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ANOTHER CONSTRUCTION OF AN EXOTIC $S^1 \underset{\sim}{\times} S^3 \# S^2 \times S^2$

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This note was motivated by Selman Akbulut's talk at this conference. (See [A].) As Akbulut pointed out, if one could construct an exotic twisted S^3-bundle over S^1, with a homotopy equivalence $g: N^4 \to S^1 \underset{\sim}{\times} S^3$, then if a transverse preimage of an S^3-fiber is a homology sphere H^3, we must have $\mu(H^3) \neq 0$. But splitting N^4 along H^3 yields an acyclic 4-manifold whose boundary is $H^3 \# H^3$. Thus searching for an exotic $S^1 \underset{\sim}{\times} S^3$ is an approach toward finding the long sought after element of order 2 in θ^3_H.

Akbulut's construction is suggested by the fact that the complement of a tubular neighborhood $E(\mathbb{RP}^2)$ of $\mathbb{RP}^2$ in $\mathbb{RP}^4$ is $S^1 \underset{\sim}{\times} B^3$. His idea was to look for an $\mathbb{RP}^2$ in Q^4, Cappell and Shaneson's exotic $\mathbb{RP}^4$([CS]), such that $\pi_1(Q^4 - \mathbb{RP}^2) = \mathbb{Z}$, and then form $Q^4 - E(\mathbb{RP}^2) \cup S^1 \underset{\sim}{\times} B^3$. Unable to find such an $\mathbb{RP}^2$ embedded in Q^4, Akbulut was nonetheless able to find an $\mathbb{RP}^2$ in $Q^4 \# S^2 \times S^2$ with $\pi_1(Q^4 \# S^2 \times S^2 - \mathbb{RP}^2) = \mathbb{Z}$ and he was then able to form $Q^4 \# S^2 \times S^2 - E(\mathbb{RP}^2) \cup S^1 \underset{\sim}{\times} B^3$ an exotic $S^1 \underset{\sim}{\times} S^3 \# S^2 \times S^2$.

After seeing Akbulut's talk we decided to see if one could construct an exotic $S^1 \underset{\sim}{\times} S^3 \# S^2 \times S^2$ using the techniques we promoted in [FS_1] and [FS_2]. As we show this is quite simple to do and the invariant ρ of these papers can be used to detect the fact that the construction is exotic. Instead of viewing $S^1 \underset{\sim}{\times} S^3$ as $S^1 \underset{\sim}{\times} B^3 \cup S^1 \underset{\sim}{\times} B^3$, it is more convenient from our point of view to think of $S^1 \underset{\sim}{\times} S^3$ as $S^2 \times MB \cup S^1 \times B^3$ (MB = Mobius band). For our construction we start with K^3 a Seifert-fibered homology $S^2 \times S^1$ obtained by surgering an exceptional fiber of $\Sigma(3,5,19)$ and form X^4, the mapping cylinder of the free involution contained in the S^1-action on K^3. If we could show that K^3 bounded a homotopy $B^3 \times S^1$ with π_1 mapping onto, we could take its union with X^4 and thus construct a fake $S^1 \underset{\sim}{\times} S^3$. We cannot do this, but we are able to show that K^3 bounds a homotopy $B^3 \times S^1 \# S^2 \times S^2$ and thus we are able to form M^4, a homotopy $S^1 \underset{\sim}{\times} S^3 \# S^2 \times S^2$. As in [$FS_1$] we can show that if M^4 were s-cobordant to $S^1 \underset{\sim}{\times} S^3 \# S^2 \times S^2$ then

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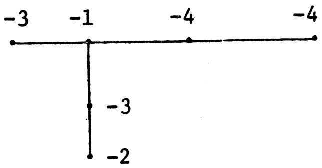
$$\mu(K/\mathbb{Z}_2) - \frac{1}{2}\alpha(K,\mathbb{Z}_2) = \rho(M^4) = \rho(S^1 \underset{\sim}{\times} S^3 \# S^2 \times S^2)$$
$$= \mu(S^2 \times S^1) - \frac{1}{2}\alpha(S^2 \times S^1, \mathbb{Z}_2) = 0 \pmod{16}$$

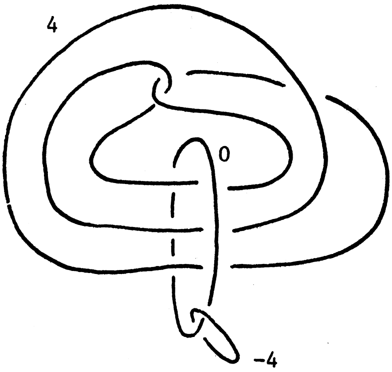
for some almost framing of $K/\mathbb{Z}_2$. However $\alpha(K;\mathbb{Z}_2) = 0$ and the two μ-invariants of $K/\mathbb{Z}_2$ are both $8 \pmod{16}$; so M^4 is exotic. Finally, we are able to show that the double cover $\tilde{M}$ is standard, i.e. $\tilde{M}$ is diffeomorphic to $S^1 \times S^3 \# S^2 \times S^2 \# S^2 \times S^2$.

We now proceed with the construction of M^4. Let K^3 be the homology $S^2 \times S^1$ which is the boundary of the plumbing manifold



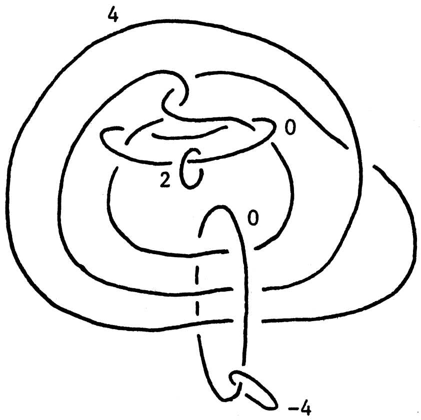
Then K is Seifert fibered with Seifert invariants $((1,1),(3,-1),(5,-2),(15,-4))$; so the involution contained in the S^1-action on K is free. Let X^4 be the mapping cylinder of the orbit map $K \to K/\mathbb{Z}_2$. As was shown in our earlier paper [FS_2, Lemma 3.1] there is a $\mathbb{Z}_2$-equivariant map $K \to S^2 \times S^1$ which induces isomorphisms on homology. (The involution on $S^2 \times S^1$ is identity $\times$ antipodal.) Taking mapping cylinders there is an induced map $f\colon X \to S^2 \times MB$ which induces isomorphisms on homology.

We have the following Kirby calculus picture for K:



(cf [FS_1; p. 362]).

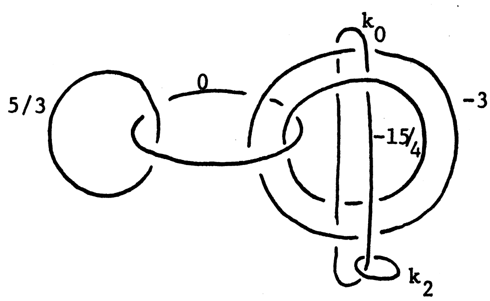
Now construct a cobordism Y^4 from K to $\partial_+ Y = \hat{K}$ by attaching the following 2-handles to $K \times I$:



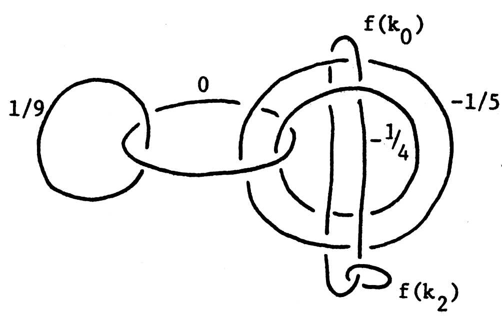
We claim that f extends over these 2-handles to a map:

$$f\colon X \cup Y \to S^2 \times MB \cup (S^2 \times S^1 \times I \# S^2 \times S^2) \quad .$$

To see this follow the 2-handles back through the Kirby calculus argument in [FS_2; p. 361-362]. The attaching circles are k_0 with 0-framing and k_2 with 2-framing:



On K-(exceptional fibers), f preserves S^1-fibers and is a 15-fold covering. The image of f (see [FS_2; Lemma 3.1]) is $S^2 \times S^1$:



In $S^2 \times S^1$, $f(k_0)$ is nullhomologous (in the above diagram we see that $f(k_0)$ bounds a genus 1 surface) therefore $f(k_0)$ is nullhomotopic in $S^2 \times S^1$. So there is a homotopy in $S^2 \times S^1$ of $f(k_0)$ to a trivial knot. By the homotopy extension property this extends to a homotopy from the identity of $S^2 \times S^1$ to a map g of $S^2 \times S^1$ to itself which takes $f(k_0)$ to a trivial knot. We can also easily arrange that $g(f(k_1))$ be a meridian of $g(f(k_0))$. Composing f with the above ambient homotopy, we extend $f: X \cup K \times I \to S^2 \times S^1 \times I$ so that $f|K \times \{1\} \to S^2 \times S^1 \times \{1\}$ maps tubular neighborhoods of k_1 and k_2 onto tubular neighborhoods of the components of a trivial Hopf link in $S^2 \times S^1 \times \{1\}$.

For some framings a_1 on $f(k_1 \times 1)$ and a_2 on $f(k_2 \times 1)$, f will extend over $Y = K \times I \cup h^2(k_1) \cup h^2(k_2) \to S^2 \times S^1 \times I \cup h^2(f(k_1)) \cup h^2(f(k_2))$. Because $f|K$ induces isomorphisms on homology the naturality of the Mayer-Vietoris sequence and the 5-lemma imply that the intersection form of these two manifolds is the same. The intersection form of Y has matrix

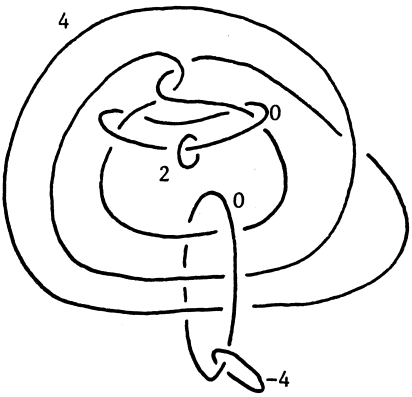
$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix},$$

and therefore is even unimodular with signature 0. Hence the same is true for the intersection form

$$\begin{pmatrix} a_1 & 1 \\ 1 & a_2 \end{pmatrix}$$

of $S^2 \times S^1 \times I \cup h^2(f(k_1)) \cup h^2(f(k_2))$. This means that this intersection form is the same as the intersection form of $S^2 \times S^2$. Hence $S^2 \times S^1 \times I \cup h^2(f(k_1)) \cup h^2(f(k_2)) \cong S^2 \times S^1 \times I \# S^2 \times S^2$.

Another 5-lemma argument shows that $f|\hat{K} \to S^2 \times S^1$ induces isomorphisms on homology. $\hat{K}$ is:



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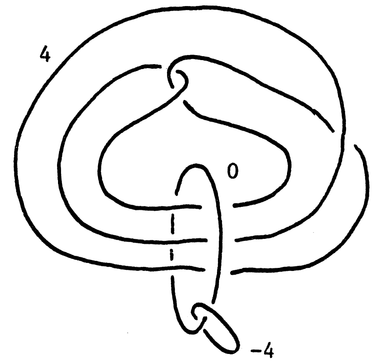

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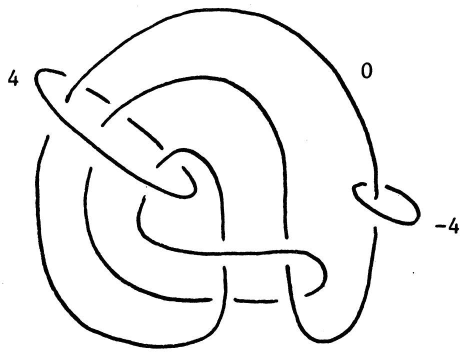
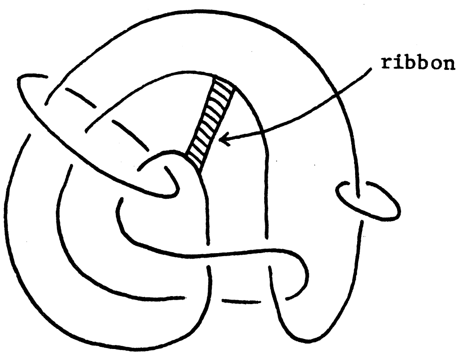
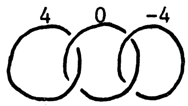


But the link



in S^3 is concordant by the ribbon move shown to

Hence there is a homology cobordism Z from $\hat{K}$ to $S^2 \times S^1 =$

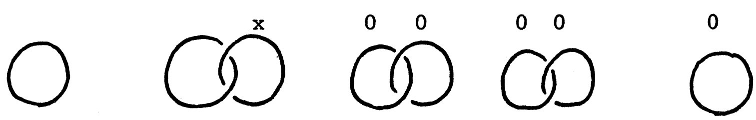


with $\pi_1(\hat{K}) \to \pi_1(Z)$ and $\pi_1(S^2 \times S^1) \to \pi_1(Z)$ onto. Let $\bar{f}: S^2 \times S^1 \to S^2 \times S^1$ be a diffeomorphism inducing on homology the same homomorphism as $(f|\hat{K})_*$. (Here we identify $H_*(\hat{K})$ with $H_*(S^2 \times S^1)$ using the homology cobordism Z.) Then by obstruction theory $f \cup \bar{f}$ extends to $f: Z \to S^2 \times S^1 \times I$. Since $\bar{f}$ extends over $B^3 \times S^1 \to B^3 \times S^1$ we obtain a homology equivalence

$$f: M = X \cup Y \cup Z \cup B^3 \times S^1 \to S^2 \times MB \cup S^2 \times S^1 \times I \# S^2 \times S^2 \cup S^2 \times S^1 \times I \cup B^3 \times S^1$$
$$= S^1 \underset{\sim}{\times} S^3 \# S^2 \times S^2 .$$

Using Van Kampen's theorem one checks that $\pi_1(M^4) = \mathbb{Z}$ and hence f induces an isomorphism on fundamental groups. Let $\tilde{f}: \tilde{M} \to S^1 \times S^3 \# S^2 \times S^2 \# S^2 \times S^2$ be the induced map on oriented double covers. As $\tilde{f}$ is degree one, the induced homomorphisms on homology with $\mathbb{Z}[\mathbb{Z}]$ coefficients split [W; Lemma 2.2]. However, all homology groups are free and in any dimension are the same rank, so $\tilde{f}$, hence f_1 induces an isomorphism on homology with local coefficients. So f is a homotopy equivalence. It is easy to compute that $\rho(M) \equiv 8 \pmod{16}$ (see [FS_2; proof of Prop. 5.5]); hence M is not s-cobordant to $S^1 \underset{\sim}{\times} S^3 \# S^2 \times S^2$.

We now show that the double cover $\tilde{M}$ is standard. Note that $\tilde{M}$ is obtained by gluing together two copies of $Y \cup Z \cup B^3 \times S^1$ by the involution $t: K \to K$. Since t is contained in an S^1 action, t is isotopic to the identity. Hence $\tilde{M}$ is the double of $Y \cup Z \cup B^3 \times S^1$. A handle decomposition for $Y \cup Z \cup B^3 \times S^1$ consists of a 0-handle, two 1-handles, and three 2-handles. (The cobordism Z is constructed by attaching algebraically cancelling 2 and 3-handles to $\hat{K} \times I$.) So the framed link picture for $\tilde{M}$ is obtained by adding a meridional circle labelled "0" to each circle representing a 2-handle. Using these it is easy to slide 2-handles to obtain



i.e. $\tilde{M} \cong S^3 \times S^1 \# S^2 \times S^2 \# S^2 \times S^2$.

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