Constructing Lens Spaces by Surgery on Knots.
Fintushel, Ronald; Stern, Ronald J.
in: Mathematische Zeitschrift, volume: 175
pp. 33 - 52

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Math. Z. 175, 33–51 (1980)

Mathematische Zeitschrift
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Constructing Lens Spaces by Surgery on Knots

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§ 1. Introduction

The lens space $L(p, q)$ is the manifold obtained from Dehn surgery on the unknot with surgery coefficient p/q. It was known to Seifert in the 1930's that lens spaces could also be obtained from surgery on non-trivial torus knots. In [4] Moser conjectured that surgery on a nontorus knot could not yield a lens space. Subsequently, Bailey and Rolfsen [1] presented a counterexample to Moser's conjecture by showing that the lens space $L(23, 7)$ could be obtained by -23-surgery on the 11,2-cable on the trefoil knot ($C(2, 11; T(2, 3))$ in our notation; see § 3). Simon also discovered similar examples, and the preponderance of such examples led Bailey and Rolfsen ([1]) to ask whether a lens space can be constructed from a knot which is not an iterated cable (i.e. a cable on a ... cable on a torus knot).

In § 4 we present several techniques for constructing infinitely many knots which are not iterated cables, but upon which appropriate surgeries yield lens spaces. In order to recognize that our examples are not iterated cables, we classify in § 3 the lens spaces which can be obtained from Dehn surgery on an iterated cable. In particular, we show that if a lens space is obtained from a Dehn surgery on a cable on a torus knot then the surgery must actually be integer surgery and the lens space must have odd order fundamental group. Further, no Dehn surgery on an iterated cable with more than one iteration can give a lens space.

Our proofs are based on the Kirby-Rolfsen calculus of links ([5], [8]), and the key to our point of view on cable knots is the surgery description of a cable knot given in Proposition 2.

Our interest in constructing lens spaces by surgery on knots comes from the following proposition.

Proposition 1. *If $L(p, q)$ bounds a simply-connected 4-manifold W^4 with $H_2(W^4) = \mathbb{Z}$ then $\pm q$ is a quadratic residue* (mod p).

* Supported in part by NSF grant MCS-7900244
** Supported in part by NSF grant MCS-76-06393

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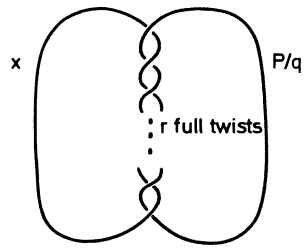
Hence this condition must be satisfied if $L(p,q)$ arises from integer surgery on a knot. We shall only sketch the proof.

Proof sketch. If $L(p,q)$ bounds such a 4-manifold W, remove a 4-ball from Int W to obtain a cobordism V^4 from $L(p,q)$ to S^3. Consider a handle decomposition of V based on $L(p,q)$. Because $\pi_1(V)=H_3(V,L(p,q))=H_4(V,L(p,q))=0$ and $H_2(V,L(p,q))=\mathbb{Z}$, we can find such a handle decomposition where all the handles cancel algebraically, except for a single 2-handle. This means that $S^3=\partial_+ V$ is the same homologically as the result of performing an integer surgery on a knot in $L(p,q)$. Up to homology we may as well assume that this knot is "unknotted." Hence there are integers r and x such that



is a homology sphere. But the order of H_1 for the above-described 3-manifold is $|px-r^2q|$. Hence $px-r^2q=\pm 1$, and it follows that $L(p,q)\cong L(p,r^2)$. □

Note that the condition that $\pm q$ be a quadratic residue (mod p) is equivalent to the condition that $L(p,q)$ be homotopy equivalent to $L(p,1)$.

If p is odd it can be arranged for the homology sphere obtained in the above proof to have μ-invariant 0. Thus if one can find an $L(p,q)$ with p odd which is homotopy equivalent to $L(p,1)$ and which is not the boundary of a simple-connected 4-manifold of 2nd betti number 1, the corresponding μ-invariant 0 homology sphere could not bound a contractible 4-manifold.

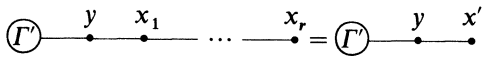
§ 2. Rational Plumbing Diagrams

As we have already mentioned, our proofs are based on the Kirby-Rolfsen calculus of links which will here often take the guise of a plumbing diagram calculus. In this section we shall describe this calculus. We assume that the reader is familiar with the Kirby-Rolfsen calculus.

Suppose L is a link in S^3 composed of unknotted circles L_i which are weighted by rational numbers x_i and suppose that the linking matrix for L has as its entries only 0's and ± 1's. The weighted link L provides a Dehn surgery description for a 3-manifold M^3. We shall often abbreviate this description by replacing L by the weighted graph Γ with a vertex v_i of weight x_i corresponding to each L_i and such that v_i and v_j bound an edge of Γ if and only if L_i and L_j are lenked. We are interested in the special case where Γ is a weighted tree, and we call Γ a *(Dehn) plumbing diagram* for M^3. Often we shall abuse notation by writing $\Gamma=M^3$. If Γ_1 and Γ_2 are weighted trees, $\Gamma_1=\Gamma_2$ will mean that they

correspond to homeomorphic 3-manifolds. Notice that unless all the wieghts of Γ are integers we obtain only a 3-manifold, not a 4-manifold with boundary. Now in a sequence of lemmas we shall translate the Kirby-Rolfsen calculus into this setting. If $x_1, \ldots, x_{r-1}$ are integers and x_r is a rational number, $[x_1, \ldots, x_r]$ denotes the continued fraction $x_1 - 1/x_2 - \ldots - 1/x_r$.

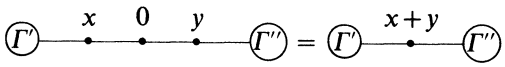
Lemma 1. *If $x_1, \ldots, x_{r-1}$ are integers then*



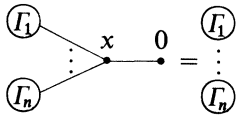
where $x' = [x_1, \ldots, x_r]$.

Proof. This is well-known. See, for example [8; p. 272] and our proof of Proposition 2. □

Lemma 2. *If x is an integer then*



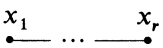
and



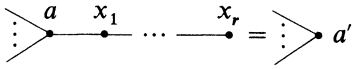
(the disjoint union). □

Since these are trivial exercises in Kirby-Rolfsen calculus, we omit the proofs. The next lemma is a restatement of moves RI and RII of [9].

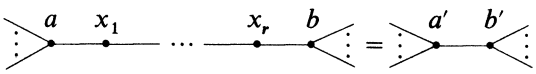
Lemma 3. *Suppose $x_1, \ldots, x_{r-1}$ are integers and*



is S^3. (This will be true if and only if $[x_1, \ldots, x_r]^{-1}$ is an integer.)

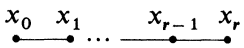


for some a', and if x_r is also an integer



for some b'. □

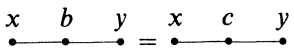
In his thesis [9] Scharf defines a plumbing diagram to be in *reduced form* when no more moves arising from Lemmas 2 and 3 can be performed on it. A plumbing diagram is *linear* if it has the form



where $x_1, \ldots, x_{r-1}$ are integers. Our next lemma is a particular case of a theorem of Scharf.

Lemma 4. (Scharf [9; 5.6]). *Let M^3 be the 3-manifold corresponding to a plumbing diagram Γ. If $\pi_1(M^3)$ is a direct sum of cyclic groups the reduced form of Γ is a disjoint union of linear plumbing diagrams.* □

Lemma 5. *Let $S^3 =$*



for integers $b \neq c$. Then either $1/x$ or $1/y$ is an integer and $b-c=\pm 1$. If $x \neq 0$ and $1/x$ is not an integer then $x=\pm 2/q$ for some odd integer q.

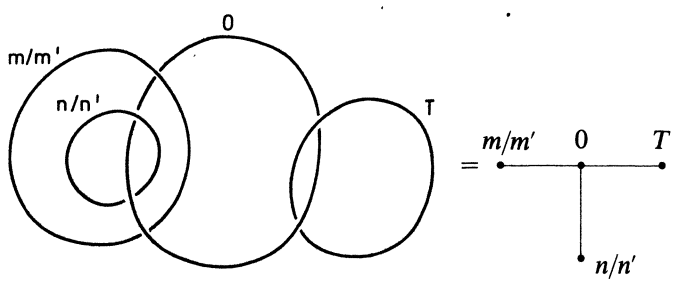
Proof. If $x=0$, then it follows from Lemma 2 that $1/y$ is an integer. So suppose $x, y \neq 0$. Then these plumbing diagrams correspond to Seifert fiberings of S^3 with Seifert invariants $(0; (1, -b), (x_1, x_2), (y_1, y_2))$ and $(0; (1, -c), (x_1, x_2), (y_1, y_2))$ if $x = x_1/x_2$, $y = y_1/y_2$ as fractions in lowest terms with $x_1, y_1 > 0$. (See [6].) Because these are Seifert invariants for S^3 we have $-bx_1x_2 + x_1y_2 + x_2y_1 = \varepsilon = \pm 1$ and $-cx_1x_2 + x_1y_2 + x_2y_1 = \varepsilon' = \pm 1$. Because $b \neq c$, $\varepsilon = -\varepsilon'$. Hence $(b+c)x_1y_1 = 2(x_1y_2 + x_2y_1)$. Since x_1 is relatively prime to both x_2 and y_1, $x_1 = 1$ or 2. Similarly $y_1 = 1$ or 2; so if $x_1 = 2$ then $y_1 = 1$. We then have $\varepsilon = -2b + 2y_2 + x_2$, $\varepsilon' = -2c + 2y_2 + x_2$; so $\varepsilon' - \varepsilon = 2(b-c)$ and $b - c = \pm 1$. □

§ 3. Surgery on Cable Knots

Let K be a knot in S^3 with tubular neighborhood N. Any knot K' lying on ∂N is called a *cable* of K. The *winding number* of K' is the integer $p \in \mathbb{Z} = H_1(N; \mathbb{Z})$ with $p \geqq 0$ and K' representing p. A cable is *nontrivial* if its winding number is not 0 or 1. The *linking number* q of K' is $q = lk(K, K')$ where K' is oriented so that its class in $H_1(N) = \mathbb{Z}$ is nonnegative. The cable knot K' is determined by these integers and we write $K' = C(p, q; K)$.

We begin this section by giving a surgery presentation for a cable knot and then use this presentation to describe precisely those lens spaces and connected sums of lens spaces which can be obtained from surgery on iterated cables of torus knots.

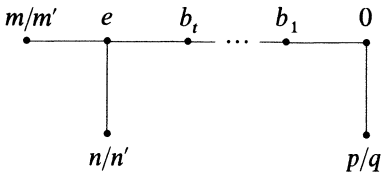
Let $m, n > 1$ be integers and m', n' integers such that $mn' + m'n = \varepsilon = \pm 1$. Then



is a surgery description for the (m, n)-torus knot $\varepsilon T(m, n)$, right- or left-handed as $\varepsilon = +1$ or -1. This means that Dehn surgery on the circles labelled 0, m/m', and n/n' turns T into $\varepsilon T(m, n) \subseteq S^3$.

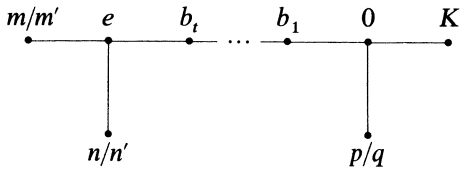
Proposition 2. *Let $m, n, p > 0$ and $m', n', e, q, b_1, \ldots, b_t$ be integers with $mn' + m'n = \varepsilon = \pm 1$ such that*

(*)

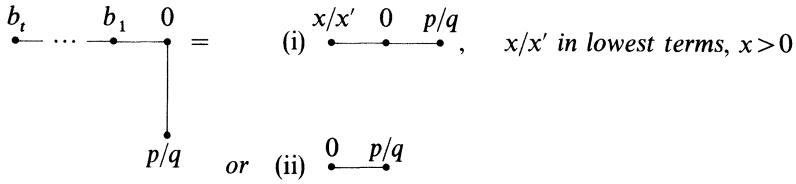


is S^3. Then

(**)

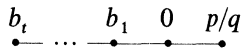


is a surgery presentation for $C(p, \varepsilon pmn + \delta x;\ \varepsilon T(m, n))$ where

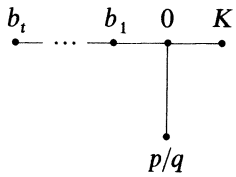


In case (i) $\delta = px' + qx = \pm 1$, *and in case* (ii) $\delta = q = \pm 1$ *and* $x = 1$.

Proof. Since (*) is S^3 and $m, n \geqq 2$, it follows from Lemma 4 that

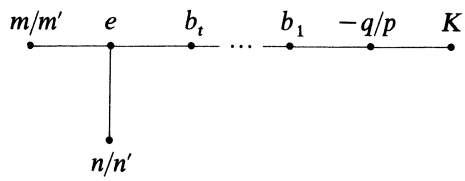


is S^3. Thus



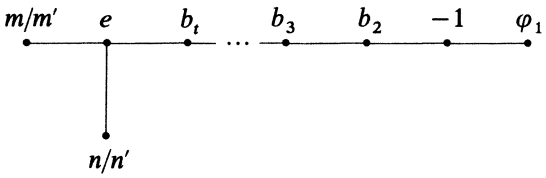
is a surgery presentation for $K = \delta T(p, x)$.

We shall now apply the Kirby-Rolfsen calculus to (**) to obtain the link in S^3 consisting of the torus knot K and the circles labelled e, m/m', and n/n'. Consider first case (i). The diagram (**) is equivalent to



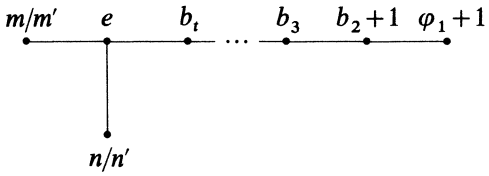
Twist the complement of the $-q/p$-circle $(b_1 + 1)$-times in the left-handed

direction. Our surgery diagram becomes

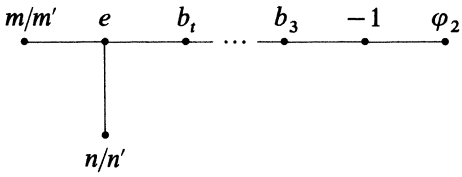


for some rational number φ_1. Let $N(\varphi_1)$ denote a tubular neighborhood of the φ_1-circle. The knot K now lies on $\partial N(\varphi_1)$ with winding number b_1+1 and linking number -1. Abbreviate this by $K=C(b_1+1,-1)$ on $\partial N(\varphi_1)$.

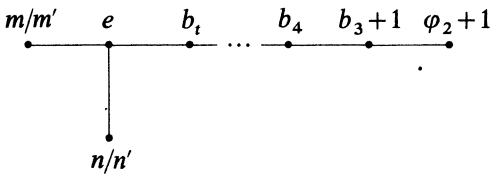
Next blow down the -1-circle; i.e. twist its complement once in the right-handed direction. We obtain



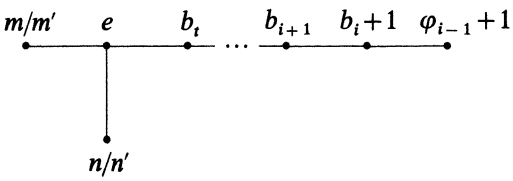
and $K=C(b_1+1,b_1)$ on $\partial N(\varphi_1+1)$. Then twist the complement of the (φ_1+1)-circle b_2+2 times in the left-hand direction to obtain



and $K=C(-b_1(b_2+2)+b_1+1,b_1)$ on $\partial N(\varphi_2)$ (with the convention that we allow the winding number to be negative; so that here $lk(K,\varphi_2\text{-circle})$ $=\operatorname{sign}((-b_1b_2-b_1+1)\cdot b_1)$. Blowing down -1:

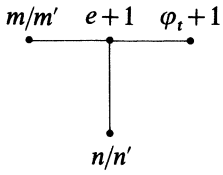


and $K=C(-b_1b_2-b_1+1,\ -b_1b_2+1)$ on $\partial N(\varphi_2+1)$. Notice that $lk(K,(\varphi_2+1)$-circle$)=\pm(-b_1b_2+1)=\pm$(numerator of $[b_1,b_2]$). Continuing inductively, at each stage



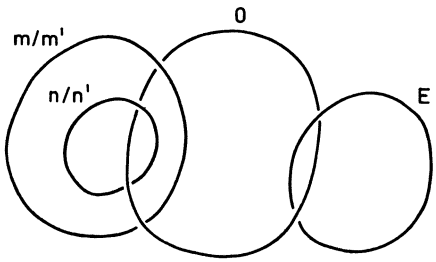
K lies on $\partial N(\varphi_{i-1}+1)$ and its linking number with $\varphi_{i-1}+1$ is $\pm(\text{numerator}\,[b_1,\ldots,b_{i-1}])$.

Eventually we reach



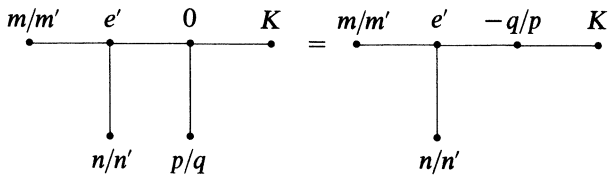
and K lies on $\partial N(\varphi_t+1)$ with linking number $\pm x$. Since this diagram yields S^3 we must have $\varphi_t+1=\dfrac{1}{e+1}$, and twisting the complement of the (φ_t+1)-circle $e+1$ times in the left-hand direction we obtain

(***)

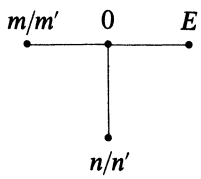


where K lies on $\partial N(E)$ with linking number $\pm x$. Since $K=\delta T(p,x)\subseteq S^3$, the winding number is p and the linking number is δx.

In case (ii) first apply Lemma 3 to obtain



and since K is a knot in S^3, $-q/p=\dfrac{1}{e'}$, so $p=|e'|$ and $q=\pm 1$. If we twist the complement of the $\dfrac{1}{e'}$-circle, e' times in the left-hand direction we obtain



and $K=C(-e',1;E)$. Since $\delta=q=-\text{sign}(e')$, and $x=1$ in this case, $K=\delta T_{(p,x)}\subseteq S^3$ in case (ii) as well.

Notice that E becomes the torus knot $\varepsilon T(m,n)$ after the labelled Dehn surgeries are performed. This means that K is a cable on $\varepsilon T(m,n)$ with winding number p. If we give E any framing then performing these Dehn surgeries increases the framing on E by εmn. Thus the curve L on $\partial N(E)$ which becomes a longitude after the surgery links E $-\varepsilon mn$ times in the surgery description (***). The intersection number of K and L on $\partial N(E)$ is $\varepsilon pmn+\delta x$. Hence $K = C(p, \varepsilon mnp+\delta x;\ \varepsilon T(m,n))$. □

Definition. *Given a surgery description S_K of a knot K, let L be a loop on the boundary of a tubular neighborhood for K in S_K such that L becomes a longitude for K after the surgery. Define $\varphi(S_K)=lk(L,K)$. This is well-defined since L lies in a uniquely determined homology class.*

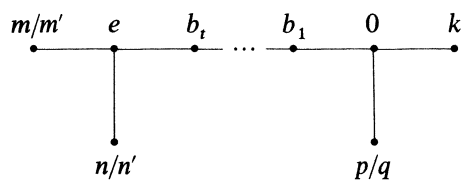
Remarks. i) Given a surgery presentation $S_{K'}$ for a knot K', Proposition 2 shows how to obtain $C(p, p\varphi(S_{K'})+\delta x, K')$.

ii) All cables are obtained in this fashion since x is restricted only to be relatively prime to p.

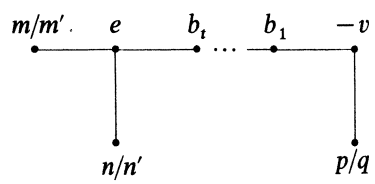
We now shall use the above surgery description to determine those lens spaces obtained by surgery on iterated cables of torus knots. Consider first the case of a single cable on a torus knot.

Theorem 1. *A lens space is obtained from a Dehn surgery on a nontrivial cable on a nontrivial torus knot if and only if the surgery is $4\varepsilon mn\pm 1$-surgery on $C(2, 2\varepsilon mn \pm 1;\ \varepsilon T(m,n))$. The result is then the lens space $L(4\varepsilon mn\pm 1, 4n^2)$.*

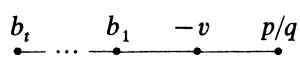
Proof. Consider the Dehn surgery diagram



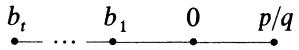
for some rational number k. (We are adopting the notation of Proposition 2.) Since the diagram represents a lens space L^3, Lemma 4 implies that $k=\frac{1}{v}$ for some nonzero integer v. Thus L^3 is given by



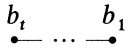
Lemma 3 and 4 imply that



is S^3. But also (as in Proposition 2)

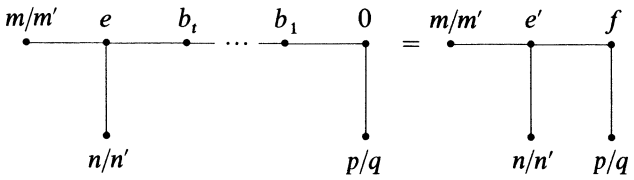


is S^3. From Lemma 5



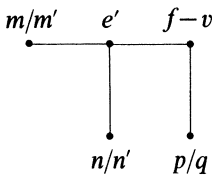
is S^3. Hence in the formula given in Proposition 2, $x=1$.

Using Lemma 3 we have $S^3=$



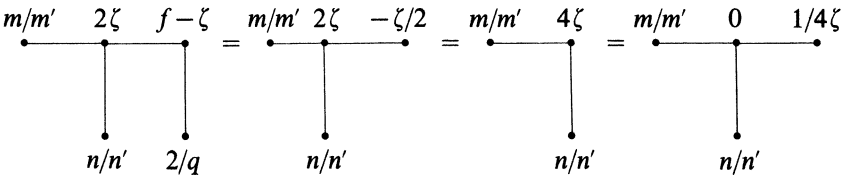
where $f-\frac{q}{p}=\frac{1}{e'}$; so $q=fp\pm 1$.

Now L^3 is given by



Since L^3 is a lens space $f-v-\frac{q}{p}=\frac{1}{l}$, some $l\in\mathbb{Z}$. Substituting the expression for q already obtained, we see that $vp=0$ or ± 2. Since $p\geqq 2$ and $v\neq 0$ are integers this means that $v=\pm 1=\zeta$ and $p=2$.

From the expression $f-v-\frac{q}{p}=\frac{1}{l}$ we now obtain $f=\frac{2\zeta+q\pm 1}{2}$. And since $f-\frac{q}{p}=f-\frac{q}{2}=\frac{1}{e'}$, we have $\frac{2\zeta\pm 1}{2}=\frac{1}{e'}$. It follows easily from the fact that $\zeta=\pm 1$ that $e'=2\zeta$. Hence $\frac{2f-q}{2}=\frac{1}{2\zeta}$; i.e. $2f-q=\zeta$. So L^3 is



since $f-\zeta-\frac{q}{2}=-\frac{\zeta}{2}$. The last diagram shows that L^3 is a lens space obtained

from $\left(\frac{1}{4\zeta}+\varepsilon mn\right)$-surgery on $\varepsilon T(m,n)$. So Moser's formula shows that L^3 is just $L(4\varepsilon mn+\zeta, 4n^2)$. $\square$

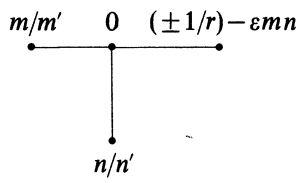
If we consider cables on arbitrary knots rather than just torus knots, our argument proves the following more general result.

Theorem 2. (i) *Let S_K be a surgery presentation for a knot K. The manifold obtained by $\varphi(S_K)\pm\frac{1}{4}$ surgery on K can also be obtained by $4\varphi(S_K)\pm 1$ surgery on $C(2, 2\varphi(S_K)\pm 1; K)$.*

(ii) *If the manifold obtained by a Dehn surgery on an (iterated, see Theorem 3) cable about K can also be obtained by a Dehn surgery on K, then it must be a cable (with only one iteration) with winding number ± 2 and Dehn surgery coefficient $\varphi(S_K)\pm\frac{1}{4}$ for some surgery presentation S_K for K.*

Each knot K has a surgery presentation S_K with $\varphi(S_K)=0$ because K can be unknotted by a finite number of ± 1-surgeries on unknotted circles which have algebraic linking number 0 with K [8; p. 159]. In this special case Theorem 2(i) states that the manifold obtained by 1/4-surgery on K is the same as the manifold obtained by $+1$-surgery on $C(2,1;K)$. This result was first proved by Neuzil [7].

Example. It is a question of some interest whether all Brieskorn spheres arise from surgery, and in particular integer surgery, on knots. It is easy to see that $\Sigma(m,n,|\pm 1-\varepsilon rmn|)$ comes from $\pm\frac{1}{r}$ surgery on $\varepsilon T(m,n)$ since $\Sigma(m,n,|\pm 1-\varepsilon rmn|)$ is

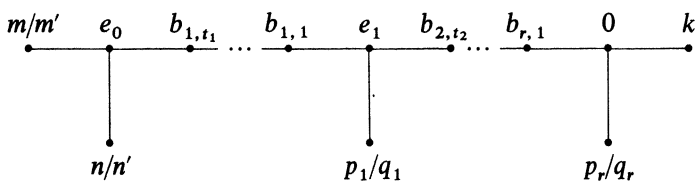


Theorem 2 then implies that $\Sigma(m,n,4mn\pm 1)$ can be obtained by integer surgery on a cable of a torus knot.

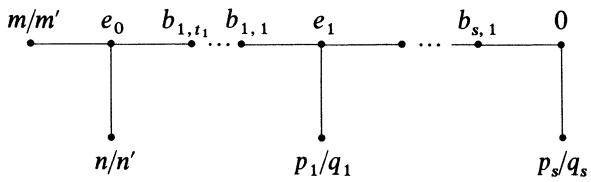
Next we consider surgery on iterated cables on knots. Say that a cable on $a\ldots$ on a cable on K is a nontrivial iterated cable on K if there are at least two nontrivial cables (winding number $\geqq 2$) in the above sequence.

Theorem 3. *No Dehn surgery on a nontrivial iterated cable on a nontrivial torus knot yields a lens space.*

Proof. The result of such a surgery is given by the Dehn plumbing diagram:

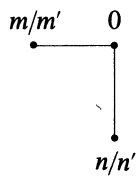


where each subdiagram:



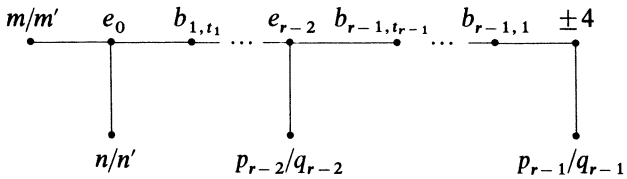
for each s

and

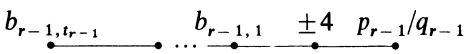


is S^3.

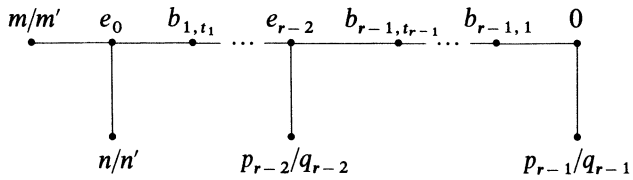
As in the proof of Theorem 1 we see that to obtain a lens space we must have $k=\pm 1$ and then the surgery diagram reduces to



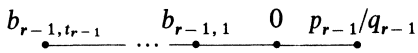
Since $m, n, p_i \geqq 2$ for all i, Lemmas 3 and 4 imply that



is S^3. But also



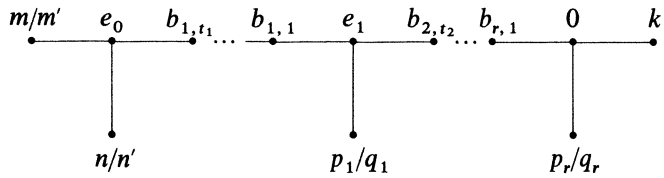
is S^3; so the same argument shows



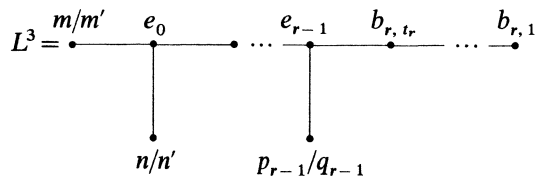
is S^3. But this is a contradiction since Lemma 5 would imply that $\pm 4-0=\pm 1$. □

Theorem 4. *A Dehn surgery on a nontrivial iterated cable on a nontrivial torus knot yields a nontrivial connected sum of lens spaces if and only if it is* $(\varepsilon mnp \pm 1)$-*surgery on* $C(p, \pm 1; \varepsilon T(m,n))$. *The result of this surgery is* $L(p,1) \# L(\varepsilon mnp \pm 1, 4n^2)$.

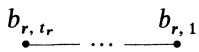
Proof. The surgery diagram in question is



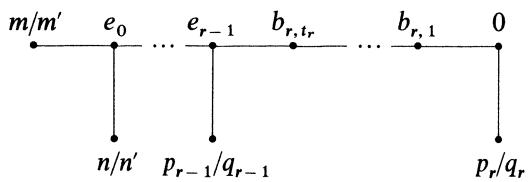
as described in the proof of Theorem 3. Since we wish the result of the surgery to be a connected sum of lens spaces, Lemma 4 implies $k=1/v$, $v\in\mathbb{Z}$ or $k=0$. If $k=1/v$ the arguments of Theorems 1 and 3 show $v=\pm 1$ and at most one lens space is obtained. Thus $k=0$, and we obtain $L(p_r,q_r)\# L^3$ where



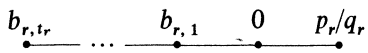
Since L^3 is a lens space,



is S^3. But also



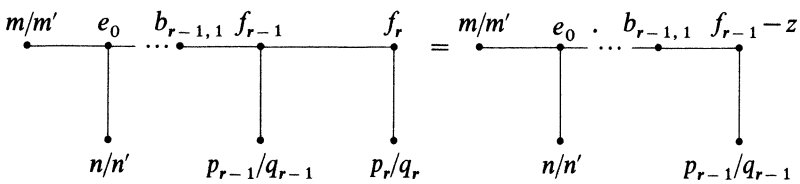
is S^3, so



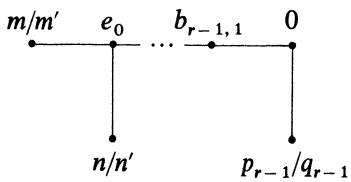
is S^3. This is the plumbing diagram for a Seifert fibration of S^3 with just one singular fiber of order p_r. It is then a well-known fact that $L(p_r,q_r)\cong L(p_r,1)$.

Now S^3 is given by

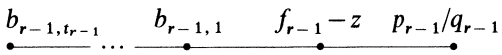
(*)



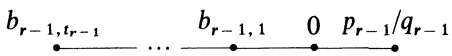
where $f_r-\dfrac{q_r}{p_r}=\dfrac{1}{z}$; so $z=\pm p_r$. But also



is S^3. Thus both

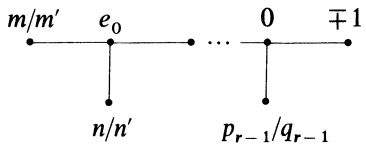


and



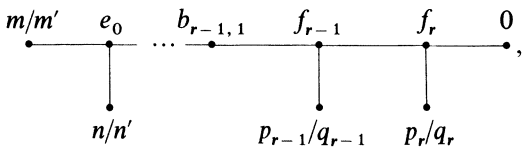
are S^3. Lemma 5 implies that either $z=f_{r-1}$ or $f_{r-1}-z=\pm 1$; and we already know $z=\pm p_r$.

If $f_{r-1}-z=\pm 1$ the diagram (*) shows S^3 is

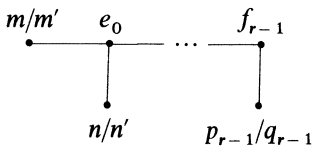


This is the diagram of a surgery on an iterated cable on a torus knot. Since the result is $S^3=L(1,-)$, a lens space, Theorems 1 and 3 imply that $r=1$. (Since $m,n\geqq 2$, $|4\varepsilon mn\pm 1|>1$.) Hence $\pm p_r=z=f_{r-1}$.

But our original surgery diagram is equivalent to

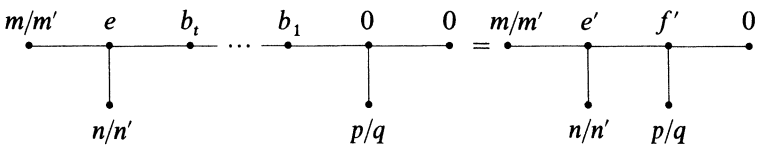


so L^3 is given by

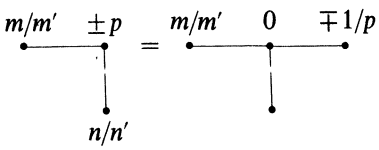


Applying our last argument to this expression for L^3 we see that $f_{r-1}=0$ or ± 1. Hence $\pm p_r=z=0$ or ± 1. This is a contradiction since it implies the last cable is trivial.

Thus our original surgery diagram must be:



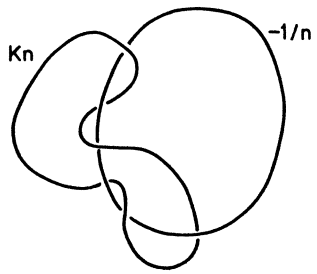
where $e'=\pm p$ and $f'p-q=\pm 1$. So L^3 is



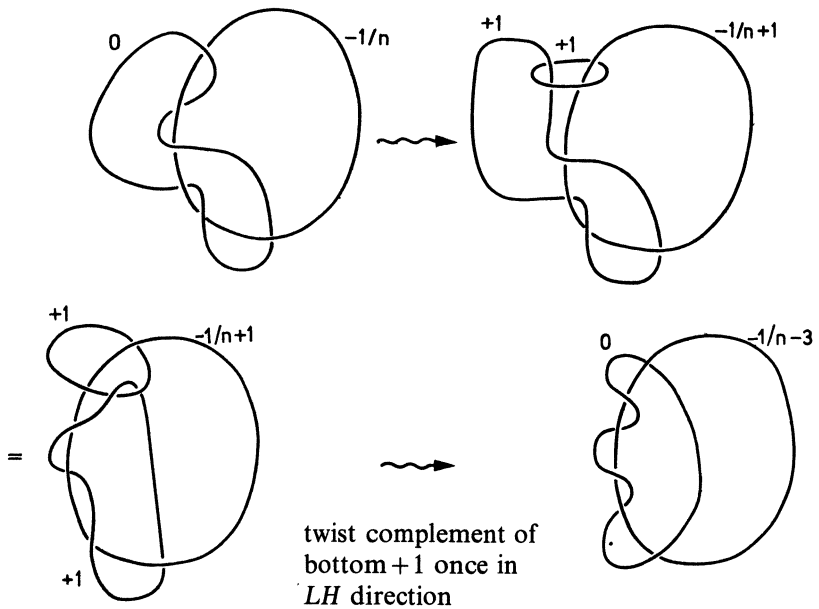
which is the result of $\varepsilon mn\mp\frac{1}{p}$ surgery on $\varepsilon T(m,n)$. Using Moser's formula gives $L^3=L(\varepsilon mnp\mp 1, 4n^2)$. $\square$

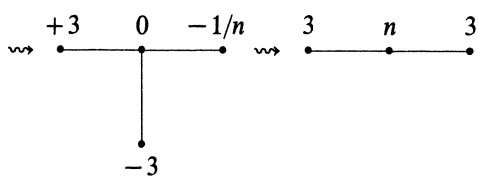
§ 4. More Unexpected Surgery Constructions of Lens Spaces

Consider the surgery description for the knot K_n:



Perform the surgery described by labelling K_n with "0" in the above figure. The following sequence of Kirby-Rolfsen calculus moves shows that the result of the surgery is the lens space $L(9n, 3n+1)$.





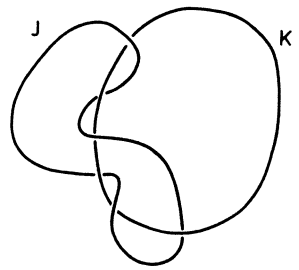
which is a plumbing picture for $L(9n, 3n+1)$.

For even n, Theorems 1 and 3 imply that K_n cannot be a nontrivial cable or iterated cable on a torus knot.

Proposition 3. *For $n \neq 0$ even, K_n is not a torus knot.*

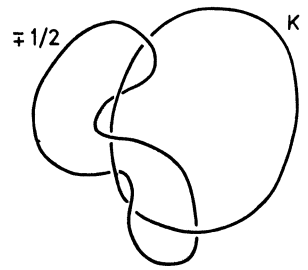
Proof. Suppose that K_n is the torus knot $T(r,s)$. Moser's formula [4] implies that if $L(9n, 3n+1)$ is obtained from $\pm\frac{9n}{p}$ surgery on $T(r,s)$ then $prs = 9n \pm 1$. Since n is even, rs must be odd. This means that the 2-fold cover of S^3 branched over $T(r,s)$ is the Brieskorn sphere $\Sigma(2,r,s)$.

The link in our surgery description for K_n is symmetric, so we have the surgery presentation for K_n:



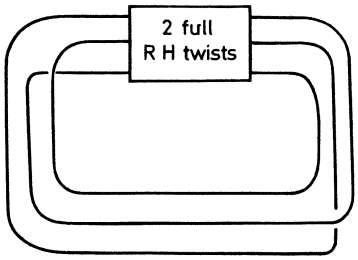
where $-1/n$-surgery is performed on J. Now the 2-fold cover of S^3 branched over K is again S^3 and the preimage of J in this cover is the figure-eight knot. (See [8; p. 337].) A theorem of Goldsmith [3] now implies that the 2-fold branched cover of S^3, branched over K_n, is obtained by $-1/m$-surgery on the figure-eight knot, where $n = 2m$. Thurston [10; Thm. 4.7] has studied the 3-manifolds obtained from Dehn surgery on the figure-eight knot and has shown in particular if $m \neq 0, \pm 1$ that the result of $1/m$-surgery has a hyperbolic structure. The resultant manifold cannot be Seifert fibered; thus it is not $\Sigma(2,r,s)$. (A more elementary proof would simply show that if $m \neq 0, \pm 1$ the result of $1/m$-surgery on the figure-eight knot has fundamental group with a trivial center.)

For $m = \pm 1$ we have the surgery description



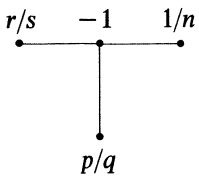
for $K_{\pm 2}$, and performing the surgery obtained by labelling K with a "0" yields $L(18,7) \cong L(18,5)$. (Actually, it is easy to see that $K_2 = -K_{-2}$.) Moser's formula implies that if $L(18,7)$ is obtained by $-18/p$-surgery on $T(r,s)$ then $prs = 18 \pm 1$. Hence either $r = 1$ or $s = 1$ so $T(r,s)$ is the unknot. $\square$

Remark. The Brieskorn sphere $\Sigma(2,3,7)$ is the 2-fold cover of S^3 branched over $T(3,7)$. Also $\Sigma(2,3,7)$ is obtained from -1-surgery on the figure-eight knot. So as in the proof of the proposition above, $\Sigma(2,3,7)$ is the 2-fold branched cover of S^3 branched over K_2. Actually Birman, Gonzales-Acuña, and Montesinos have shown that each $\Sigma(2,p,q)$ is the 2-fold cover of S^3 branched over 2 distinct knots. It is interesting to note that K_2 is

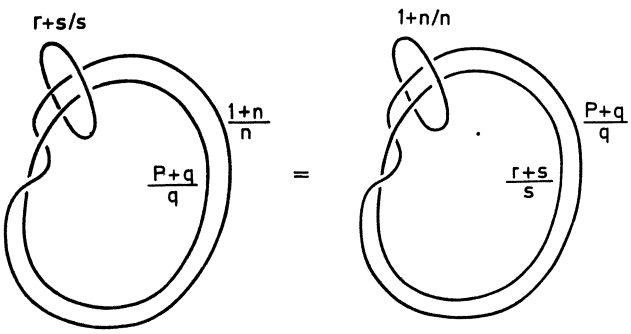


and that if the crossing on the upper left-hand corner is changed $T(3,7)$ is obtained.

Many more examples of knots K, which are not torus knots or cables on torus knots but for which some surgery on K yields a lens space, can be obtained from the following scheme. The plumbing diagram

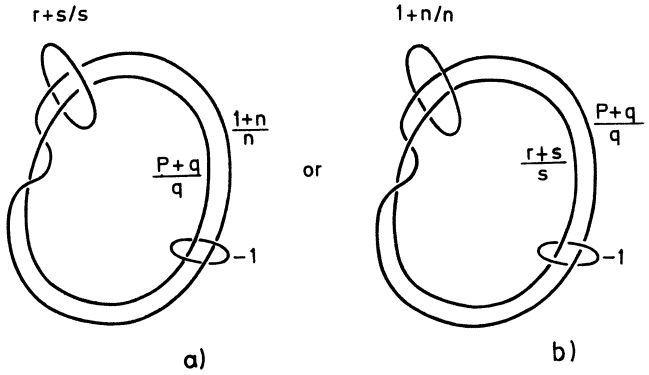


for $n \in \mathbb{Z}$ describes a lens space. By giving the complement of the -1-circle in the corresponding link diagram a right-hand twist we obtain

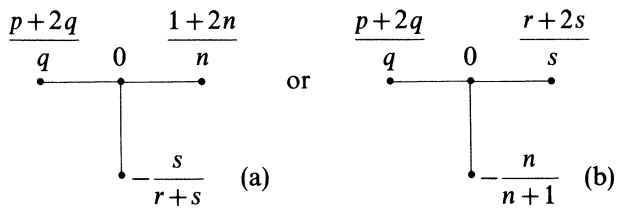


(a) (b)

Add the 2-handle labelled "-1"

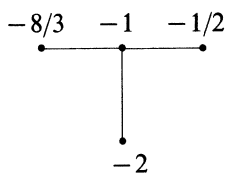


to obtain a cobordism from the lens space to

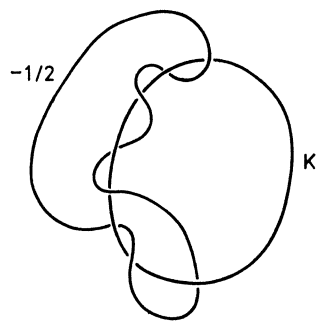


By placing restrictions on the various integers involved we can assure that this last manifold be S^3, so that the original lens space is obtained from surgery on a knot in S^3.

For example if we start with $L(30,11) \cong$

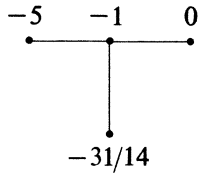


and follow procedure (a), we see that $L(30,11)$ is obtained from $+30$-surgery on the knot with surgery description



Theorems 1 and 3 imply that this knot is not a nontrivial iterated cable on a torus knot, and Moser's formula implies that this is not a torus knot.

Theorem 4 and the above construction also allow the construction of infinitely many knots, which are not iterated cables, upon which appropriate surgeries yield a nontrivial connected sum of two lens spaces. For example, if we start with $L(5,4)\#L(31,17)\cong$

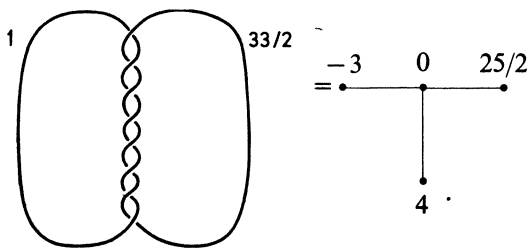


and follow procedure (a) above we see that $L(5,4)\#L(31,17)$ is obtained by a 155-surgery on some knot K. Up to orientation the only surgery on a torus knot which could possibly yield $L(5,4)\#L(31,17)$ is a 155-surgery on $T(31,5)$ which does yield $L(5,1)\#L(31,6)$. However, $L(31,6)$ is not diffeomorphic to $L(31,17)$ so K is not a torus knot. By Theorem 4, if K were a nontrivial iterated cable on a torus knot then $L(5,4)\#L(31,\ 17)$ would have to be diffeomorphic to $L(5,1)\#L(31,5)$ or to $L(5,1)\#L(31,16)$, which is impossible. Thus K is not an iterated cable knot.

We close with two questions.

(1) Is it possible to obtain a connected sum of 3 or more prime 3-manifolds by surgery on a knot in S^3? We are especially interested when these manifolds are lens spaces whose fundamental groups have relatively prime orders.

(2) Does $L(33,2)$ bound a simply connected 4-manifold of 2nd betti number 1? This would be the case if $L(33,2)$ were obtainable from integer surgery on a knot. Notice that $L(33,2)\cong L(33,16)$ satisfies the criterion of Proposition 1. If we add the handle described in the proof of Proposition 1 we obtain the homology sphere



which is $\Sigma(3,4,25)$. Now $\Sigma(3,4,25)$ has μ-invariant 0; so if the answer to question (2) is negative, it could not bound a contractible 4-manifold. Of course, there are no known examples of μ-invariant 0 homology spheres which are not boundaries of contractible 4-manifolds.

References

1. Bailey, J., Rolfsen, D.: An unexpected surgery construction of a lens space. Pacific J. Math. **71**, 295–298 (1977)
2. Birman, J., Gonzales-Acuña, F., Montesinos, J.: Heegard splittings of prime 3-manifolds are not unique. Michigan J. Math. **23**, 97–103 (1976)

3. Goldsmith, D.: Symmetric fibered links. In: Knots, Groups, and 3-Manifolds, (L.P. Neuwirth Ed.), pp. 3–23. Annals of Mathematics Studies **84**. Princeton: Princeton University Press 1975
4. Moser, L.: Elementary surgery along a torus knot. Pacific J. Math. **38**, 737–745 (1971)
5. Kirby, R.: A calculus for framed links in S^3. Invent. Math. **45**, 35–56 (1978)
6. Neumann, W., Raymond, F.: Seifert manifolds, plumbing, μ-invariant, and orientation-reversing maps. In: Algebraic and Geometric Topology, Proceedings of a Symposium (Santa Barbara 1977), pp. 163–196. Lecture Notes in Mathematics **664**. Berlin-Heidelberg-New York: Springer 1978
7. Neuzil, J.: Surgery on a curve in a solid torus. Trans. Amer. Math. Soc. **204**, 385–406 (1975)
8. Rolfsen, D.: Knots and Links. Boston: Publish or Perish 1976
9. Scharf, A.: Faserungen von Graphenmannigfaltigkeiten. Math. Ann. **215**, 35–45 (1975)
10. Thurston, W.: The Geometry and Topology of 3-Manifolds. Mimeographed notes. Princeton University 1978

Received November 14, 1979