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An exotic free involution on S^4

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Dedicated to Deane Montgomery on the occasion of his seventieth birthday.

1. Introduction

Dimension 4 seems to be the pivotal dimension in the study of many of the properties of manifolds. For example a theorem of Livesay [L] states that any free involution on S^3 is smoothly equivalent to the antipodal map, while in contrast to this many exotic free involutions exist on S^n in dimensions $n \geqq 5$ [Lo]. Until quite recently nothing was known about the situation in dimension 4. The first important step in understanding free involutions on S^4 was taken by Cappell and Shaneson [CS] who exhibited a smooth 4-manifold which is homotopy equivalent to real projective 4-space RP^4, but not smoothly (or PL) s-cobordant to RP^4. Thus this showed that there is a smooth homotopy 4-sphere which admits a free involution which is not smoothly equivalent to a linear involution on S^4. Actually, Cappell and Shaneson's construction gives a collection of manifolds which are homotopy equivalent to RP^4 but not s-cobordant to it. The double covers of these manifolds, however, are not known to be diffeomorphic to S^4. Akbulut and Kirby have shown that the homotopy 4-sphere that double covers at least one of these examples is obtained by removing a tubular neighborhood of a 2-sphere in S^4 and sewing back in via the nontrivial bundle map [AK_1].

In this paper we shall construct a smooth exotic free involution T on S^4 whose quotient S^4/T is not smoothly s-cobordant to RP^4. Our construction utilizes the Brieskorn homology 3-sphere $\Sigma(3, 5, 19)$ and the properties which it enjoys as a Seifert fiber space. We show, using the Kirby calculus, that $\Sigma(3, 5, 19)$ is the boundary of a contractible 4-manifold U^4 whose double is the 4-sphere. If t is the involution contained in the natural S^1-action on $\Sigma(3, 5, 19)$, then t is free and the manifold $M^4 = U \cup_t U$ obtained from the union of two copies of U glued together by t admits a free involution T which extends t. This will give our example since we shall easily see that

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M^4 is diffeomorphic to S^4.

In the next section we shall develop a technique for showing that certain free involutions on homotopy 4-spheres have exotic orbit spaces. In Section 3 we use this technique to show that the involution T is exotic.

2. Detecting exotic orbit spaces

Let M^4 be a smooth homotopy 4-sphere supporting a free involution T which desuspends to an involution t on a homology 3-sphere Σ^3. Then there is an acyclic 4-manifold U^4 with $\partial U^4 = \Sigma^3$ such that T is equivariantly diffeomorphic to the involution on $U^4 \cup_t U^4$ which sends a point x in one copy of U^4 to x in the other copy of U^4. (This is compatible with the given involution t on Σ^3.) We shall identify the involution T on M^4 with this last involution. The quotient M^4/T is homotopy equivalent to RP^4. Suppose that M^4/T is s-cobordant to RP^4 and let W^5 be such an s-cobordism.

Now classify the involution T on M^4 as follows. First choose N large enough so that there is a pullback diagram of double covers:

$$\begin{array}{ccc} \Sigma^3 & \xrightarrow{\tilde{\varphi}} & S^{N-1} \\ \downarrow & & \downarrow \\ \Sigma^3/t & \xrightarrow{\varphi} & RP^{N-1} \end{array} .$$

Equivariantly extend $\tilde{\varphi}$ over M^4 so that $\tilde{\varphi}(U) \subseteq B_+^N$, the upper hemisphere of S^N, and $\tilde{\varphi}(TU) \subseteq B_-^N$:

$$\begin{array}{ccc} M^4 & \xrightarrow{\tilde{\varphi}} & S^N \\ \downarrow & & \downarrow \\ M^4/T & \xrightarrow{\varphi} & RP^N \end{array} .$$

This can be done so that φ is a smooth map transverse to RP^{N-1} and $\varphi^{-1}(RP^{N-1}) = \Sigma^3/t$.

The antipodal involution on S^4 is similarly classified by maps ψ, $\tilde{\psi}$:

$$\begin{array}{ccc} S^4 & \xrightarrow{\tilde{\psi}} & S^N \\ \downarrow & & \downarrow \\ RP^4 & \xrightarrow{\psi} & RP^N \end{array} .$$

Since N is large, the only possible obstruction to extending $\varphi \cup \psi$ to $W^5 \to RP^N$ lies in $H^2(W^5, \partial W^5; \pi_1(RP^N)) = \mathbf{Z}_2$; but it is easy to see that this obstruction vanishes since both double covers, M^4 and S^4, are connected. So there is a smooth extension Φ: $W^5 \to RP^N$ which we make transverse to RP^{N-1} (rel ∂W^5). We have a pullback diagram:

$$\begin{array}{ccc} \widetilde{W}^5 & \xrightarrow{\widetilde{\Phi}} & S^N \\ \downarrow & & \downarrow \\ W^5 & \xrightarrow{\Phi} & RP^N . \end{array}$$

Let I denote the induced free involution on $\widetilde{W}^5$ which is an equivariant cobordism from (M^4, T) to $(S^4$, antipodal).

By transversality $\widetilde{\Phi}^{-1}(S^{N-1})$ is a proper 4-dimensional submanifold of $\widetilde{W}$ which is invariant under the involution I. Let $\widetilde{Y}$ be the component of $\widetilde{\Phi}^{-1}(S^{N-1})$ which contains Σ^3. Because $I(\Sigma^3) = t(\Sigma^3) = \Sigma^3$, we have $I(\widetilde{Y}) = \widetilde{Y}$. An easy point-set argument then shows that $\partial \widetilde{Y} = \Sigma^3 \cup S^3$. Let $Y = \widetilde{Y}/I$ be the corresponding component of $\Phi^{-1}(RP^{N-1})$.

LEMMA. $w_1(Y) = w_2(Y) = 0$.

Proof. Let $Y' = \Phi^{-1}(RP^{N-1})$. We show that $w_1(Y') = w_2(Y') = 0$; then it is also true for each component. Consider the diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\Phi|_{Y'}} & RP^{N-1} \\ i\big\downarrow & & \big\downarrow j \\ W^5 & \xrightarrow{\Phi} & RP^N . \end{array}$$

Let α be the nontrivial element of $H^1(RP^N; \mathbf{Z}_2)$. Then

$$\begin{aligned} i^* w_1(W) &= w_1(Y') + w_1(\nu(Y' \hookrightarrow W)) \\ &= w_1(Y') + \Phi|_{Y'}^* w_1(\nu(RP^{N-1} \hookrightarrow RP^N)) = w_1(Y') + \Phi|_{Y'}^* j^*(\alpha) . \end{aligned}$$

But $\Psi = \Phi|_{RP^4}$ is just inclusion and $w_1(RP^4) \neq 0$, so $w_1(RP^4) = \Psi^*(\alpha)$, and since $k: RP^4 \hookrightarrow W^5$ is a homotopy equivalence, $w_1(W) = \Phi^*(\alpha)$. From our above formula, $i^*\Phi^*(\alpha) = w_1(Y') + \Phi|_Y^* j^*(\alpha)$; so it follows from commutativity of the above diagram that $w_1(Y') = 0$. Thus $w_2(Y') = i^*\ w_2(W) = i^* k^{*-1} w_2(RP^4) = 0$ since $w_2(RP^4) = 0$. □

Now consider $\widetilde{Y} \subset \widetilde{W}$. We have already seen that $\widetilde{Y}$ is a connected proper codimension 1 submanifold of $\widetilde{W}$ with $\partial \widetilde{Y} = \Sigma^3 \cup S^3$, and since Y is orientable, the restriction of the involution I to $\widetilde{Y}$ is orientation-preserving. Now $\widetilde{Y}$ separates $\widetilde{W}$, and a component of $\widetilde{W} - \widetilde{Y}$ has closure which is a 5-manifold with boundary $U^4 \cup \widetilde{Y} \cup B^4$. So the signature $\sigma(U^4 \cup \widetilde{Y} \cup B^4) = 0$; hence $\sigma(\widetilde{Y}) = 0$.

Since $\widetilde{Y}$ has a free involution I extending $t \cup$ (antipodal) on $\Sigma^3 \cup S^3 = \partial \widetilde{Y}$, a standard formula for the α-invariant ([W; p. 198]) yields

$$2\sigma(Y) - \sigma(\widetilde{Y}) = \alpha(S^3, \text{antipodal}) - \alpha(\Sigma^3, t) ,$$

and $\alpha(S^3$, antipodal$) = 0$; so $\sigma(Y) = -1/2\alpha(\Sigma, t)$. Because $w_1(Y) = w_2(Y) = 0$

we may choose a framing for Y, and this in turn induces an almost-framing on the RP^3-component of ∂Y. The manifold RP^3 admits two almost-framings and each extends to a framing on the total space of the tangent or cotangent disk bundle of S^2. Let E be the total space of the disk bundle whose framing extends the almost-framing induced on RP^3 from Y.

Now $X^4 = Y \cup E$ is an almost-framed, hence framed 4-manifold with $\partial X^4 = \Sigma^3/t$. A framing on X^4 induces an almost-framing $\mathcal{F}$ on Σ^3/t whose μ-invariant is

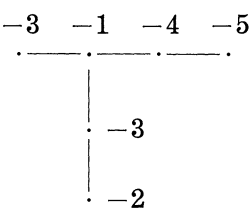
$$\mu(\Sigma^3/t; \mathcal{F}) \equiv \sigma(Y) + \sigma(E) \equiv -1/2\alpha(\Sigma^3, t) \pm 1 \pmod{16} .$$

This proves:

PROPOSITION 1. *Let T be a free involution on a homotopy 4-sphere whose quotient is s-cobordant to RP^4 and which desuspends to an involution t on a homology 3-sphere Σ^3. Then there is an almost-framing $\mathcal{F}$ for Σ^3/t such that $\mu(\Sigma^3/t; \mathcal{F}) + 1/2\alpha(\Sigma^3, t) \equiv \pm 1 \pmod{16}$.* □

3. The example

The Brieskorn homology 3-sphere $\Sigma(3, 5, 19)$ is the intersection of the variety in $\mathbf{C}^3$ described by $x^3 + y^5 + z^{19} = 0$ with a 5-sphere centered at the origin. It can also be conveniently described as the boundary of the plumbing manifold



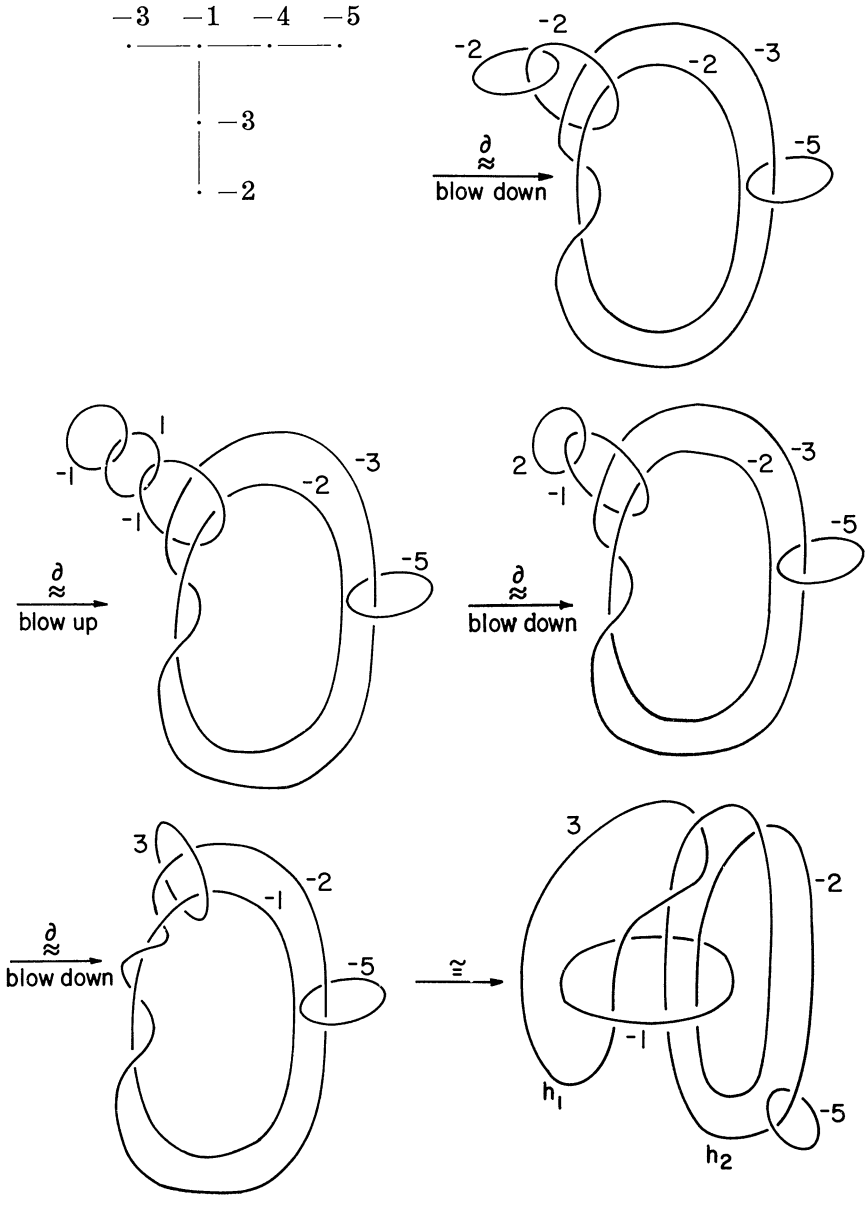
or alternatively as the Seifert fiber space with Seifert invariants $((1, 1), (3, -1), (5, -2), (19, -5))$ (see [NR], § 4).

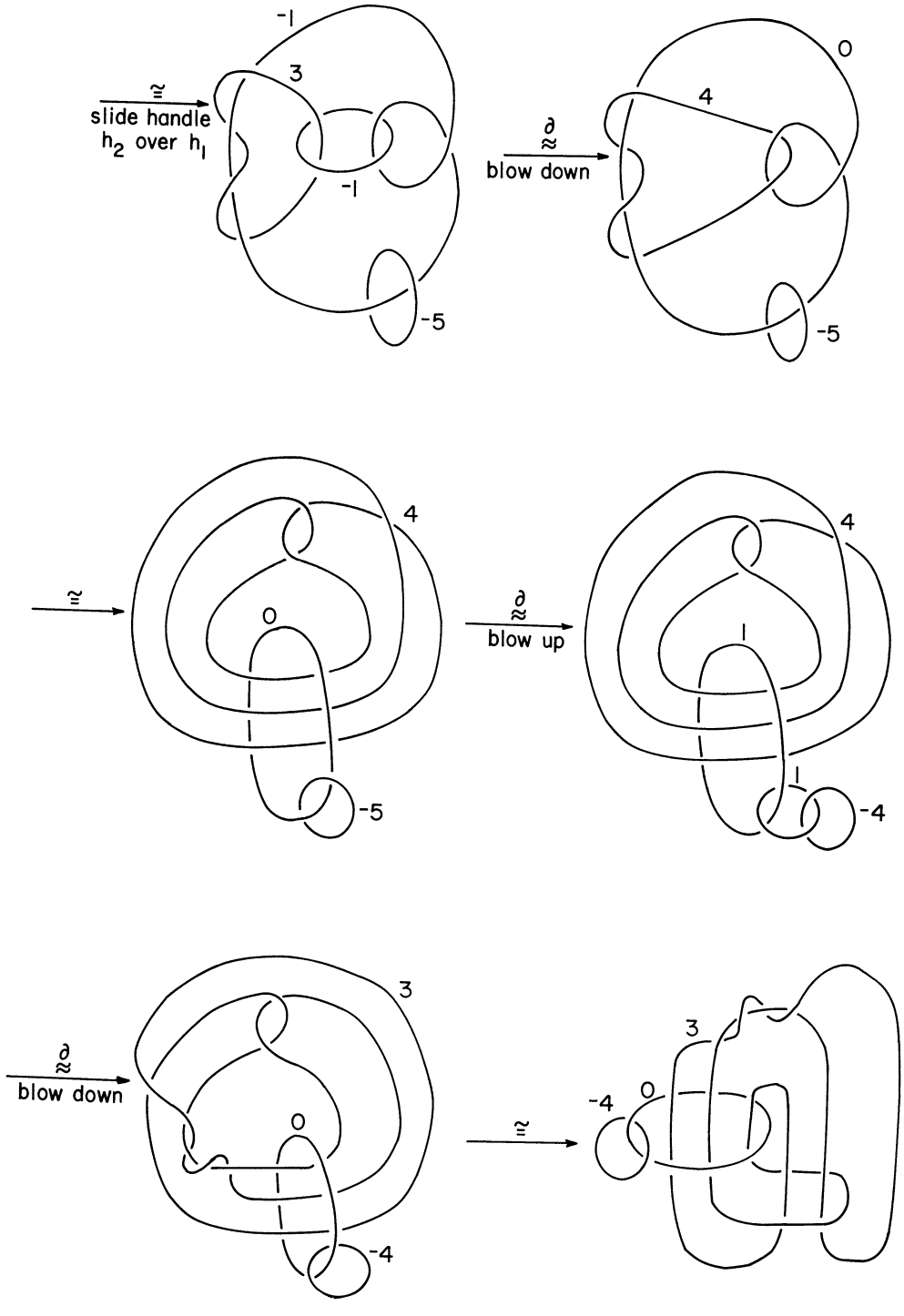
PROPOSITION 2. $\Sigma(3, 5, 19)$ *is the boundary of a contractible 4-manifold U^4 whose double is S^4.*

Proof. We shall show that there is a cobordism from $\Sigma(3, 5, 19)$ to $S^2 \times S^1$ built by adding exactly one 2-handle to $\Sigma(3, 5, 19) \times I$. We then add a 3-handle to $S^2 \times S^1$ to obtain S^3 and cap off with a 4-ball to obtain U^4. Then, dually, U^4 has a handle decomposition with exactly one 0, 1, and 2-handle, and it is easy to see that U^4 is contractible. Now $U^4 \times I$ has a handle decomposition with exactly one handle of index 0, 1, and 2; the attaching circle for the 2-handle lies in $\partial(B^4 \times S^1)$ and is homotopic, therefore isotopic, to a generator of $\pi_1(S^3 \times S^1)$. This means that the 1 and 2-handles

cancel; so $U^4 \times I \cong B^5$ and $2U^4 \cong \partial(U^4 \times I) \cong S^4$.

We shall now use the Kirby calculus of links [K] to show that one may add a 2-handle to $\Sigma(3, 5, 19)$ to obtain $S^2 \times S^1$. We use the notation "$\underset{\approx}{\partial}$" to mean that the 4-manifolds in question have the same boundary and "$\cong$" to mean that they diffeomorphic. The first part of this proof is motivated by Akbulut and Kirby's proof that $\Sigma(3, 4, 5)$ bounds a contractible manifold [AK2].




≅
slide handle
h_2 over h_1
-1
3
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blow down
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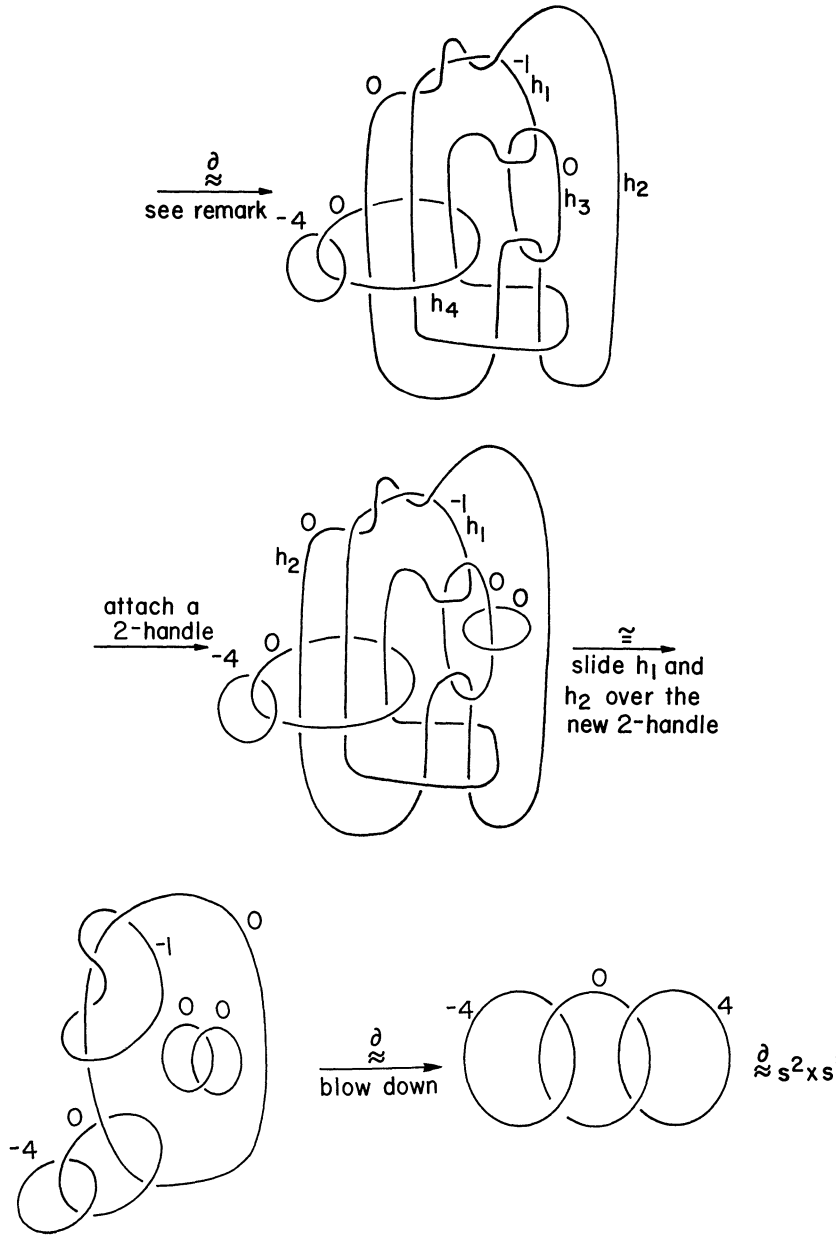
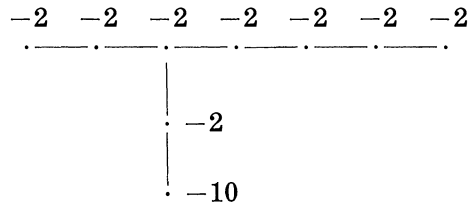


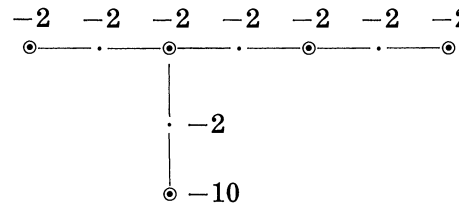
Remark. To see that these two 4-manifolds have diffeomorphic boundaries, slide h_1 over h_2 so that h_3 links h_2 geometrically once. Now slide h_1 and h_4 over h_3 so as to unlink h_2 from h_1 and h_4. $\square$

Let t be the free involution on $\Sigma(3, 5, 19)$ which is contained in the S^1-action on $\Sigma(3, 5, 19)$. Thus t is isotopic to the identity; so $U^4 \cup_t U^4 \cong$

$U^4 \cup_{\mathrm{id}} U^4 \cong 2U^4 \cong S^4$. This gives a free involution T on S^4 as described in the introduction, and T desuspends to t on $\Sigma(3, 5, 19)$. Walter Neumann's thesis [N] gives a formula for the α-invariant of $\Sigma(3, 5, 19)$ in terms of its Seifert invariants. (See also [NR], p. 184.) Using this formula we obtain $\alpha(\Sigma(3, 5, 19), t) = 0$. The Seifert invariants for $\Sigma(3, 5, 19)/t$ are $((1, 2), (3, -2), (5, -4), (19, -10))$ (see [NR], 4.2), so $\Sigma(3, 5, 19)/t$ is the boundary of the plumbed 4-manifold V^4:



with $\sigma(V^4) = -9$. There are exactly two almost-framings of $\Sigma(3, 5, 19)/t$ since its almost-framings are in $1-1$ correspondence with $H^1(\Sigma(3, 5, 19)/t; \mathbf{Z}_2) \approx \mathbf{Z}_2$. These two almost-framings, $\mathcal{F}_0$ and $\mathcal{F}_1$, correspond to the two characteristic homology classes of V^4, the zero class and the class represented by the 0-sections of the plumbing which are circled below:



So the two μ-invariants of $\Sigma(3, 5, 19)/t$ are: $\mu(\Sigma(3, 5, 19)/t, \mathcal{F}_0) \equiv -9$ and $\mu(\Sigma(3, 5, 19)/t, \mathcal{F}_1) \equiv -9 - (-18) \equiv 9 \pmod{16}$ (see [CS]). Thus it follows from Proposition 1 that S^4/T is not smoothly s-cobordant to RP^4.

THEOREM. *There is an exotic free involution on S^4 whose quotient S^4/T is not smoothly s-cobordant to RP^4.* □

Remark. There are many Brieskorn spheres other than $\Sigma(3, 5, 19)$ which we could have used in our construction (for example, $\Sigma(3, 5, 49)$). We do not know if the orbit spaces are diffeomorphic.

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