or better, several neural correlates for several types of pain (shooting, burning, aching). We then discover a subject with one of these neural correlates - who isn't just numb because they can locate the sensation and describe the sort of sensation it is - but who doesn't find the sensation unpleasant and doesn't act to avoid the stimulus. Suppose finally that when faced with this subject who has the neural/sensory markers of pain but not the affective/behavioral markers, we find ourselves inclined to describe them as having pain without pain behavior. Wittgenstein concludes: 'You have now introduced a new, a physiological, criterion' (PPF §236); what were once symptoms of pain are now being treated as criteria, as what it is to be in pain. But in so doing, we've changed the subject, 'the psychological concept hangs out of reach of this explanation' (PPF §236). The scientific work tells us nothing philosophical or conceptual about pain as we understand it.

This much should sound familiar. Among the reasons we might give for applying a concept to a particular case, the tendency to favor some over others - like criteria over symptoms - echoes through Wittgenstein's earlier thought, from that-which-cannot-be-said to verification conditions as part of the sense to grammatical connections. The trouble is that his mature view doesn't allow for a unified account of sense, so it's hard to specify what counts as a

²⁹ This passage is about seeing an aspect, but given the generality of *PI* §580, it seems apt to pain as well.

criterion. James Klagge makes this point in a fascinating paper, 'The evolution of concepts';³⁰ he cites evidence of Wittgenstein's discomfort with a firm link between change in criteria and change of concept as early as 1939.

This worry apparently grew during the post-*PI* period, at least according to hearsay from his conversation partners, for example, this from Rush Rhees:

In 1944 he talked with me for several weeks about the relations of grammatical propositions and empirical propositions. He was working with the idea that the division between them was not a sharp one, and that his own earlier suggestions about this had been wrong or misleading. (quoted in Klagge (2017), p. 198)

In 1951, days before his death, Wittgenstein set this down in writing:

it would be conceivable that a connection could be established with someone else through which I would feel the same pain (i.e. the same kind of 'pain'), and in the same place, as the other person ... And if this way of getting to know someone else's pain were to have proved its worth, it's conceivable that one would apply it against a person's expression of pain, and thus would mistrust his expression if it contradicted that test. (*LWPP*, pp. 92-93)

Presumably, he'd say the same for the more likely case that a psychophysical connection is established between a person's neural state and their pain experience, that what was the neural correlate could become criterial. In his final days, then, it seems Wittgenstein would have admitted that science can help mold our concepts, that it isn't irrelevant to philosophical inquiry, after all.

As a footnote to this discussion, it's worth noting that the imagined example of a moment ago - a subject with the

³⁰ See Klagge (2017), p. 195.

neurological/sensory aspects of pain without the affective/behavioral aspects - isn't, in fact, imaginary.³¹ The condition is called 'pain asymbolia' or 'pain dissociation'; sufferers are said to feel pain without being disturbed by it. Philosophers can and do debate the correctness of this usage, but as far as I can tell, English speakers find it the natural way to describe the situation, and it's become entrenched.

7. Closing note

I've been wrestling with various aspects and stages of Wittgenstein's thought since I was an undergraduate math major, and I don't expect this to end until I do. Here I've described the general outlines of what's been for me the most interesting and productive way of reading his views and their development - noting along the way what I take to be illuminating areas of agreement and perhaps less illuminating areas of disagreement with my own views on logic, mathematics, and the relation of science to philosophy. But I'd like to close by acknowledging a moment of direct influence when my intellectual direction was fundamentally changed by his thought.

The question that drove me into philosophy and that's motivated much of my career comes from mathematics: how are the axioms of our basic set theory properly justified? In *Realism in Mathematics*, I tried to develop Gödel's analogy between mathematics and science so as to model justification in set theory on the confirmation of theory by

³¹ See Grahek (2007).

evidence in natural science. Unfortunately, soon after the book appeared, I began to doubt the underlying Quine/Putnam indispensability arguments that I'd appealed to along the way. I found myself at a loss - what now?! - what can replace the set-theoretic realism I'd counted on to underwrite the justificatory methods at work in the search for new axioms? Some form of conceptualism maybe ... something else? Then I thought of this passage:

The first step is the one that altogether escapes notice ... [it's] what commits us to a particular way of looking at the matter ... The decisive movement in the conjuring trick has been made, and it was the very one that seemed to us quite innocent. (*PI* §308)

Thinking the failure of one metaphysical view meant I needed another - maybe *this* was the false step! But where could set-theoretic justifications be rooted if not in a pre-theoretic account of its subject matter? Then another passage occurred to me:

If I am inclined to suppose that a mouse comes into being by spontaneous generation out of grey rags and dust, it's a good idea to examine those rags very closely ... But if I am convinced that a mouse cannot come into being from these things, then this investigation will perhaps be superfluous. But what it is in philosophy that resists such an examination of details, we have yet to come to understand. (*PI* §52)

As it happened, I'd just spent months among the set theorists collecting scores of considerations actually offered for or against new axiom candidates - roughly mathematical considerations. Why was I so convinced that a mouse of proper justification couldn't arise from the grey rags of these materials that I turned to extra-mathematical metaphysics for help?

What was actually needed, I realized, was a clear-eyed investigation of these intra-mathematical arguments in their own terms, to figure out how they work and how to separate the good ones

from the bad ones. That's been my project in the philosophy of set theory ever since. Without those decisive prods from Wittgenstein, I truly don't know where I'd have ended up.

One last note. There's a sense of mismatch when I take myself to be advancing a position - for example, that logic is contingent - while the mature Wittgenstein claims only to be dissolving problems - the problem of accounting for the necessity of logic dissolves because logic is actually contingent. This difference is largely packaging, though my posture may be what motivates me to follow the inquiry into the science. But there's also this: Wittgenstein dissolves philosophical problems by returning us to our ordinary beliefs and methods - but insisting that those ordinary beliefs and methods are philosophically relevant is, all by itself, a highly controversial philosophical posture. It's where my naturalism begins.³²

³² Thanks to Sorin Bangu, Adam Chen, Chris Mitsch, Stella Moon, Jeffrey Schatz, and Evan Sommers for helpful comments on an earlier draft.

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