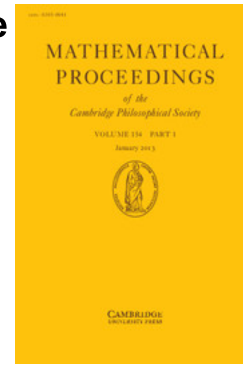

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Orientation-reversing involutions on homology 3-spheres

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1. *Introduction.* An important problem in geometric topology is to find a homology 3-sphere M with Rohlin invariant $\mu(M) = \frac{1}{2}$ such that $M \# M$ bounds an acyclic 4-manifold. If such an M exists then, for instance, all closed topological manifolds of dimension $\geqslant 5$ support a polyhedral structure(3, 4). One way of producing such examples is to find a homology 3-sphere M with $\mu(M) = \frac{1}{2}$ such that M admits an orientation-reversing diffeomorphism h, for then $M \# M$ would be diffeomorphic to $M \# -M$ which bounds an acyclic manifold. In this paper we observe by elementary means that if h is further assumed to be an involution, then this is not possible, namely:

THEOREM. *Let M be a homology* 3*-sphere which admits an orientation reversing involution. Then M bounds a parallelizable rational* 4*-ball. In particular,* $\mu(M) = 0$.

We remark that Joan Birman and also W. C. Hsiang and Peter Pao have independently shown that $\mu(M) = 0$.

2. *Almost-framings.* Suppose M^3 is an oriented 3-manifold. A *framing* for M is a particular trivialization of the tangent bundle of M. An *almost-framing* for M is a particular trivialization for the tangent bundle of M-{point} and the almost framings of M are in one-to-one correspondence with the elements of $H^1(M; Z_2)$. If $t \in H^1(M; Z_2)$ is a particular almost framing of M, then let

$$\mu(M^3, t) = \text{index}\,(W)/16 \in Q/Z,$$

where W is a 4-manifold with $\partial W = M^3$ and W is framed so as to extend the almost framing t. This invariant of a 3-manifold was investigated by J. Eells and N. H. Kuiper(2) and is known as the Rohlin invariant of M as it is well-defined by Rohlin's theorem (5).

Our only use of almost framings is the following. Note that $RP^3 \# RP^3$ has four almost framings. Two of these almost framings have Rohlin invariant zero, and the other two have non-zero Rohlin invariant, namely $\pm\frac{1}{8}$. Now $RP^3 \# RP^3$ bounds a parallelizable rational homology ball W, i.e. $RP^3 \times I$ minus a tube joining the boundary components. However, the framing on W extends an almost framing t on $RP^3 \# RP^3$ *precisely* when $\mu(RP^3 \# RP^3, t) = 0$.

Also, RP^3 itself has two almost framings. Note that RP^3 with either almost framing bounds a 4-manifold, framed so as to extend the given almost framing on RP^3, gotten by adding a two handle to the 4-ball along an unknotted circle with framing $+2$ or possibly -2.

3. *Proof of Theorem.* Let M be a homology 3-sphere with an orientation-reversing involution $h: M \to M$. By Smith theory and the Lefschetz fixed point theorem the

fixed point set $F(h)$ of h must be either S^2 or S^0, as $F(h) \neq S^1$ for an orientation-reversing involution. If $F(h) = S^2$, then $M = P \# Q$ with $h(P) = Q$ and $h(Q) = P$, so that $M = P \# -P$ which bounds an integral homology 4-ball. So we assume

$$F(h) = x_0 \cup x_1.$$

Let D_0^3, D_1^3 be disjoint equivariant disk neighbourhoods of x_0 and x_1, respectively, and let $X = cl(M - (D_0^3 \cup D_1^3))$. Let X^* denote the orbit space of h and $p: X \to X^*$ the quotient map. Note that the mapping cylinder $M(p)$ of p is a manifold and that $\partial M(p)$ is diffeomorphic to $M \# RP^3 \# RP^3$ since $h|\partial D_i^3$, $i = 0, 1$, can be assumed to be the antipodal map. We first show that $M(p)$ is a parallelizable rational 4-ball.

To show that $M(p)$ is rationally acyclic it suffices to show that X^* is rationally acyclic. Let $X_0 = cl(M - D_0^3)$ and let X_0^* be the orbit space of $h|X_0$. Since X_0 is acyclic, it follows that X_0^* is acyclic (Theorem 5·4 of (1)), so that since $\partial X_0^* = RP^2$, X^* is rationally acyclic, in fact $H_*(X^*; Z) = H_*(RP^2 \times I;\, Z)$ with the inclusion of the boundary components of X^* inducing an isomorphism on integral homology.

To show that $M(p)$ is a parallelizable 4-manifold it suffices to show that all the Stiefel–Whitney classes of $M(p)$ vanish. Observe that the mapping cylinder projection $q: M(p) \to X^*$ is a non-trivial line bundle ξ over X^*. Thus the tangent bundle of $M(p)$ splits as the Whitney sum $q^*(\xi) \oplus q^*\tau(X^*)$. Since $w_1(\xi) = 1$, $w_2(\xi) = 0$,

$$w_1(\tau(X^*)) = 1 \quad \text{and} \quad w_2(\tau(X^*)) = 1,$$

the Whitney sum formula implies that $w_1(M(p)) = w_2(M(p)) = 0$.

Now a framing of $M(p)$ induces an almost framing t of $\partial M(p)$, hence an almost framing t_1 of $\mathbb{R}P^3 \# RP^3$. By attaching to $M(p)$ the framed manifolds given in § 2 that RP^3 bounds along the two copies of $RP^3 - D^3$ in $M(p)$, we see that M bounds a parallelizable 4-manifold with the homology of $S^2 \times S^2$ minus a 4-disc. Thus, $\mu(M) = 0$ and since $\mu(M \# RP^3 \# RP^3, t) = \mu(M) + \mu(RP^3 \# RP^3, t_1) = 0$, we have that

$$\mu(RP^3 \# RP^3, t_1) = 0.$$

By § 2, t_1 extends to a framing on a rational ball W with $\partial W = RP^3 \# RP^3$. By attaching W to $\mathbb{R}P^3 \# RP^3$ minus a disk, there results a parallelizable rational homology ball B with $\partial B = M$. Note that $H_1(B; Z) = H_2(B; Z) = Z_2$ and $H_3(B; Z) = 0$.

4. *Questions.* (1) Does every homology 3-sphere which admits an orientation-reversing involution (diffeomorphism) bound an acyclic manifold?

(2) Does every homology 3-sphere which bounds a parallelizable rational 4-ball bound an acyclic manifold?

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