Instantons and the Topology of 4-Manifolds

Ronald J. Stern

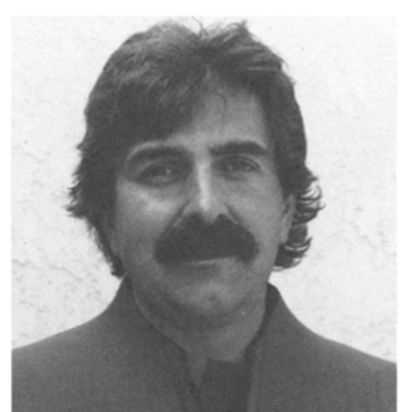
Geometric topology is the study of metric spaces which are locally homeomorphic to Euclidean n-space $\mathbf{R}^n$; that is, it studies topological (TOP) n-manifolds. The customary goal is to discover invariants, *usually* algebraic invariants, which classify all manifolds of a given dimension. This is separated into an existence question—finding an n-manifold with the given invariants—and a uniqueness question—determining how many n-manifolds have the given invariant. As is (and was) quickly discovered, TOP manifolds are too amorphous to study initially, so one adds structure which is compatible with the available topology and which broadens the available tools. Presumably, the richer the structure imposed on a manifold, the fewer objects one is forced to study.

As our first example consider $M = \mathbf{R}^n$ (or open subsets thereof). One could study continuous functions on M, but there are many more tools available for studying *smooth* functions. To extend the concept of smoothness to more general manifolds one specifies a smoothly compatible set of coordinate charts. A *differentiable atlas* on a metric space M is a cover of M by open sets $\{U_\alpha\}$ (called *coordinate charts*) and homeomorphisms $\Phi_\alpha : U_\alpha \to \mathbf{R}^n$ (so far this is just the definition of a TOP n-manifold) so that the *transition functions* $\Phi_\beta \circ \Phi_\alpha^{-1} : \Phi_\alpha(U_\alpha \cap U_\beta) \to \Phi_\beta(U_\alpha \cap U_\beta)$ are smooth. Note that both the domain and range of the transition functions are open subsets of $\mathbf{R}^n$, so smoothness makes sense. Now the ordinary calculus of $\mathbf{R}^n$ can be patched together via the transition functions to allow one to do calculus on all of M. For instance, $f : M \to \mathbf{R}$ is smooth if $f \circ \Phi_\alpha^{-1} : \Phi_\alpha(U_\alpha) \to \mathbf{R}$ is smooth for each U_α.

There are some obvious redundancies in this notion of differentiation; we say that two atlases $\{U_\alpha, \Phi_\alpha\}$ and $\{V_\beta, \Psi_\beta\}$ are equivalent if their union is an atlas on M; that is, if $U_\alpha \cap V_\beta \neq \Phi$, then $\Phi_\alpha \circ \Psi_\beta^{-1}$ is smooth. A smooth (DIFF) structure on M is an equivalence class of atlases on M.

Poincaré promoted another type of structure on a manifold, where the above transition functions $\Phi_\alpha \circ \Phi_\beta^{-1}$, instead of being smooth are required to preserve the natural combinatorial structure of $\mathbf{R}^n$ (piecewise straight-line segments are mapped to piecewise straight-line segments). Such a maximal atlas on M is called a piecewise linear (PL) structure on M; we say M is combinatorially triangulated. A PL n-manifold can be shown to be PL homeomorphic to a simplicial complex that is a so-called combinatorial n-manifold (the link of every vertex is PL homeomorphic to the $(n-1)$-sphere S^{n-1} with its standard triangulation). These various concepts of a manifold were introduced toward the end of the nineteenth century in order to better understand solutions to differential equations. It is ironic that the field of geometric topology developed, ignoring its roots, and how, as we shall shortly point out, some of the most pressing questions concerning manifolds today are being answered using modern analysis and geometry.

A basic problem for topologists became, then, to determine when a TOP manifold admits a PL structure and, if it does, whether there is also a compatible DIFF structure. In the 1930s amazingly delicate proofs of the triangulability of DIFF manifolds were given by Whitehead [24] (so that indeed DIFF $\subset$ PL). By the mid 1950s, it was known that every TOP manifold of dimension less than or equal to 3 admits a *unique* DIFF structure [10, 15, 18]. It was unthinkable that for a given TOP (or PL) manifold M there were two distinct calculuses available; that is, that M admitted more than one DIFF structure. In 1956 Milnor showed that there are 28 distinct DIFF structures on S^7. (The examples were provided by some "well-understood" S^3 bundles over S^4 [12].) Other spheres were then discovered to possess



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exotic DIFF structures. Later work by a host of major mathematicians (e.g., Thom, Kervaire, Milnor, Munkres, Hirsh, Mazur, Poenaru, Lashof, Rothenberg, Haefliger, Smale, Novikov, Browder, Wall, Sullivan, Kirby, and Siebenmann) during the period from 1956 to 1970 began to sort things out in dimensions 5 and greater—a golden era for manifold topology. In the end, the obstructions to putting a PL or DIFF structure on a TOP manifold M of dimension at least 5 became a "lifting" problem (a reduction of the structure group of the tangent bundle) and thus a discrete problem.

In 1968 Kirby and Siebenmann [11] determined that for a TOP manifold of dimension *at least 5*, there is a *single* obstruction $k(M) \in H^4(M; \mathbf{Z}_2)$ to the existence of a PL structure on M; if $k(M) = 0$, there is a PL structure, otherwise there isn't. Moreover, if there is one PL structure, then there are $|H^3(M; \mathbf{Z}_2)|$ distinct PL structures. Earlier, when it was first realized that structure problems really should be lifting problems, it was discovered that (without dimension restrictions) there are further discrete obstructions to "lifting" from PL to DIFF; these involve the homotopy groups of spheres [9]. In particular every PL n-manifold, $n \leqslant 7$, admits a compatible DIFF structure which, if $n \leqslant 6$, is unique up to DIFF isomorphism! As one might expect (or hope), our ordinary Euclidean spaces $\mathbf{R}^n$, $n \neq 4$, are indeed ordinary, in that they possess but one DIFF or PL structure.

Surprisingly, virtually none of the techniques developed during these decades has had any impact on the dimensions in which we live and operate, dimensions 3 and 4. In dimension 3, Thurston has made impressive progress in the last six years—much more remains to be done. In dimension 4, there has been equally dramatic progress in the last two years. Perhaps the most striking fact to surface is that, unlike all other Euclidean spaces, $\mathbf{R}^4$ is not so ordinary; there is an exotic DIFF structure on $\mathbf{R}^4$! The existence of this exotic $\mathbf{R}^4$, denoted $\mathscr{R}^4$, is proved using a combination of topology (the work of M. Freedman [7]), geometry, and analysis (the work of S. Donaldson [4]). This $\mathscr{R}^4$ is truly bizarre in that there exists a compact set K in $\mathscr{R}^4$ such that no smoothly embedded 3-sphere in $\mathscr{R}^4$ contains K. Since $\mathscr{R}^4$ is homeomorphic to $\mathbf{R}^4$, there are certainly continuously embedded 3-spheres in it containing K. Thus the horizon of $\mathscr{R}^4$ is extremely jagged. (After looking in the mirror some mornings I am convinced I live in $\mathscr{R}^4$).

At this time it is not known how many such exotic $\mathbf{R}^4$'s exist, although three have been found [8]. Because of the nature of the constructions, topologists speculate that there are uncountably many distinct DIFF structures on $\mathbf{R}^4$—an appealing possibility. The classification of DIFF structures, which in higher dimensions is a discrete problem, could (will) wander into the realm of geometry—a whole moduli space of DIFF structures on 4-manifolds! Whether or not this is true, the work of Donaldson points out the impossibility of characterizing DIFF structures in terms of characteristic classes; i.e., it is not a discrete problem. Taking it to the limit could have so many meanings in dimension 4! Enough mystical wanderings. Why is there such an $\mathscr{R}^4$?

The Intersection Form

We said at the beginning of this article that the goal of geometric topology is to discover algebraic invariants which classify (at least partially) all manifolds in a given dimension. Historically one of the most important of these invariants has been the *intersection form.*

Perhaps it's best to start with two-dimensional manifolds where the intersection form and intersection numbers are more familiar. We can represent one-dimensional homology classes on a smooth surface S by smooth oriented curves. Suppose $\alpha, \beta \in H_1(S; \mathbf{Z})$ are represented by curves A and B. By slightly perturbing the curves we can assume they intersect transversally

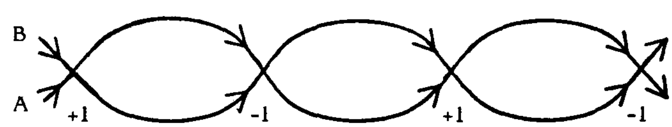

Figure 1. Intersection number zero

in isolated points. That means that at each point of intersection a tangent vector to A, together with a tangent vector to B (in that order!), forms a basis for the tangent space of S. To each point of intersection we assign $+1$ if the orientation of this basis agrees with the orientation of S; otherwise we assign -1. The (oriented) intersection number $A \cdot B$ is defined to be the algebraic sum of these numbers over all points of intersection, and the intersection form is the induced bilinear pairing defined by $I_S(\alpha, \beta) = A \cdot B$. It's easy to see that I_S is skew-symmetric $[I_S(\alpha, \beta) = -I_S(\beta, \alpha)]$ and unimodular. In fact, for any such form we can choose a basis so that the matrix of the form is

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Intersection numbers and the intersection form for a smooth 4-manifold M are defined similarly. This time we suppose *two*-dimensional homology classes $\alpha, \beta \in H_2(M; \mathbf{Z})$ are represented by smooth, oriented surfaces A and B and that the surfaces intersect transversally in isolated points. Again we assign $+1$ to a point of intersection if an (oriented) basis for the tangent space of A together with an (oriented) basis for the tangent space of B agrees with the orientation for M; otherwise we assign -1. The intersection number $A \cdot B$ is the algebraic sum of these over all points of intersection, and the intersection form is the bilinear pairing $I_M(\alpha, \beta) = A \cdot B$. This time, however, the pairing is symmetric $[I_M(\alpha, \beta) = I_M(\beta, \alpha)]$. It is still unimodular—the matrix for the form has determinant ± 1.

For smooth manifolds there is another way to define the intersection form. By Poincaré duality we can define the pairing in cohomology rather than homology. If we use DeRham cohomology $H^*_{DR}(M)$, then $\alpha, \beta \in H^2_{DR}(M)$ can be represented by 2-forms a and b. We simply let

$$I_M(\alpha, \beta) = \int_M a \wedge b$$

Defining the intersection form on cohomology allows us to extend the definition to *all* 4-manifolds, smooth or not. If $\alpha, \beta \in H^2(M; \mathbf{Z})$ and $[M] \in H_4(M; \mathbf{Z})$ is the fundamental class of M (given by an orientation on M), then $I_M(\alpha, \beta) = (\alpha \cup \beta)[M]$, where "$\cup$" is the cup product in cohomology.

Here are some examples.

1. The 4-sphere S^4. Since $H_2(S^4; \mathbf{Z}) = 0$, the intersection form is trivial: $I_{S^4} = \varnothing$.
2. The complex projective plane $\mathbf{C}P^2$. Here $H_2(\mathbf{C}P^2; \mathbf{Z}) = \mathbf{Z}$, and so the matrix for $I_{\mathbf{C}P^2}$ is (1).
3. The product of spheres $S^2 \times S^2$. In this case $H_2(S^2 \times S^2; \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$, and we can represent generators by the embedded surfaces $A = S^2 \times \{pt\}$ and $B = \{pt\} \times S^2$. Since A and B intersect in a single point, and each of them can be "pushed off" themselves, the matrix for $I_{S^2 \times S^2}$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

4. The Kummer surface

$$K = \{[Z_0, Z_1, Z_2, Z_3] \in \mathbf{C}P^3 | Z_0^4 + Z_1^4 + Z_2^4 + Z_3^4 = 0\}$$

This time things are much more complicated. The rank of $H_2(K; \mathbf{Z})$ is 22, and one can show that the matrix for I_K is given by $E_8 \oplus E_8 \oplus 3\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, where

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

(In fact, E_8 is the Cartan matrix for the exceptional Lie algebra e_8.)

The intersection form is indeed a basic invariant for compact 4-manifolds. In 1958 Milnor [13] showed that the homotopy type of a compact, simply-connected 4-manifold is completely determined by the isomorphism class of the intersection form.

The classification (up to isomorphism) of integral unimodular symmetric bilinear forms starts with three things: the rank (the dimension of the space on which the form is defined), the signature (the number of positive eigenvalues minus the number of negative values when considered as a real, rather than integral, form), and the type (the form is even if all the diagonal entries in its matrix are even, otherwise it's odd). A form is positive (negative) definite if all eigenvalues are positive (negative); otherwise it is indefinite.

For indefinite forms the rank, signature, and type form a complete set of invariants [14]. The classification of definite forms, however, is much more difficult. There is only one nontrivial restriction on an even definite form—its signature must be divisible by 8. In fact, it is known that E_8 (mentioned above) is the unique positive definite even form of rank 8; there are two even positive definite forms of rank 16 ($E_8 \oplus E_8$ and E_{16}); 24 such forms of rank 24; and many thousands of rank 40.

Given this complicated classification, it was natural to ask which forms could actually occur as the intersection form of a 4-manifold. For example, does there exist a manifold with intersection form E_8? or even $E_8 \oplus E_8$? The Kummer surface comes close for the second, but until two years ago no one knew the answer.

Topological 4-Manifolds

History had taught us that one first understands DIFF and PL manifolds and then proceeds to the more delicate questions concerning TOP manifolds. (Recall that DIFF = PL in dimensions ≤ 6). Imagine the shock when in the summer of 1981 Michael Freedman announced that a compact simply-connected (i.e., every map of the circle extends to a map of the disk) TOP 4-manifold is completely and faithfully classified by two elementary pieces of information!

The first piece of information is the *intersection form* on a 4-manifold M which we have just discussed.

The second piece of information required for Freedman's classification theorem is the Kirby–Siebenmann obstruction $\alpha(M) \in \mathbf{Z}_2$. It is completely characterized by the statement that $\alpha(M) = 0$ if and only if $M \times S^1$ admits a DIFF structure.

Freedman's Theorem. Compact, simply-connected TOP 4-manifolds are in 1-to-1 correspondence with pairs $\langle I, \alpha \rangle$, where I is an integral unimodular symmetric bilinear form, $\alpha \in \mathbf{Z}_2$, and if I is even $\sigma(I)/8 \equiv \alpha (\mathrm{mod}\ 2)$.

In particular (existence): For an integral unimodular symmetric bilinear form I, there is a TOP 4-manifold M_I realizing I as its intersection form; (uniqueness): If I is even, the homeomorphism type of M_I is uniquely determined by I. For odd I, there are exactly two non-

Comments on Freedman's Proof

For the purposes of our $\mathcal{R}^4$ we view Freedman's result as the ability to do TOP surgery. In particular, recall the Kümmer surface K whose intersection form I_K had a matrix representation as $E_8 \oplus E_8 \oplus 3\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, where $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ is the intersection form for $S^2 \times S^2$. In 1973 A. Casson, in an effort to "surger" K, showed how to represent each of the three $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ part of the homology of K by so-called embedded Casson (flexible, kinky) handles. These take the form of disjointly embedded DIFF manifolds CH_i, $i = 1, 2, 3$, each of the proper homotopy type of $S^2 \times S^2 - B^4$, and such that each CH_i contained one of the $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ part of the homology of K. If each CH_i were in fact diffeomorphic to $S^2 \times S^2 - B^4$, we could remove the CH_i, leaving a manifold with S^3 boundaries, and then cap off each S^3 by a B^4 to obtain a DIFF manifold whose intersection pairing is the $E_8 \oplus E_8$ form. By a "dual" construction of Casson each CH_i embeds in the standard $S^2 \times S^2$, in fact, $CH_i = S^2 \times S^2 - X_i$, where X_i is a compactum which is a suspension of an iterated Whitehead continuum. R. D. Edwards then observed that in fact CH_i was DIFF isomorphic to $S^2 \times S^2 - B^4$ if and only if X_i is smoothly cellular in $S^2 \times S^2$; i.e., X_i is the nested intersection of smooth 4-balls in $S^2 \times S^2$. The crux of Freedman's proof is that indeed each CH_i is (TOP) *homeomorphic* to $S^2 \times S^2 - B^4$. Thus one can topologically do the surgery and hence get a TOP manifold whose intersection form is $E_8 \oplus E_8$. Freedman proves that CH_i is standard by first reimbedding the CH_i with delicate geometric control so that the "frontier" of CH_i is geometrically small. Armed with the classical decomposition theoretic fact that, although $\mathbf{R}^3$ mod a Whitehead continuum is not a manifold, but crossed with $\mathbf{R}^1$ is indeed a manifold, Freedman artfully constructs a collar of the frontier of the CH_i, using (and developing) powerful techniques from decomposition space theory (and with a little help from R. D. Edwards, our premier shrinker).

homeomorphic M_{I} realizing I as their intersection form, characterized by the fact that one $M_{\mathrm{I}} \times S^1$ admits a DIFF structure, the other does not. So for instance, there are two manifolds realizing the form (1), namely, $\mathbf{C}P^2$ and another manifold Ch with the property that Ch is homotopy equivalent to $\mathbf{C}P^2$ but Ch admits no DIFF structure.

How beautifully simple—yet so sweeping! The uniqueness statement for $\langle \varnothing, 0\rangle$ is the four-dimensional topological Poincaré conjecture—that a 4-manifold of the homotopy type of S^4 is homeomorphic to S^4. There also is the existence of a unique TOP 4-manifold with the intersection pairing E_8—a manifold long sought after by topologists.

(Freedman's original theorem had a further hypothesis that M with a point deleted admits a DIFF structure. However, F. Quinn [17], during the summer of 1982, showed that any noncompact TOP 4-manifold admits a DIFF structure, along with many other important facts, such as the four-dimensional annulus conjecture!)

The proof of Freedman's theorem itself blends two historically independent schools of topology—the surgery and decomposition space schools. The proof utilizes the most powerful techniques and results from each school. (See the *Freedman's Proof* box.)

DIFF 4-Manifolds

After such a complete understanding of compact simply-connected TOP 4-manifolds, one is amazed at how little was still known about DIFF 4-manifolds in the *immediate* post-Freedman era. There were no new DIFF 4-manifolds (although there were some new candidates) or no old candidates that were eliminated. The earliest indication that DIFF 4-manifolds are peculiar is a result of Rochlin in 1952 [20].

Rochlin's Theorem. If a simply-connected DIFF 4-manifold has an even intersection form I, then $\sigma(I)$ is divisible by 16.

Recall that the algebraic restriction on such an I is that $\sigma(I)$ be divisible by 8, so the topology of a DIFF manifold restricts the possible intersection pairings. Now Freedman's theorem guarantees the existence of a compact simply-connected TOP 4-manifold M_{E_8} with $I_M = E_8$ and E_8 is even and $\sigma(E_8) = 8$, so that M_{E_8} does not admit a DIFF structure!

Until very recently, Rochlin's theorem and related signature invariants were the only tools available to study DIFF 4-manifolds. (It's amazing we got so far with so little!) After a blow to the head from Freedman's work, rumors were afloat at the end of 1981 that the mathematical physicists were able to detect a new obstruction to the smoothability of 4-manifolds. Then during the summer of 1982 at the first year of the AMS Summer Research Conference Series in New Hampshire (an event planned well before Freedman's work), 4-manifolds topologists were dealt a blow from the forgotten roots of our subject—we were treated to a strong dose of geometry and analysis to explain the remarkable theorem of Simon Donaldson [4], a graduate student at Oxford.

Donaldson's Theorem. Suppose M is a compact simply-connected DIFF 4-manifold with positive definite intersection pairing I. Then I is equivalent over the integers to the standard diagonal form diag(1, 1, . . . , 1).

In particular $E_8 \oplus E_8$ is *not* diagonalizable over the

Riemannian 4-Manifolds: The Proof of Donaldson's Theorem

Although the statement of Donaldson's theorem is topological in nature (i.e., a statement topologists can understand), its proof is anything but topological in nature (i.e., a proof a topologist cannot easily digest). Differential geometers have long been aware that four-dimensional space does have some remarkable properties which distinguish it from spaces of other dimensions. Perhaps at the cornerstone is that the rotation group $SO(n)$ is a simple Lie group for all $n \neq 4$ and that $SO(4)$ is locally isomorphic (in fact double-covers) $SO(3) \times SO(3)$; i.e., $SPIN(4) = SU(2) \times SU(2)$. In terms of Lie algebras this decomposition can be given as follows. The six-dimensional space $\Lambda^2(\mathbf{R}^4)$ of 2-forms on the inner product space $\mathbf{R}^4$ can be viewed as the Lie algebra of $SO(4)$. As $so(4) = so(3) \times so(3)$, $\Lambda^2(\mathbf{R}^4)$ decomposes as the sum of three-dimensional spaces $\Lambda^2(\mathbf{R}^4) = \Lambda^2_+ \oplus \Lambda^2_-$. An alternate description of this decomposition is given in terms of the Hodge star operator $*$: $\Lambda^2(\mathbf{R}^4) \to \Lambda^2(\mathbf{R}^4)$. If $(e_1, \ldots, e_4)$ is an oriented orthonormal basis for $\mathbf{R}^4$, then $*(e_i \wedge e_j) = e_k \wedge e_\ell$, where (i, j, k, ℓ) is an even permutation of $(1, 2, 3, 4)$. As $(*)^2 = 1$, $\Lambda^2(\mathbf{R}^4)$ decomposes as the ± 1 eigenspaces $\Lambda^2_\pm$ of $*$. Thus, if M admits a Riemannian metric, $\Lambda^2(T^*M) = \Lambda^2_+(M) \oplus \Lambda^2_-(M)$, and this decomposition is an invariant of the conformal class of the metric on M. An element of $\Lambda^2_+(M)$ $(\Lambda^2_-(M))$ is called a *self dual* (*anti-self dual*) 2-form. This decomposition is significant in differential geometry because curvature is a 2-form (with values in a Lie algebra) so that the curvature decomposes into its self dual and anti-self dual components. So, given a principal $SU(2)$ bundle ξ over a Riemannian 4-manifold M with a connection ∇, the curvature R^∇ of ∇ is an element of $\Lambda^2(T^*M) \otimes \mathfrak{g}$ [(where $\mathfrak{g}$ is a $su(2)$ bundle associated to ξ, the *adjoint bundle* of ξ) and $R^\nabla = R^\nabla_+ + R^\nabla_-$. If $R^\nabla_-(R^\nabla_+)$ vanishes, then ∇ is called a *self dual* (*anti-self dual*) connection; i.e., $*R^\nabla = R^\nabla(-R^\nabla)$. The existence of such self dual connections has garnered much interest in recent years, as the differential equation

$$*R^\nabla = R^\nabla$$

is the self dual Yang–Mills (YM) equation–a non-abelian [as the group is $SU(2)$, not $SO(2)$] version of Maxwell's equation. (A connection whose curvature satisfies YM is sometimes called an *instanton*.)

Let's consider a specific example. Suppose ξ is the principal $SU(2)$ bundle over S^4 with (real) first Pontrjagin class $p_1(\xi) = 4$. Does ξ have *any* self dual connections, and if so how many; i.e., what is the moduli space $\mathcal{M}$ of solutions to YM? This problem has been completely solved, and in [1] it is shown that $\mathcal{M}$ is an open 5-manifold with a collared copy of S^4 itself near infinity. In fact $\mathcal{M}$ is the five-dimensional hyperbolic space $SO(5, 1)/0(5)$, the open 5-ball!

What about other manifolds? We now come to the outline of the proof of Donaldson's theorem. Suppose M is a manifold with ξ the principal $SU(2)$ bundle on M with $p_1(\xi) = 4$. C. H. Taubes [23] has shown that, if M has a positive definite intersection form, then there is an (irreducible) solution to YM; i.e., ξ admits a self dual connection that does not reduce to a $SO(2)$ connection. The Atiyah–Singer index theorem and some deformation theory of self dual forms guarantees that $\mathcal{M}$ is a 5-manifold except there may be a closed set $C \subset \mathcal{M}$ such that in a neighborhood of any point of C, $\mathcal{M}$ is modeled as the zero set of some real analytic map $f\colon \mathbf{R}^{p+5} \to \mathbf{R}^p$. Furthermore, there are inherent point singularities in $\mathcal{M}$ whose neighborhoods are cones on $\mathbf{C}P^2$ one for each "reducible" self dual connection in ξ, i.e., one for every reduction of the group of ξ to $SO(2)$, i.e., one for every two solutions of $I_M(\alpha, \alpha) = 1$.

Utilizing the work of K. Uhlenbeck and the nature of the construction of the self dual connections by Taubes, Taubes and Uhlenbeck, and (independently) Donaldson have shown that the ends of $\mathcal{M}$ contain a collared copy of M itself. Furthermore, if $\pi_1(M) = 0$ [or no nontrivial representations of $\pi_1(M)$ in $SU(2)$], there is only one end to $\mathcal{M}$. Suppose $\mathcal{M}$ were indeed a manifold off of the inherent singular points so that there would be an orientiable DIFF cobordism from M to a disjoint union of $\mathbf{C}P^2$'s. *Again* utilizing the assumption that the intersection pairing I_M on M is positive definite, the Gram–Schmidt process allows the conclusion that I_M is equivalent to the intersection pairing of the other end of the cobordism, namely, the diagonal form diag$(1, 1, \ldots, 1)$. The bulk of Donaldson's proof is to show that, although $\mathcal{M}$ may not be a manifold off the inherent singular points, it can be perturbed (rel its end) in the space of connections to be a manifold. (Uhlenbeck [6] has recently given an argument where she perturbs the metric so that $\mathcal{M}$ itself is a manifold off the inherent singular points). In short, some new information is obtained about the topology of M by a study of the space of solutions of a particular PDE, namely, YM. Poincaré would be pleased.

integers, so that Freedman's TOP 4-manifold $M_{E_8 \oplus E_8}$ does not admit a DIFF structure—a fact that cannot be detected by characteristic classes!

The proof of Donaldson's theorem is a tight combination of topology, differential geometry, and analysis—the nondiscrete ingredient that topologists have been missing. (See the *Riemannian 4-manifold* box.)

The existence of $\mathscr{R}^4$ is now proved indirectly. Freedman provides a topological construction of $M_{E_8 \oplus E_8}$ from the Kummer surface K. It was noticed that, if $\mathbf{R}^4$ has a unique differentiable structure, then this construction can be carried out differentiably to produce a differentiable manifold. Since Donaldson showed such a manifold *cannot* exist, we must conclude that $\mathbf{R}^4$ does *not* have a unique differentiable structure. (See the *construction of* $\mathscr{R}^4$ box for more details.)

The subject of TOP and DIFF 4-manifolds is alive. There is the problem of characterizing non simply-connected TOP 4-manifolds (Freedman has made some recent progress). The entire subject of DIFF 4-mani-

The Construction of $\mathcal{R}^4$

As we pointed out above, $M_{E_8 \oplus E_8}$ exists as a TOP but not DIFF 4-manifold. In the previous section we showed that $M_{E_8 \oplus E_8}$ was obtained by "surgering" the Kümmer surface K, that is, by showing that each CH_i, $= 1, 2, 3$, was homeomorphic to $S^2 \times S^2 - B^4$. Since $M_{E_8 \oplus E_8}$ does not exist as a DIFF manifold, one of these CH_i must not be DIFF isomorphic to $S^2 \times S^2 - B^4$, say $CH = CH_1$. As R. D. Edwards pointed out, $CH = S^2 \times S^2 - X$ is DIFF (TOP) isomorphic to $S^2 \times S^2 - B^4$ if and only if X is smoothly (topologically) cellular. Thus X is the nested intersection of topological, but not smooth, 4-balls. Thus the interior of one of these topological 4-balls with the DIFF structure inherited from $S^2 \times S^2$ as an open subset is certainly homeomorphic but not DIFF isomorphic to $\mathbf{R}^4$. This is our $\mathcal{R}^4$! Note that this $\mathcal{R}^4$ cannot smoothly embed in S^4, for if it did one could perform the DIFF surgery that is not allowed. In fact, by a more judicious construction $\mathcal{R}^4$ does not smoothly embed in S^4, $\mathbf{C}P^2$ or any positive definite manifold. (A second $\mathcal{R}^4$, denoted $\mathcal{R}^4_1$, was constructed by Gompf [8] by a similar technique, however, $\mathcal{R}^4_1$ is distinct in that it embeds in $\mathbf{C}P^2$ (but not $-\mathbf{C}P^2$), so $-\mathcal{R}^4_1$ is yet a third exotic $\mathbf{R}^4$.)

folds is wide open—I hope wide open enough to assimilate the new techniques provided by analysis and geometry. We still have left unsettled the smooth 4-dimensional Poincaré conjecture (does S^4 admit exotic DIFF structures?). What is a reasonable characterization of simply-connected DIFF 4-manifolds? Perhaps all that exist are the ones we already know, namely (up to orientation) connected sums of S^4, $\mathbf{C}P^2$, and algebraic surfaces. As a starting point, can we realize the intersection form

$$E_8 \oplus E_8 \oplus 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

by a DIFF 4-manifold?

We have indicated only a flavor of the recent developments in 4-manifold topology. For further, deeper, and more complete information concerning Freedman's theorem, I recommend the reading sequence [3, 21, 7, 22]. For background material relevant to Donaldson's theorem, I recommend the reading sequence [19, 16, 2, 5, 6].

There's something that I have to prove. . . . I don't know what it is . . . or to whom I have to prove it. . . . I only know it is. . . . It keeps me busy day and night . . . reaching for the sky. . . . Working, straining, cursing out . . . seldom asking why. . . . It drives me like a raging storm . . . to whatever end . . . hoping that with conquest . . . I will comprehend.—Vas

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Department of Mathematics
University of Utah
Salt Lake City, Utah 84112