A UNIVERSAL 5-MANIFOLD WITH RESPECT TO SIMPLICIAL TRIANGULATIONS

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I. INTRODUCTION

One of the most important questions in geometric topology is whether or not every topological manifold has a locally finite simplicial triangulation. We first recall some results in this direction.

Let θ_3^H be the abelian group obtained from the set of oriented *PL* homology 3-spheres, under the operation of connected sum, modulo those which bound *PL* acyclic 4-manifolds. Also let $\mu : \theta_3^H \to \mathbb{Z}_2$ be the Kervaire-Milnor-Rohlin map given by $\mu[H^3] = I(W)/8 \bmod 2$ where $I(W)$ is the index of any parallelizable *PL* 4-manifold W which H^3 bounds. We note that μ is well defined and surjective.

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THEOREM 1.1. (Galewsik-Stern [3], [4]). Let M^n be a compact topological n-manifold with $n \geq 6$ ($n \geq 5$ if ∂M simplicially triangulated). Then there exists an element $t_M \in H^5(M, \partial M; \ker \mu)$ such that $t_M = 0$ y and only if there exists a simplicial triangulation $K|\partial M$ compatible with the given triangulation on ∂M. Moreover, the number of concordance classes of simplicial triangulations rel ∂M *is in 1-1 correspondence with $H^4(M, \partial M; \ker \mu)$, where two triangulations K_1 and K_2 of M are concordant* rel ∂M *if there is a triangulation K of $M \times I$ such that $K|_{M\times 0}$, $K|_{M\times 1}$, and $K_{\ \ M\times I}$ are compatible with K_0, K_1 and $K|_{\partial M} \times I$, respectively.*

THEOREM 1.2. (Galewski-Stern [4]). If $\sigma_M \in H^4(M, \partial M; \mathbb{Z}_2)$ is the Kirby-Siebenmann obstruction [5] to a PL triangulation of M rel ∂M *then $\beta(\sigma_M) = t_M$ where β is the Bockstein associated with the exact sequence*

$$0 \to \ker \mu \to \theta_3^H \xrightarrow{\mu} \mathbb{Z}_2 \to 0.$$

THEOREM 1.3. (Galewski-Stern [3], [4]; T. Matumoto [7]). If there exists an element $x \in \theta_3^H$ with $\mu(x) = 1$ and $2x = 0$, then all compact topological n-manifolds with $n \geq 6$ ($n \geq 5$ if ∂M simplicially triangulated) have a simplicial triangulation.

Our original classification theorem in [3] had two possibly non-zero obstructions. However, the solution of the double suspension conjecture ([1],[2]) implies that one of these obstructions vanish.

In this paper we give a geometric construction of a closed non-orientable topological 5-manifold N^5 with the property that N^5 has a simplicial triangulation if and only if *every* compact topological n-manifold M^n $n \geq 6$ ($n \geq 5$ if ∂M simplicially triangulated) has a simplicial triangulation. Note that Siebenmann's Theorem B of [9] and the double suspension theorem ([1],[2]) show that all open or oriented closed 5-manifolds can be simplicially triangulated.

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II. PRELIMINARY RESULTS

Recall S_q^1 is the Bockstein associated with the short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{r} \mathbb{Z}_2 \longrightarrow 0.$$

THEOREM 2.1. If there exists a closed simplicially triangulated topological n-manifold N^n for any $n \geq 5$ with $S_1^1\sigma_N \neq 0$ where σ_N is the Kirby-Siebenmann obstruction then <u>*all*</u> *compact topological m-manifolds M^m with $m \geq 6$ ($m \geq 5$ if ∂M simplicially triangulated) have a simplicial triangulation.*

Proof. Let N^n be a closed simplicially triangulated topological n-manifold with $n \geq 5$ and assume that there exists a compact m-manifold M^m with $m \geq 6$ ($m \geq 5$ if ∂M simplicially triangulated) that does not have a simplicial triangulation. They by 1.3 there does not exist an element $x \in \theta_3^H$ with $\mu(x) = 1$ and $2x = 0$. Let θ be the finitely generated subgroup of θ_3^H generated by the 3-dimensional links of a triangulation of N^n. We now construct a homeomorphism $\gamma : \theta \to \mathbb{Z}_4$ so that the following diagram commutes

$$\begin{array}{ccccccccc} & & & & \theta & \xrightarrow{i} & \theta_3^H & & \\ & & & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\mu} & & \\ 0 & \to & \mathbb{Z}_2 & \xrightarrow{\times 2} & \mathbb{Z}_4 & \xrightarrow{r} & \mathbb{Z}_2 & \to & 0 \end{array}$$

where r is reduction mod 2. By the fundamental theorem for finitely generated abelian groups there exists elements $x_1, x_2, \ldots, x_k$

of θ such that $\theta = \langle x_1\rangle \oplus \langle x_2\rangle \oplus \ldots \oplus \langle x_k\rangle$ where $\langle x_i\rangle$ is the cyclic group generated by $\langle x_i\rangle$. If $\mu(x_i) = 0$, define $\gamma(x_i) = 0$; if $\mu(x_i) = 1$ and $\langle x_i\rangle \approx \mathbb{Z}$, define $\gamma(x_i) = 1$; and if $\mu(x_i) = 1$ and $\langle x_i\rangle \approx \mathbb{Z}_{2^j p}$ then $j \geq 2$ by our assumption, so define $\gamma(x_i) = 1$. It is easy to check that γ is well defined and that $\mu i = r\gamma$.

Note that N with its triangulation is a homology manifold so there exists an obstruction [6] $\tilde{\sigma}_N \in H^4(N;\ \theta_3^H)$ to resolving N to a PL manifold. Now $\tilde{\sigma}_m$ assigns to every 4-dimensional dual cell of N, its boundary, which is PL homeomorphic to a 3-dimensional link in N and hence in θ. So there exists a $\tilde{\sigma}_N \in H^4(N;\ \theta)$ so that $i_* \tilde{\tilde{\sigma}}_N = \tilde{\sigma}_N$. Also by [4] $\mu_* \tilde{\sigma}_N = \sigma_N$. Hence $S_q^1\ \sigma_N = S_q^1\ \mu_*\tilde{\sigma}_N = S_q^1\ \mu_* i_* \tilde{\sigma}_N = S_q^1 r_* \gamma_* \sigma_N = 0$ since $S_q^1 r_* = 0$. A contradiction to the assumption that there exists a manifold M^m, $m \geq 6$ ($m \geq 5$ if ∂M has a simplicial triangulation) which is not trianguable. Thus the theorem follows.

We note that Siebenmann [8] has shown the existence of a topological 5-manifold N with $S_q^1\ \sigma_N \neq 0$. In the next section we will explicitly construct such a manifold.

III. THE CONSTRUCTION

We first recall theorem 1.4 of [4].

THEOREM 3.1. A homology manifold H^n with $\dot{H} = \emptyset$ and $n \geq 5$ is a topological n-manifold if and only if the links of vertices are 1-connected.

We note this theorem was a consequence of the double suspension conjecture recently proved by J. Cannon [1] and R. D. Edwards [2].

Now we use this result to geometrically construct a closed topological 5-manifold N with $S^1_q\ \sigma_N \neq 0$. Thus if there existed a simplicial triangulation of N, all compact topological m-manifolds with $m \geq 6$ ($m \geq 5$ if ∂M simplicially triangulated) would be simplicially triangulable!

Let H^3 be any oriented PL homology 3-sphere that bounds an oriented parallelizable PL 4-manifold W with index 8. Let $X = W \cup_H c(H)$ where $c(H)$ is the cone on H and x the cone point of $c(H)$. Attach a PL 1-handle $D^3 \times [0,1]$ to $c(H) \times 0 \cup c(H) \times 1 \subset X \times [0,1]$ so that $\partial S = H \# H$ (not $H \# -H$) where $S = c(H) \times 0 \cup_{D^3 \subset H^3} D^3 \times I \cup_{D^3 \subset H^3} c(H) \times 1$. Let $Y = X \times I \cup S \cup_{H \# H} c(H \# H)$ where z is the cone point of $c(H \# H)$. Note that the polyhedron Y contains the sub polyhedron $T = S \cup_{H \# H} c(H \# H)$ and is a homology 4-manifold with the same homotopy type as S^4. Let $P = Y \cup_T c(T)$ where y is the cone point of $c(T)$. Now P is a homology 5-manifold with ∂P PL homeomorphic to $W \quad W \cup c(H \# H)$, where y denotes connected sum along the boundary. Note that all of the Steiefel-Whitney numbers of ∂P are zero. Next add an exterior collar $C = \partial P \times [0,1)$ to P along ∂P and call the resulting homology 5-manifold Q.

We first observe that the only 4-dimensional links i.e. $z, y, x \times 0$ and $x \times 1$, which are not PL homeomorphic to S^3 are simply connected, hence by 3.1 Q is a simplicially triangulated 5-manifold.

We next observe that the only 3-dimensional links of Q which are not PL homeomorphic to S^3 occur as links of the subpolyhedron $L = x \times [0,1] \cup y * (x \times [0,1]) \overset{PL}{\approx} S^1$ and $M = y * z \cup z \times [0,1) \overset{PL}{\approx} [0,1)$ of Q. The links of 1-simplexes of L are PL homeomorphic to H and the links of 1-simplexes of M are PL homeomorphic to $H \# H$. Thus by Theorem C of [9] there exists a PL structure Σ of $Q - L$ since $\mu[H \# H] = 0$. Note that Σ does not agree with the polyhedral structure of Q.

We can now use *PL* transversality with respect to $\Sigma|_{\partial P \times (0,1)}$ to get a compact connected orientable 4-dimensional submanifold V in $\partial P \times (0,1)$, with trivial normal bundle, which separates $\partial P \times [0,1)$ into two components A and B, with the closure of one of them, say A, containing ∂P. Now $P \cup c\ell[A]$ is a topological manifold with $\partial(Q \cup c\ell[A]) = V$. Since all of the Stiefel-Whitney numbers of V are zero, V bounds a *PL* 5-manifold $\overline{W}$. Finally define $N^5 = P \cup_{\partial P} c\ell[A] \cup_V \overline{W}$.

Now since $N - L$ is a *PL* manifold it is clear that the Poincare dual of σ_N is represented by L. Also the Poincare dual of the first Stiefel-Whitney class of N, $w_1(N)$, restricted to P is represented by $X \times \frac{1}{2}$. Therefore by the $W\mu$ formula, $S_q^1\ \sigma_N = W_1(N) \cup \sigma_N$ = intersection number of L and $X \times \frac{1}{2}$ which is non-zero. Therefore N is the desired 5-manifold.

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