# Shorter Notes: The Tangent Microbundle of a Suitable Manifold
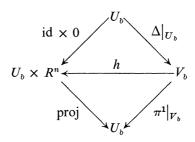
Ronald J. Stern

## SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

# THE TANGENT MICROBUNDLE
# OF A SUITABLE MANIFOLD

RONALD J. STERN[1]

ABSTRACT. The purpose of this note is to generalize to the topological category the fact that a suitable differentiable manifold is parallelizable (Theorem 4 of [1]). This result has a "folk-theorem" status in some quarters, but I believe that in view of the recent interest in $H$-manifolds [2], it would be desirable to have the result on record.

Let $M$ be an $n$-manifold. Define $\Delta: M \to M \times M$ to be the diagonal map, and $\pi^1, \pi^2: M \times M \to M$ to be the projections on the first and second factor respectively. Milnor [3] calls the diagram $\Delta: M \rightleftarrows M \times M: \pi^1$ the *tangent microbundle* of $M$, where for each point $b \in M$ there exists an open set $U_b$ in $M$ containing $b$, an open set $V_b$ in $M \times M$ containing $\Delta(b)$, and a homeomorphism $h: V_b \to U_b \times R^n$ such that the following diagram commutes:
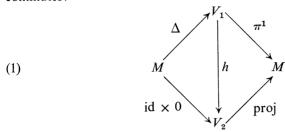
An $n$-manifold $M$ is *topologically parallelizable* if there exists an open set $V_1$ in $M \times M$ containing $\Delta(M)$, an open set $V_2$ in $M \times R^n$ containing $M \times 0$, and a homeomorphism $h: V_1 \to V_2$ so that the following diagram commutes:

(1)

$$
\begin{array}{ccc}
 & V_1 & \\
\Delta \nearrow & \big\downarrow h & \searrow \pi^1 \\
M & & M \\
\text{id} \times 0 \searrow & \big\downarrow & \nearrow \text{proj} \\
 & V_2 &
\end{array}
$$

Pick $e \in M$. $M$ is *suitable* if there is a continuous map $\Phi: M \to G(M)$ such that $\Phi(x)(x) = e$ and $\Phi(e) = $ identity, where $G(M)$ is the group of all homeomorphisms of $M$ onto itself with the compact-open topology. By Theorem 2 of [1], $M$ is suitable iff there exists a $\theta \in G(M \times M)$ such that

$$\theta(M \times (M - e)) = \{(x, y) \in M \times M : x \neq y\} \quad \text{and} \quad \pi^1 \theta = \pi^1.$$

Note that a suitable manifold supports an $H$-space structure [1].

THEOREM. *A suitable n-manifold $M$ is topologically parallelizable.*

PROOF. Let $U_b$ and $V_b$ be as in the definition of the tangent microbundle of $M$. Let $W$ be an open set in $M$ such that $e \in W \subset \mathrm{cl}\ W \subset U_e$. Choose the $V_b$'s so that $\pi^2 \theta^{-1}(x, y) \in W$ for $(x, y) \in V_b$. Let $k: U_e \to R^n$ be a co-ordinate map such that $k(e) = 0$. Define $\lambda: M \to [0, 1]$ so that $\lambda$ is 1 on a neighborhood of $\mathrm{cl}\ W$ and 0 on a neighborhood of $M - U_e$.

Let $V = \bigcup_{b \in M} V_b$ and define $h: V \to M \times R^n$ by

$$h(x, y) = (x, \lambda(\pi^2 \theta^{-1}(x, y)) k(\pi^2 \theta^{-1}(x, y))).$$

$h$ is a local homeomorphism, i.e. for $b \in M$, $h: V_b \to$ image $h|_{V_b}$ is a homeomorphism, for define $h'$: image $h|_{V_b} \to V_b$ by

$$h'(x, r) = (x, \pi^2 \theta(x, k^{-1}(r))).$$

Then on $V_b$, $\pi^2 \theta^{-1}(x, y) \in W$ so $h'h = \mathrm{id}$, and on image $h|_{V_b}$, $k^{-1}(r) \in W$ so $hh' = \mathrm{id}$.

However, $h: \Delta(M) \to M \times 0$ homeomorphically, so by Lemma 4.1 of [4], there is a neighborhood $V_1$ in $M \times M$ of $\Delta(M)$ and a neighborhood $V_2$ in $M \times R^n$ of $M \times 0$ such that $h: V_1 \to V_2$ is a homeomorphism. As $h$ commutes in (1) this proves our result.

## REFERENCES

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024