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SEIFERT FIBERED 3-MANIFOLDS AND NONORIENTABLE 4-MANIFOLDS

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1. INTRODUCTION

Let X^4 be a smooth connected closed nonorientable 4-manifold. In this paper we illustrate a technique for constructing manifolds Q^4 and simple homotopy equivalences $f: Q^4 \to X^4$ whose smooth normal invariants are nonzero.

For a specific X consider its orientable double cover $\tilde{X}$ with covering translation I. The free involution I desuspends in the sense that $\tilde{X}$ is the union of two copies of a compact 4-manifold W, $\tilde{X} = W \cup_i W$ glued along their boundaries by a free involution $i: \partial W \to \partial W$ so that I is equivariantly diffeomorphic to the involution on $W \cup_i W$ which sends a point x in one copy of W to x in the other copy of W. (This is compatible with the given involution i on ∂W.) Our goal is to replace ∂W with a Seifert-fibered 3-manifold M^3 which admits a fixed point free involution $t: M^3 \to M^3$ and an equivariant homology equivalence $h: M^3 \to \partial W$ and to replace W by a manifold U^4 with boundary M^3 and with a simple homotopy equivalence $H: U^4 \to W$ extending h. If we let $\tilde{Q} = U^4 \cup_t U^4$, $\tilde{Q}$ admits the free involution T which sends a point x in one copy of U^4 to x in the other copy of U^4. There results an equivariant homotopy equivalence $\tilde{f}: \tilde{Q} \to \tilde{X}$ covering a homotopy equivalence $f: Q = \tilde{Q}/T \to \tilde{X}/I = X$. With suitably chosen U^4, f will have the desired smooth normal invariants.

This technique was successfully employed in [FS] to construct a smooth closed 4-manifold Q^4 having the homotopy type of real projective 4-space RP^4 but not smoothly s-cobordant to RP^4. Furthermore $\tilde{Q}$ is diffeomorphic to S^4. In this paper we further illustrate this technique by constructing homotopy smoothings of $S^2 \times RP^2$ with any desired smooth normal invariants.

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At this point two remarks are in order. First, Cappell and Shaneson [CS] have shown that if X is any closed 4-manifold such that $\pi_1(X)$ has an orientation-reversing element of order 2, there is a 4-manifold Q and a simple homotopy equivalence $f: Q \to X$ whose smooth normal invariant $\eta(f)$ is the only nontrivial element in the kernel of $[X; G/O] \to [X, G/TOP]$. (See also [K; Problem 4.14].) Our construction for the case $X = S^2 \times RP^2$ will illustrate that our technique can provide examples with any given normal invariant (in particular, with nontrivial topological normal invariant). Also, our technique can often further identify the orientable cover $\tilde{Q}$ as being standard, $S^2 \times S^2$ in the case at hand.

Second, T. Matumoto [M] has shown that all homotopy smoothings of $S^2 \times RP^2$ can be realized by self-homotopy equivalences of $S^2 \times RP^2$ (see §2). In particular, all homotopy smoothings $f: Q \to S^2 \times RP^2$ have the property that Q is s-cobordant to $S^2 \times RP^2$. However, we cannot show that our manifolds Q constructed in §5 are diffeomorphic to $S^2 \times RP^2$.

2. HOMOTOPY SMOOTHINGS OF $S^2 \times RP^2$

A homotopy smoothing of $S^2 \times RP^2$ is a pair (Q, f) where Q is a closed smooth 4-manifold and $f: Q \to S^2 \times RP^2$ is a simple homotopy equivalence. Let $\mathcal{S}(S^2 \times RP^2)$ denote the set of s-cobordism classes of homotopy smoothings of $S^2 \times RP^2$ and let $\eta: \mathcal{S}(S^2 \times RP^2) \to [S^2 \times RP^2; G/O]$ denote the map which associates to each homotopy smoothing (Q, f) its normal cobordism class $\eta(f)$. The smooth normal invariant $\eta(f)$ induces the topological normal invariant $\eta_{TOP}(f)$ by the natural map $[S^2 \times RP^2; G/O] \to [S^2 \times RP^2; G/TOP]$.

T. Matumoto [M] has shown that $\mathcal{S}(S^2 \times RP^2)$ has four elements represented by the self homotopy equivalences $\mathrm{id}, \sigma, \tau, \sigma \circ \tau$ described below.

Let $D^4 \subset S^2 \times RP^2$ be a disk and shrink ∂D^4 to a point to obtain a map $c: S^2 \times RP^2 \to S^2 \times RP^2 \vee S^4$. Let $\eta^2: S^4 \to S^2$ be an essential map, $p: S^2 \to RP^2$ the covering map, and consider the composites

$$s: S^2 = S^2 \times * \hookrightarrow S^2 \times RP^2$$

and

$$t: S^2 \xrightarrow{p} RP^2 = * \times RP^2 \hookrightarrow S^2 \times RP^2.$$

Define σ and τ to be the composites

$$S^2 \times RP^2 \xrightarrow{c} S^2 \times RP^2 \vee S^4 \xrightarrow{\mathrm{id} \vee \eta^2} S^2 \times RP^2 \vee S^2 \xrightarrow{(\mathrm{id},\, x)} S^2 \times RP^2$$

where for σ we take $x = s$ and for τ take $x = t$, and id is the identity.

These homotopy smoothings are characterized by the fact that $\eta_{TOP}(\sigma) \neq 0$, $\eta_{TOP}(\tau) = 0$, $\eta(\tau) \neq 0$, $\eta_{TOP}(\sigma \circ \tau) \neq 0$ and $\eta(\sigma) \neq \eta(\sigma \circ \tau)$. Note that η_{TOP} is detected by the characteristic variety $* \times RP^2$ of $S^2 \times RP^2$. As is pointed out in [M], $\sigma^{-1}(* \times RP^2)$ can be assumed to be $W \cup (* \times RP^2)$ with W framed, where W is the pre-image of a point under the generator $\eta^2: S^4 \to S^2$ of $\pi_4(S^4) = \mathbb{Z}_2$. The splitting invariant is then the Arf invariant of the framed manifold W, which is equal to one.

3. CONSTRUCTING HOMOTOPY SMOOTHINGS OF $S^2 \times RP^2$ VIA SEIFERT FIBRATIONS

The orientation cover of $S^2 \times RP^2$ is $S^2 \times S^2$ and the induced involution A on $S^2 \times S^2$ desuspends to an involution on $S^2 \times S^1$; i.e., $S^2 \times S^2 = S^2 \times D^2 \cup_a S^2 \times D^2$ where a is the involution on $S^2 \times S^1$ which is the identity on the S^2-factor and the antipodal map on the S^1-factor. In keeping with the program outlined in §1 we seek Seifert fibered 3-manifolds M^3 having the homology of $S^2 \times S^1$, which admit free involutions and which are obtained by surgery on a knot in a homology sphere which bounds a contractible manifold. Such M bound homotopy $S^2 \times D^2$'s.

<u>LEMMA</u> 3.1: Let M^3 be a Seifert fibered homology $S^2 \times S^1$. Then there is a fiber-preserving map $h: M^3 \to S^2 \times S^1$ which induces an isomorphism on homology.

<u>PROOF</u>: We refer to [O] for notation and terminology concerning Seifert manifolds. M^3 is an S^1-manifold with orbit map $p: M^3 \to S^2$. Suppose M^3 has n exceptional orbits with Seifert invariants (α_i, β_i), $i = 1, \dots, n$ which can be chosen so that

$$\sum_{i=1}^{n} \beta_i/\alpha_i = 0.$$

Let D_i, $i = 2, \dots, n$, be small 2-disk neighborhoods of the exceptional points in $S^2 = p(M^3)$ and let

$$S_0 = S^2 - \bigcup_{i=1}^{n} D_i.$$

Let S^1 act by translation in $S_0 \times S^1$ and via $t \times (u, v) \to (t^{\nu_i} u, t^{\alpha_i} v)$ in $D_i \times S^1$ where $0 < \nu_i < \alpha_i$ such that $\beta_i \nu_i \equiv 1 \pmod{\alpha_i}$. Then the S^1-action on M^3 is equivalent to

$$(S_0 \times S^1) \cup_{\varphi} \left(\bigcup_{i=1}^{n} D_i \times S^1\right)$$

where $\varphi | \partial D_i \times S^1 \to \partial(S_0 \times S^1)$ is defined by the matrix

$$\begin{bmatrix} \alpha_i & -\nu_i \\ \beta_i & -\rho_i \end{bmatrix}$$

where $\rho_i = (\beta_i \nu_i - 1)/\alpha_i$.

Let γ be the least common multiple of $\{\alpha_1, \ldots, \alpha_n\}$ and define $h_0\colon S_0 \times S^1 \to S_0 \times S^1$ by $h_0(u, v) = (u, v^{\gamma})$. Note that h_0 is fiber-preserving. Next define $h_i\colon D_i \times S^1 \to D_i \times S^1$ by

$$\begin{bmatrix} \alpha_i & -\nu_i \\ 0 & \gamma/\alpha_i \end{bmatrix} ,$$

and $\phi_i\colon \partial D_i \times S^1 \to \partial D_i \times S^1$ by

$$\begin{bmatrix} 1 & 0 \\ \beta_i \gamma/\alpha_i & 1 \end{bmatrix} .$$

Then h_i is fiber-preserving and on $\partial D_i \times S^1$ we have $h_0 \varphi_i = \phi_i h_i$. Hence the h_i together define a fiber-preserving map

$$h\colon (S_0 \times S^1) \cup_{\varphi} \left(\bigcup_{i=1}^{n} D_i \times S^1\right) \to (S_0 \times S^1) \cup_{\varphi} \left(\bigcup_{i=1}^{n} D_i \times S^1\right)$$

where the domain manifold is M^3 and the range has Seifert invariants (unnormalized) $\{(1, \beta_i \gamma/\alpha_i);\ i = 1, \ldots, n\}$. Since $\sum_{i=1}^{n} \beta_i \gamma/\alpha_i = 0$, the range is $S^2 \times S^1$ with the Seifert structure it inherits as the total space of an S^1-bundle over S^2. Thus $h\colon M^3 \to S^2 \times S^1$ is fiber-preserving.

For each i let $\mu_i\alpha_i = \gamma$, then the μ_i have no common divisor, hence there are δ_i, $i = 1, \ldots, n$, with $\sum_{i=1}^{n} \delta_i\mu_i = 1$; so $\sum_{i=1}^{n} \delta_i/\alpha_i = 1/\gamma$. If we identify $H_1(M) \approx \mathbb{Z}$ then a regular orbit of M represents $\gamma \in \mathbb{Z}$. (This can easily be seen by identifying the abelianization of the standard presentation of $\pi_1(M)$ (see [O]) with $\mathbb{Z}$.) Thus the ith exceptional orbit E_i represents $\gamma/\alpha_i \in \mathbb{Z}$; and so $\sum_{i=1}^{n} \delta_i[E_i]$ generates $H_1(M^3)$. With the Seifert structure on $S^2 \times S^1$ arising from a free S^1-action, the orbits are all homologous and the orbit class $[R]$ generates $H_1(S^2 \times S^1)$. Now $h_*([E_i]) = \gamma/\alpha_i\,[R]$; so $h_*(\sum_{i=1}^{n} \delta_i[E_i]) = \sum_{i=1}^{n} (\delta_i\gamma/\alpha_i)\,[R] = [R]$. So h_* is an isomorphism on H_1. Thus it follows from Poincaré duality and the universal coefficient theorem that h_* is an isomorphism in all dimensions. $\square$

Note that h has degree $\gamma = \ell.\text{c.m.}\{\alpha_1, \ldots, \alpha_n\}$ on a regular fiber and degree γ/α_i on an exceptional fiber of order α_i. If all the α_i are odd it follows that h is equivariant with respect to the free involutions contained in the S^1-actions.

Now assume that all the exceptional fibers of M^3 have isotropy groups of odd order (i.e, all α_i are odd). Then the involution in the S^1-action on M is free. Let $\tilde{M}^3$ be the S^1-equivariant double cover of M^3; note that $\widetilde{S^2 \times S^1} = S^2 \times S^1$. Then h lifts to $\tilde{h}$:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{h}} & S^2 \times S^1 \\ \pi\downarrow & & \downarrow \text{id} \times \alpha \\ M^3 & \xrightarrow{h} & S^2 \times S^1 \end{array} \qquad \text{where} \quad \alpha(z) = z^2.$$

Then $\tilde{M}$ is again a Seifert fibered homology $S^2 \times S^1$ and $\tilde{h}$ is an isomorphism on homology. We have an induced map

$\bar{h}$: Mapping cylinder $(\pi) \to$ Mapping cylinder $(\text{id} \times \alpha) = S^2 \times$ Mobius band.

Suppose that $\tilde{M}^3$ is obtained by surgery on a knot in the boundary of a homotopy 4-ball V^4; so that $\tilde{M}^3 = \partial U^4$, $U^4 = V^4 \cup$ (2-handle), or dually, $U^4 = (\tilde{M} \times I) \cup$ (2-handle) $\cup V^4$. This last 2-handle is attached to $\tilde{M} \times \{1\}$ along a knot K which must represent a generator of $H_1(\tilde{M})$

since $\partial(\tilde{M} \times I \cup (2\text{-handle})) = \tilde{M} \cup \partial V^4$. So $\tilde{h}(K)$ is homologous, hence homotopic, to $\{*\} \times S^1$. Hence we may assume that $\tilde{h}(K) = \{*\} \times S^1$. So h extends to

$$(\tilde{M} \times I) \cup (2\text{-handle}) \to S^2 \times S^1 \times I \cup (2\text{-handle added along } * \times S^1)$$
$$= S^2 \times S^2 - D^4 .$$

Extending over $V^4 - D^4$ we get $H: U^4 \to S^2 \times D^2$ extending h and inducing isomorphisms on homology. So we obtain

$$f: Q^4 = U^4 \cup \text{Mapping cylinder } (\pi) \to S^2 \times D^2 \cup S^2 \times \text{Mobius band} = S^2 \times RP^2$$

which again induces isomorphisms on homology.

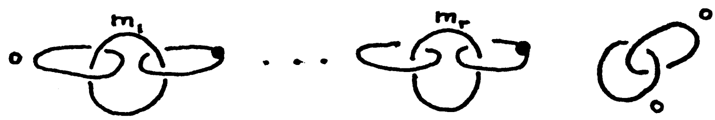
In the double covers f is covered by $\tilde{f}: \tilde{Q}^4 \to S^2 \times S^2$. But $\tilde{Q}^4 = U^4 \cup_t U^4$ where t is the covering translation of $\tilde{M}^3$. Since U^4 is simply-connected so is $\tilde{Q}$; so f is a map of universal covers inducing isomorphisms on homology. Hence $\tilde{f}$ is a homotopy equivalence. But f is an isomorphism on π_1, hence f is a homotopy equivalence.

<u>LEMMA</u> 3.2: Suppose the contractible manifold V has handle decomposition with only 0, 1, and 2-handles. Then $\tilde{Q}$ is diffeomorphic to $S^2 \times S^2$.

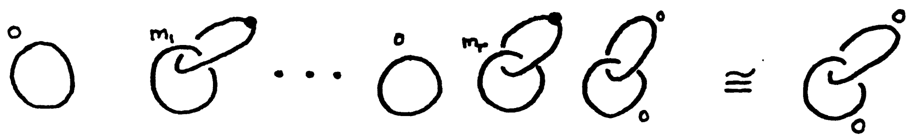
<u>PROOF</u>: Consider a handle decomposition of V with one 0-handle, r 1-handles, and r 2-handles. Since V-(0-handle) is an h-cobordism we can slide 2-handles until the intersection matrix of the attaching circles of the 2-handles with the belt spheres of the 1-handles is the identity matrix.

Now consider a framed link picture for this handle decomposition of V. (See [S].) It consists of r unknotted, unlinked dotted circles representing the 1-handles and r circles representing the 2-handles. The framed link picture for U is obtained by adding one more circle corresponding to K, the attaching circle for the added 2-handle.

Since the involution t of $\tilde{M}^3$ lies in the S^1-action on $\tilde{M}$, t is isotopic to the identity. Hence $\tilde{Q} = U \cup_t U \cong U \cup_{id} U$, the double of U. The framed link picture for the double of U is obtained from the framed link picture for U by adding a meridional circle labelled "0" to each undotted circle in the framed link picture for U. Utilizing these it is easy to slide 2-handles to obtain the following framed link picture for $\tilde{Q}$:



Sliding 2-handles over 1-handles yields:



which is the framed link picture for $S^2 \times S^2$. □

4. DETECTING EXOTIC HOMOTOPY SMOOTHINGS OF $S^2 \times RP^2$

Let X_i^4, $i = 0, 1$, be a smooth homotopy $S^2 \times S^2$ supporting a free involution T_i which desuspends to an involution t_i on a homology $S^2 \times S^1$, say M_i^3. Then there is a homology $S^2 \times D^2$, U_i^4, with $\partial U_i^4 = M_i^3$ and $X_i = U_i \cup_{t_i} U_i$. Also $X_i/T_i = U_i \cup$ Mapping cylinder $(M_i \to M_i/t_i)$. Suppose there is a homotopy equivalence $f_i\colon X_i/T_i \to S^2 \times RP^2 = S^2 \times D^2 \cup$ Mapping cylinder $(S^2 \times S^1 \to S^2 \times S^1/\mathrm{id} \times \alpha)$ with $f_i^{-1}(S^2 \times S^1) = M_i^3$. Suppose further that U_i^4 is obtained by surgery on a knot in the boundary of a homotopy 4-ball V_i^4. Let K_i be the transverse preimage under f_i of $* \times S^1$ chosen so that K_i does not intersect the cocore of the handle attached to ∂V_i^4. Now suppose there is an annulus from M_i^3 to ∂V_i^4 bounded by $K_i \subset M_i^3$ and $K_i' \subset \partial V_i^4$ and let $\mathrm{Arf}(K_i')$ denote the Arf invariant of $K_i' \subset \partial V_i^4$. Since $* \times RP^2 \subset S^2 \times RP^2$ is a characteristic variety we have:

<u>LEMMA</u> 4.1: $\eta_{TOP}(f_i) \neq 0$ if and only if $\mathrm{Arf}(K_i) \neq 0$. □

Suppose that $(X_0/T_0, f_0)$ and $(X_1/T_1, f_1)$ represent the same element of $\mathcal{S}(S^2 \times RP^2)$. Then there is an s-cobordism W^5 between X_0/T_0 and X_1/T_1 and a simple homotopy equivalence $F\colon W \to S^2 \times RP^2$ with $f|\partial W = f_0 \cup f_1$. By relative transversality we assume that

$Y = F^{-1}(S^2 \times S^1)$ is a connected cobordism between M_0^3/t_0 and M_1^3/t_1, and one easily checks that $w_1(Y) = w_2(Y) = 0$.

Let $\tilde{W}$ be the orientable double cover of W; so $\tilde{W}$ is an s-cobordism between X_0 and X_1. Let $\tilde{F}: \tilde{W} \to S^2 \times S^2$ be a lift of F. Then $\tilde{Y} = \tilde{F}^{-1}(S^2 \times S^1)$ is a proper codimension 1 submanifold of $\tilde{W}$ with $\partial\tilde{Y} = M_0^3 \cup M_1^3$. Now $\tilde{Y}$ separates $\tilde{W}$, and a component of $\tilde{W}-\tilde{Y}$ has closure which is a 5-manifold with boundary $U_0^4 \cup \tilde{Y} \cup U_1^4$. So the signature $\sigma(U_0^4 \cup \tilde{Y} \cup U_1^4) = 0$, hence $\sigma(\tilde{Y}) = 0$.

Since $\tilde{Y}$ has a free involution extending $t_0 \cup t_1$ on $\partial\tilde{Y} = M_0^3 \cup M_1^3$, a standard formula for the α-invariant [W; p. 198] yields

$$2\sigma(Y) - \sigma(\tilde{Y}) = \alpha(M_1, t_1) - \alpha(M_0, t_0).$$

As $w_1(Y) = w_2(Y) = 0$, we may choose a framing for Y which in turn induces an almost-framing F_i on the component M_i/t_i of ∂Y. So computing the μ-invariant of the almost framed 3-manifold ∂Y:

$$\sigma(Y) \equiv \mu(M_1/t_1;\ F_1) - \mu(M_0/t_0;\ F_0) \pmod{16}$$

so that

$$\mu(M_1/t_1;\ F_1) - \tfrac{1}{2}\,\alpha(M_1, t_1) \equiv \mu(M_0/t_0;\ F_0) - \tfrac{1}{2}\,\alpha(M_0, t_0) \pmod{16}.$$

Let $\rho(f_i, F_i)$ denote $\mu(M_i/t_i;\ F_i) - \frac{1}{2}\,\alpha(M_i, t_i) \pmod{16}$. We then have:

<u>PROPOSITION</u> 4.2: Let X_i^4, $i = 0,1$, be a homotopy $S^2 \times S^2$ admitting a free involution T_i which desuspends to a free involution t_i on M_i^3, a homology $S^2 \times S^1$. If $f_i: X_i/T_i \to S^2 \times RP^2$ is a homotopy equivalence with $f_i^{-1}(S^2 \times S^1) = M_i$ and if $(X_0/T_0, f_0)$ and $(X_1/T_0, f_1)$ represent the same element of $S(S^2 \times RP^2)$ there are almost-framings F_i of M_i/t_i such that

$$\rho(f_0, F_0) \equiv \rho(f_1, F_1) \pmod{16}. \qquad \square$$

Note that if $X/T = S^2 \times RP^2$ and f = identity, then $\rho(\text{identity}, F) \equiv 0 \pmod{16}$ for any almost-framing on $S^2 \times S^1$.

5. THE EXAMPLES

Before giving our examples we shall list techniques for computing invariants of the examples. These techniques are given in the three lemmas below which should be considered as folklore.

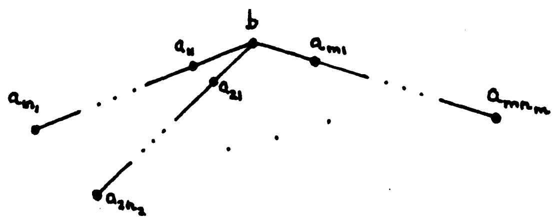
LEMMA 5.1 [CS]: Let M^3 be a 3-manifold obtained from surgery on the framed link L in the boundary of a homology 4-ball. Let L' be the characteristic sublink corresponding to the almost-framing F of M^3. Then

$$\mu(M^3; F) \equiv \sigma + 8 \operatorname{Arf}(L') - \ell(L', L') \pmod{16}$$

where σ is the signature of the linking matrix for L. □

LEMMA 5.2: Let K be a knot in the boundary of a homology 4-ball and let Σ^3 be a homology 3-sphere obtained from framed surgery on K. Then $\operatorname{Arf}(K) \equiv 1/8\mu(\Sigma^3) \pmod 2$.

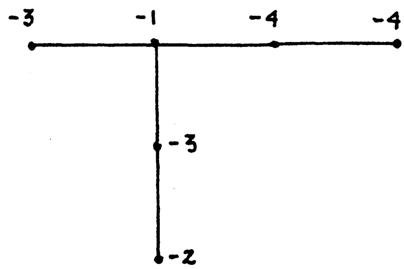
LEMMA 5.3: Let P^4 be the starlike plumbed 4-manifold with $m \geq 3$



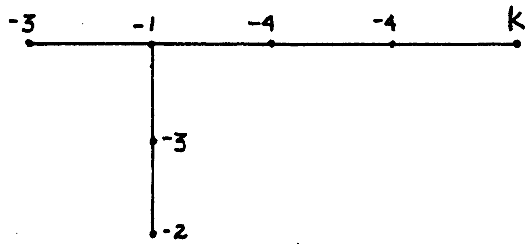
and let M^3 be the Seifert fibered 3-manifold $M^3 = \partial P^4$. Suppose that the involution t in the S^1-action on M^3 is free. Then $\alpha(M^3, t) = (P^4) - F \cdot F$ where F is a characteristic homology class composed of 0-sections of the plumbing which includes the central vertex (labelled b).

PROOF: It follows from the g-signature theorem that $\alpha(M^3, t) = \sigma(P^4, T) - P^T \cdot P^T$ where T is the involution in the natural S^1-action on P which extends the S^1-action on M. Since T is isotopic to the identity, $\sigma(P, T) = \sigma(P)$. Now given $x \in H_2(P)$, $x \cdot T_* x \equiv x \cdot P^T \pmod 2$. Let $F = P^T$ be the fixed point set; note that F is a union of 0-sections and $x \cdot F = x \cdot T_* x = x^2 \pmod 2$ since T_* = identity. Hence F is characteristic and clearly contains the central vertex when $m \geq 3$. □

Now consider the Seifert fibration M^3 which is the boundary of the plumbed 4-manifold



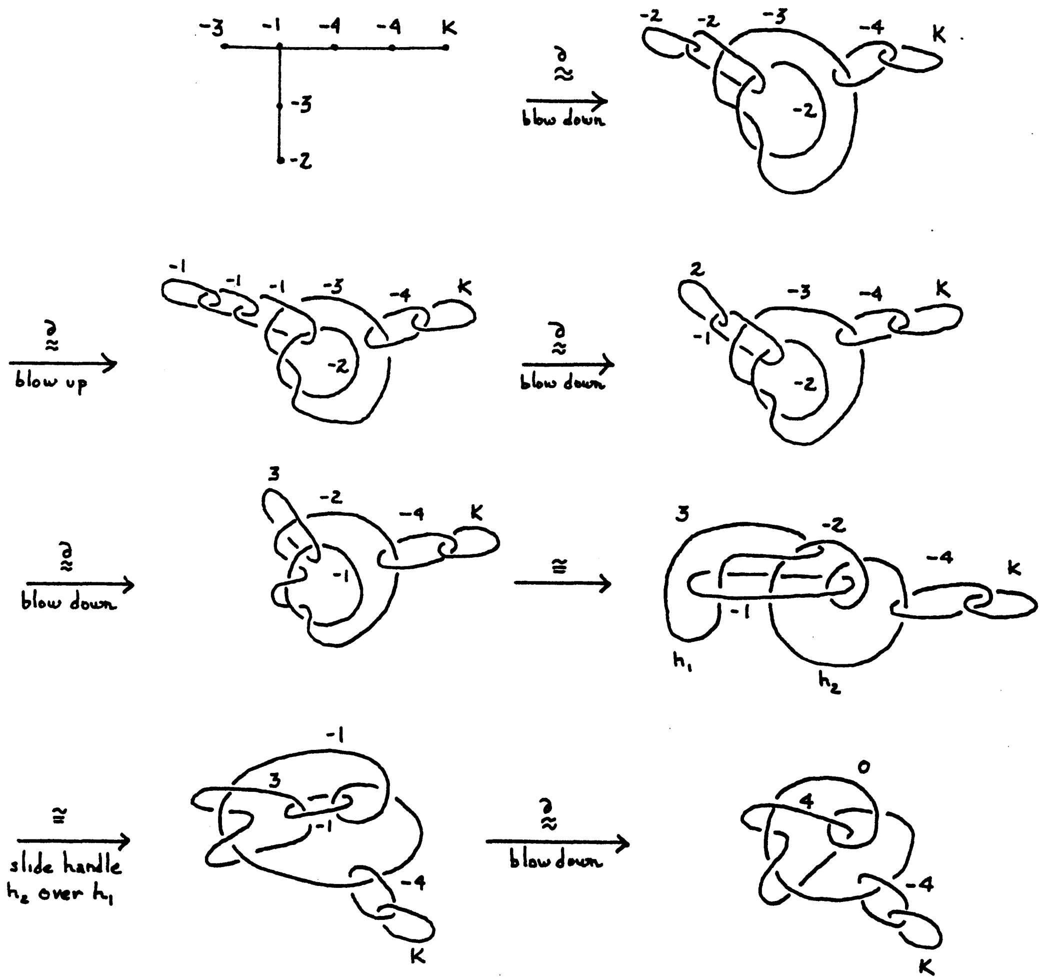
which is a homology $S^2 \times S^1$ with Seifert invariants $\{(1,1), (3,-1), (5,-2), (15,-4)\}$. By Lemma 3.1 there is a map $\tilde{h}: M^3 \to S^2 \times S^1$ which induces isomorphisms on homology such that $K = \tilde{h}^{-1}(* \times S^1)$ is the exceptional fiber of order 15, namely



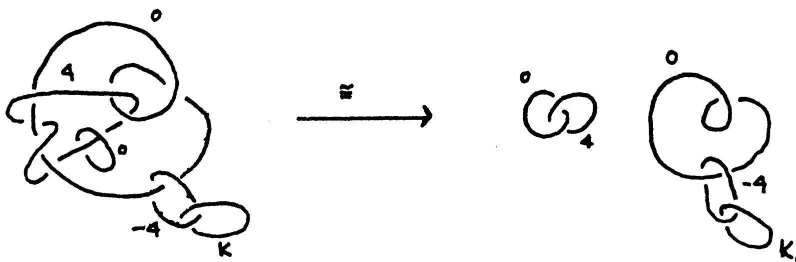
<u>PROPOSITION</u> 5.4: M^3 is the boundary of manifolds U_i^4, $i = 0, 1$, obtained by attaching a 2-handle to the boundary of the 4-ball B^4. Furthermore, there is an annulus from $K \subset M^3$ to $K_i' \subset \partial B^4$ such that $\mathrm{Arf}(K_i) \equiv i \pmod 2$. Also the double of U_i^4 is diffeomorphic to $S^2 \times S^2$.

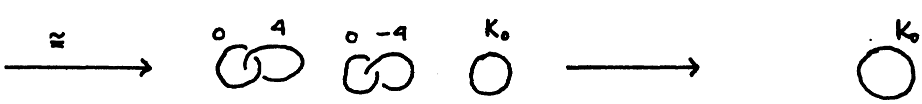
<u>PROOF</u>: We shall show that there are cobordisms from M^3 to S^3 built by adding a 2-handle to $M^3 \times I$. Then add a 4-handle to obtain U_i^4. The double of U_i^4 is $S^2 \times S^2$ by Lemma 3.2.

We use the Kirby calculus of links to show that one may add a 2-handle to M^3 to obtain S^3. We use the notation "$\underset{\approx}{\partial}$" to mean that the 4-manifolds in question have the same boundary and "$\cong$" to mean that they are diffeomorphic.



We obtain U_0^4 by adding the 2-handle

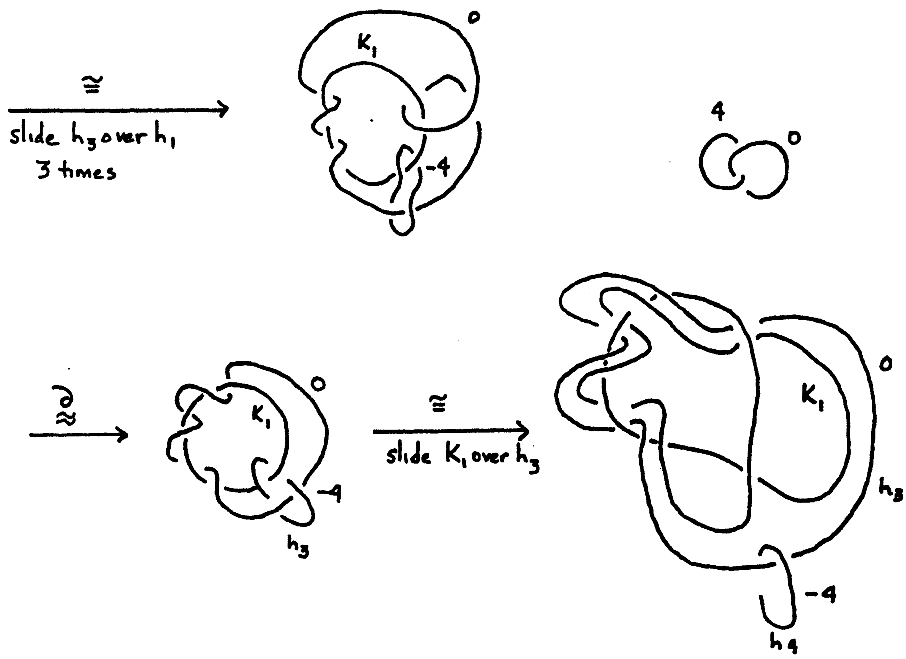




Note that $K_0 \subset S^3$ is the unknot, so $\mathrm{Art}(K_0) = 0$.

We obtain U_1 (which is diffeomorphic to U_0) by adding the 2-handle

(Note that this 2-handle is the same as the 2-handle attached to obtain U_0. However, it is different as a 2-handle attached to the pair (M^3, K). Thus the construction of §3 will give a different homotopy equivalence to $S^2 \times RP^2$).



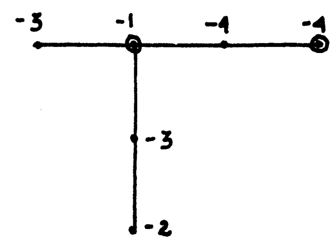
If we do a $+37$ surgery on K_1 we obtain a homology sphere Σ^3 and the framed link has characteristic sublink $L' = K_1 \cup h_4$. Since K_1 and h_4 are unlinked and K_1 is the (2,5)-torus knot, $\mathrm{Arf}(L') = \mathrm{Arf}((2,5)\text{-torus knot}) = 1$. So $\mu(\Sigma^3) = 1 + 8 - (37-4) \equiv 8 \pmod{16}$ by Lemma 5.1. So by Lemma 5.2, $\mathrm{Arf}(K_1) = 1$. □

Since the orders of the exceptional fibers of M^3 are odd, M^3 admits a free involution t isotopic to the identity. The construction in §3 yields manifolds $X_i^4 = U_i \cup_t U_i \cong S^2 \times S^2$, $i = 0, 1$, involutions T_i on X_i^4 and homotopy equivalences $f_i: X_i/T_i \to S^2 \times RP^2$.

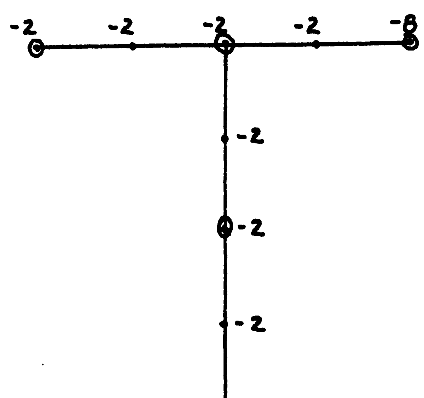
PROPOSITION 5.5: $\eta(f_i) \neq 0$, $i = 0, 1$, and $\eta_{TOP}(f_0) = 0$, $\eta_{TOP}(f_1) \neq 0$.

PROOF: Since $\mathrm{Arf}(K_i) = i \pmod 2$, Lemma 4.1 implies that $\eta_{TOP}(f_0) = 0$ and $\eta_{TOP}(f_1) \neq 0$. To show that $\eta(f_i) \neq 0$ it suffices to show that $\rho(f_i, F) \not\equiv 0 \pmod{16}$ for any almost-framing F of M^3/t, for by Proposition 4.2 this implies that $\{f_i\}$ does not equal $\{\text{identity}\}$ in $S(S^2 \times RP^2)$. But since $L_5(\mathbb{Z}_2, -) = 0$, $S(S^2 \times RP^2) \xrightarrow{\eta} [S^2 \times RP^2; G/O]$ is 1-1.

Recall M^3 is the boundary of the plumbed 4-manifold P^4:



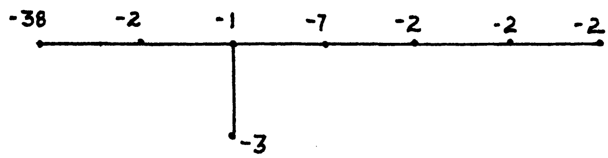
where circled vertices correspond to the class F described in Lemma 5.3. Hence $\alpha(M^3, t) = -5-(-5) = 0$. The Seifert invariants for M^3/t are $\{(1,2), (3,-2), (5,-4), (15,-8)\}$ (see [NR]) so that M^3/t is the boundary of the plumbed 4-manifold V^4:



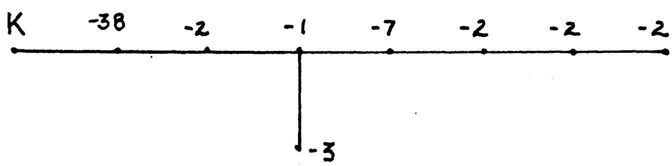
The two almost-framings F_1 and F_2 of M^3 correspond to the two characteristic homology classes of V^4, the zero class, and the class represented by the 0 sections of the plumbing which are circled above. So by Lemma 5.1 we compute the two μ-invariants of M^3/t: $\mu(M^3/t;\ F_1) \equiv -8 \pmod{16}$ and $\mu(M^3/t;\ F_2) \equiv -8 + 16 \equiv 8 \pmod{16}$. Hence $\rho(f_i,\ F) \equiv 8 \pmod{16}$ for $i = 0, 1$ and any F. $\square$

We have now constructed nontrivial homotopy smoothings of $S^2 \times RP^2$ with nontrivial smooth normal invariants. To construct the other nontrivial homotopy smoothing, by the discussion in §2, we need to find a homotopy smoothing f_2 with $\eta_{TOP}(f_2) \neq 0$ and $\eta(f_2) \neq \eta(f_1)$.

Consider the Seifert fibration N^3 which is the boundary of the plumbed manifold

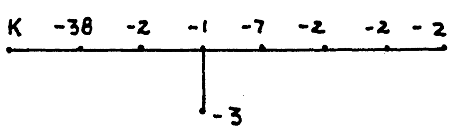


with Seifert invariants $\{(1,1),\ (75,-38),\ (3,-1),\ (25,-4)\}$. By Lemma 3.1 there is a map $\tilde{h}\colon N^3 \to S^2 \times S^1$ which induces isomorphisms on homlogy such that $K = \tilde{h}^{-1}(* \times S^1)$ is the exceptional fiber of order 75, namely:

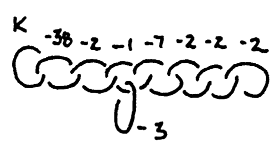


<u>PROPOSITION</u> 5.6: N^3 is the boundary of a manifold U^4 obtained by attaching a 2-handle to the boundary of a homotopy 4-ball C^4. Furthermore, there is an annulus from $K \subset N^3$ to $K' \subset N^4$ such that $\mathrm{Arf}(K') \neq 0$. Also the double of U^4 is $S^2 \times S^2$.

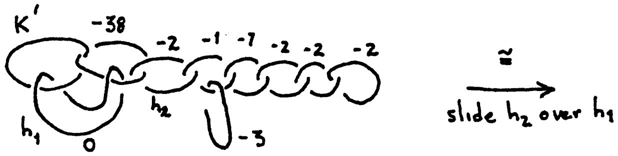
<u>PROOF</u>:

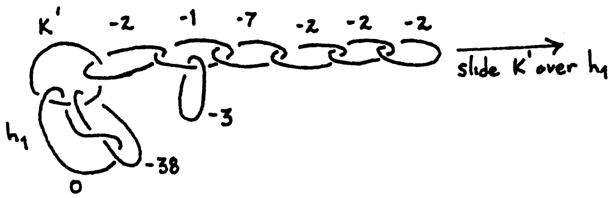


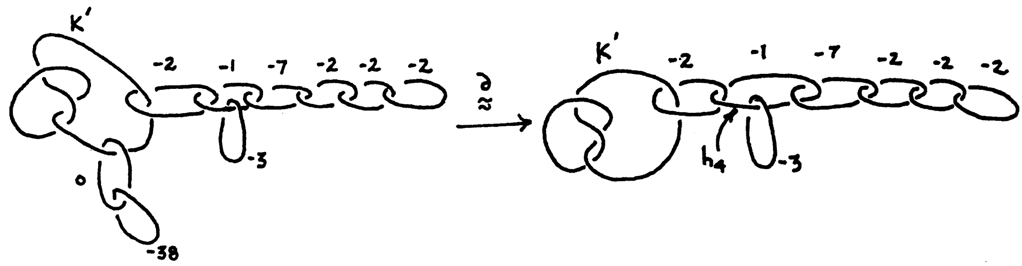
$\cong$



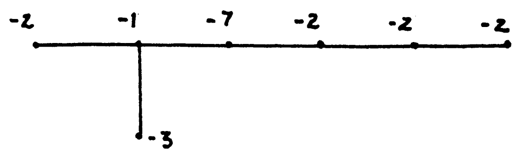
Attach a 2-handle h_1:







Now K' is a knot in the boundary of the plumbed manifold



whose boundary is the Brieskorn homology sphere $\Sigma(2, 3, 25)$ which is known to bound a contractible 4-manifold C^4 built with only 0, 1 and 2-handles. (See [K; Problem 4.2]). Let $U^4 = N^3 \times I \cup$ (attached 2-handle) $\cup C^4$. Then Lemma 3.2 implies that the double of U^4 is $S^2 \times S^2$.

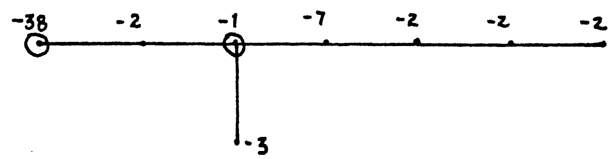
If we perform -39-surgery on K' we obtain a homology 3-sphere Σ^3 and the framed link has characteristic sublink $L' = K' \cup h_4$ with $\mathrm{Arf}(L') = \mathrm{Arf}((2,3)$ torus knot$) = 1$. So by Lemma 5.1 $\mu(\Sigma^3) \equiv -8 + 8(1) - (39-1) \equiv 8 \pmod{16}$. Hence Lemma 5.2 implies that $\mathrm{Arf}(K') \neq 0$. $\square$

Since the orders of the exceptional fibers of N^3 are all odd, we obtain as before a free involution s on N^3 which is isotopic to the identity, a free involution S on $X^4 \triangleq U \cup_s U \approx S^2 \times S^2$, and a homotopy equivalence $f_2\colon X^4/S \to S^2 \times RP^2$.

PROPOSITION 5.7: $\eta_{TOP}(f_2) \neq 0$ and $\eta(f_2) \neq (f_1)$.

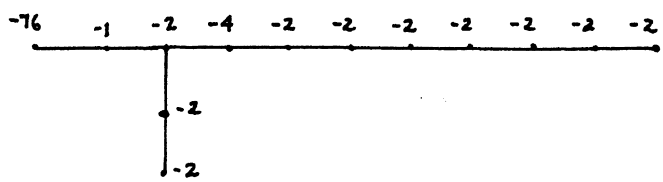
PROOF: By Lemma 4.1 $\eta_{TOP}(f_2) \neq 0$ since $\mathrm{Arf}(K') \neq 0$. We show that $\eta(f_2) \neq \eta(f_3)$ by showing that $\rho(f_2, F) \equiv 0 \pmod{16}$ for any almost-framing F of N^3. (So Proposition 4.2 will imply that $\{f_2\} \neq \{f_1\} \in S(S^2 \times RP^2)$ and hence $\eta(f_2) \neq \eta(f_1)$ since $L_5(\mathbb{Z}_2, -) = 0$.)

Now N^3 is the boundary of the plumbed 4-manifold

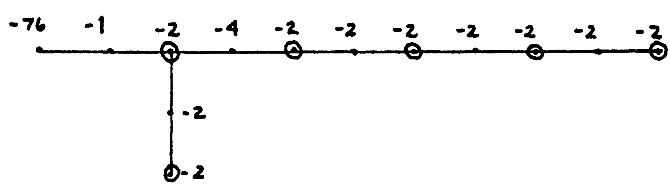


where the circled vertices correspond to the class F described in Lemma 5.3. So $\alpha(N^3, s) = -7-(-39) = 32$.

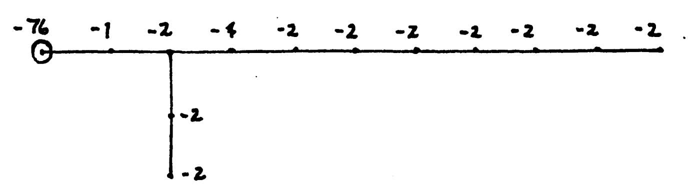
The Seifert invariants for N/s are $\{(1,2), (75,-76), (3,-2), (25,-8)\}$ so that N/s is the boundary of the plumbed 4-manifold



The two almost-framings F_3 and F_4 of N^3 correspond to the two characteristic homology classes represented by the 0-sections circled below:



and



Hence the two μ-invariants of N^3/s are $\mu(N^3/s;\ F_3) \equiv -12-(12) \equiv 0$ (mod 16) and $\mu(N^3/s;\ F_4) \equiv -12-(-76) \equiv 0$ (mod 16). Thus $\rho(f_2, F_i)$ $\equiv 0 - 1/2(32) \equiv 0$ (mod 16) for $i = 3, 4$. $\square$

As is pointed out in the introduction, each of these homotopy $S^2 \times RP^2$'s is s-cobordant to $S^2 \times RP^2$.

REFERENCES

[CS] S. Cappell and J. Shaneson, Some new four manifolds, Ann. of Math. 104 (1976), 61-72.

[FS] R. Fintushel and R. Stern, An exotic free involution on S^4, Ann. of Math. 113 (1981), 357-365.

[K] R. Kirby, Problems in low dimensional manifold theory, Proc. Sympos. Pure Math., Vol. 32, A.M.S., 273-312.

[M] T. Matumoto, On homotopy equivalence of $S^2 \times RP^2$ to itself, J. Math. Kyoto Univ. 19 (1979), 1-17.

[NR] W. Newmann and F. Raymond, Seifert manifolds, plumbing, μ-invariant, and orientation-reversing maps, pp. 163-196 in Algebraic and Geometric Topology, Proceedings, Springer Notes #664.

[O] P. Orlik, Seifert Manifolds, Springer Notes #291.

[S] E. César de Sá, A link calculus for 4-manifolds, in Topology of low-dimensional manifolds, Proceedings, Sussex 1977, Lecture Notes in Math. No. 722, 16-30.

[W] C. T. C. Wall, Surgery on Compact Manifolds, London Math. Soc. Monograph #1, Academic Press, 1970.

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