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Gauge Theories as a Tool for Low Dimensional Topologists*

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The 1980s have experienced a tremendous growth in our understanding of four dimensional smooth manifolds. This has been principally through the work of Simon Donaldson [D] with his celebrated theorem.

Theorem: Let M be a smooth closed oriented simply-connected 4-manifold with positive definite intersection form Φ. Then Φ is "standard", i.e. over the integers.

$$\Phi \cong (1) \oplus \cdots \oplus (1)$$

Although this is a theorem about 4-dimensional topology, its proof is differential geometric and analytical in nature. The main theme of Donaldson's work is to study the space of solutions of the self-dual Yang-Mills equations on an $SU(2)$ bundle over the Riemannian manifold M and relate it to the topology of M.

At first (or even second) glance, the use of techniques from Mathematical Physics to solve an important problem in topology must be ad hoc and there must be a more "topological" proof! The purpose of this note is to argue that gauge theories and Yang-Mills connections *naturally* arise in the study of smooth 4-manifolds. (Why didn't low dimensional topologists discover them sooner?)

1 The Intersection Pairing

The traditional goal of geometric topology is to discover algebraic invariants which classify (at least partially) all manifolds in a given dimension. Historically, one of the most important of these invariants has been the intersection form.

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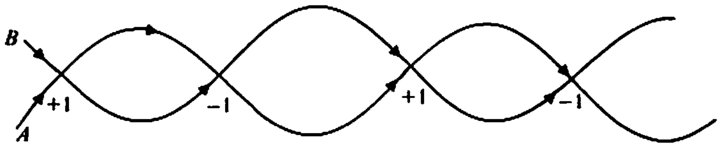
Perhaps it's best to start with two-dimensional manifolds where the intersection form and intersection numbers are more familiar. We can represent one-dimensional homology classes on a smooth surface S by smooth oriented curves. Suppose $\alpha, \beta \in H_1(S, \mathbb{Z})$ are represented by curves A and B. By slightly perturbing the curves we can assume that they intersect transversally in isolated points. This means that at each point of



Intersection number zero

intersection a tangent vector to A, together with a tangent vector for B (in that order), form a basis for the tangent space of S. To each point of intersection we assign $+1$ if the orientation of this basis agrees with the orientation of S; otherwise we assign -1. The (oriented) intersection number $A \cdot B$ is defined to be the algebraic sum of these numbers over all points of intersection, and the intersection form is the induced bilinear pairing defined by $I_s(\alpha, \beta) = A \cdot B$. It's easy to see that I_s is skew-symmetric $[I_s(\alpha, \beta) = -I_s(\beta, \alpha)]$ and unimodular. In fact, for any such form we can choose a basis so that the matrix of the form is

$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Intersection numbers and the intersection form for a smooth 4-manifold M are defined similarily. This time we suppose 2-dimensional homology classes $\alpha, \beta \in H_2(M, \mathbb{Z})$ are represented by smooth oriented surfaces A and B and that the surfaces intersect transversally in isolated points. Again we assign $+1$ to a point of intersection if an (oriented) basis for the tangent space of A together with an (oriented) basis for the tangent space of B agrees with the orientation for M; otherwise we assign -1. The intersection number $A \cdot B$ is the algebraic sum of these over all points of intersection, and the intersection form is the bilinear pairing $I_M(\alpha, \beta) = A \cdot B$. This time, however, the pairing is symmetric $[I_M(\alpha, \beta) = I_M(\beta, \alpha)]$. It is still unimodular - the matrix for the form has determinant ± 1.

For smooth manifolds there is another way to define the intersection form. By Poincaré duality we can define the pairing in cohomology rather than homology. If we use DeRham cohomology $H^*_{DR}(M)$, then $\alpha, \beta \in H^2_{DR}(M)$ can be represented by 2-forms a and b. We simply let

$$I_M(\alpha, \beta) = \int_M a \wedge b.$$

Defining the intersection pairing on cohomology allows us to extend the definition to all 4-manifolds, smooth or not. If $\alpha, \beta \in H_2(M; \mathbb{Z})$ and $[M] \in H_4(M; \mathbb{Z})$ is the fundamental class of M (given by the orientation on M), then $I_M(\alpha, \beta) = \alpha \cup \beta$, where "$\cup$" is the cup product in cohomology.

Here are some examples.

1. The 4-sphere S^4. Since $H_2(S^4, \mathbb{Z}) = 0$, the intersection form is trivial $I_{S^4} = 0$.
2. The complex projective plane $\mathbb{C}P^2$. Here $H_2(\mathbb{C}P^2; \mathbb{Z}) = \mathbb{Z}$, and so the matrix for $I_{\mathbb{C}P^2}$ is (1).
3. The product of spheres $S^2 \times S^2$. In this case $H_2(S^2 \times S^2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, and we can represent generators by the embedded surfaces $A = S^2 \times \{pt\}$ and $B = \{pt\} \times S^2$. Since A and B intersect in a single point, and each of them can be "pushed off" themselves, the matrix for $I_{S^2 \times S^2}$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
4. The Kummer surface $K = \{[Z_0, Z_1, Z_2, Z_3] \in \mathbb{C}P^3 \mid Z_0^4 + Z_1^4 + Z_2^4 + Z_3^4 = 0\}$. This time things are much more complicated. The rank of $H_2(K; \mathbb{Z})$ is 22, and one can show that the matrix for I_K is given by $E_8 + E_8 + 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

(In fact, E_8 is the Cartan matrix for the exceptional Lie algebra e_8.)

The intersection form is indeed a basic invariant for closed 4-manifolds. In 1949 Whitehead [W] showed that the homotopy type of a closed, simply-connected 4-manifold is completely determined by the isomorphism class of the intersection form.

The classification (up to isomorphism) of integral unimodular symmectic bilinear forms starts with three things: the rank (the dimension of the space on which the form is defined), the signature (the number of positive eigenvalues minus the number of negative eigenvalues considered as a real, rather than an integral form), and the type (the form is even if all the diagonal entries in its matrix are even, otherwise it's odd). A form is positive (negative) definite if all eigenvalues are positive (negative); otherwise it's indefinite.

For indefinite forms the rank, signature, and type form a complete set of invariants [MH]. The classification of definite forms, however, is much more difficult. There is only one nontrivial restriction on an even form — it's signature must be divisible by 8. In fact it is known that E_8 (mentioned above) is the unique positive definite form of rank 8; there are two even positive definite forms of rank 16 ($E_8 \oplus E_8$ and Γ_{16}); 24 such forms of rank 24; $\geqslant 10^7$ such forms of rank 32; $\geqslant 10^{51}$ such forms of rank 40.

The first indication that the intersection form of a smooth 4-manifold had more than algebraic restrictions was Rohlin's Theorem [R] which asserts that any even form coming from a smooth simply-connected 4-manifold has signature divisible by 16. In particular the form E_8 cannot occur as the intersection pairing of a simply-connected smooth 4-manifold.

2 A Study of $H^2(M;\mathbb{Z})$

In order to motivate the use of gauge theories to better understand the intersection form on a smooth 4-manifold M, we begin by studying the second cohomology group $H^2(M;\mathbb{Z})$ from the vantage point of an algebraic topologist, differential topologist, differential geometer, and then an analyst. This material is classical and appears in various (although rarely one) standard graduate courses. For simplicity we assume that $H_1(M;\mathbb{R}) = 0$.

When an algebraic topologist is confronted with the group $H^2(M;\mathbb{Z})$, obstruction theory comes to mind, whence $H^2(M;\mathbb{Z}) = [M, CP^\infty]$. But CP^∞ is the classifying space for $SO(2) = U(1)$ bundles, so that the have a $1-1$ correspondence

$$H^2(M;\mathbb{Z}) \leftrightarrow \{\text{Isomorphism classes of } SO(2) \text{ bundles over } M\}$$

where $x \in H^2(M;\mathbb{Z})$ corresponds to that $SO(2)$ bundle L over M with Euler class $= e(L) = x$.

Now assuming that M is smooth, a differential topologist would consider $H^2(M;\mathbb{R})$ rather than $H^2(M;\mathbb{Z})$ and then use the DeRham theorem to identify $H^*(M;\mathbb{R})$ with $H^*_{DR}(M)$ i.e., the homology of the DeRham complex

$$(*)\ 0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \cdots$$

where Ω^p are the p-forms on M and d is the exterior derivative. Since $d^2 = 0$, $(*)$ is indeed a complex and we can form its homology groups $H^*_{DR}(M)$. Thus any element $x \in H^2(M;\mathbb{R}) = H^2{}_{DR}(M) = \ker d/\operatorname{im} d$ is represented by an element $x \in \Omega^2$ with $dx = 0$.

Let's now assume that M is endowed with a Riemannian metric. This induces a metric $\langle , \rangle$ on p-forms, so that the differential operator d in $(*)$ has a formal adjoint δ; i.e., $\langle \delta a, b\rangle = \langle a, db\rangle$. An analyst would form the Laplacian $\Delta = d\delta + \delta d$ and then use Hodge theory to identify $H^2_{DR}(M)$ with the space of harmonic 2-forms, i.e., those 2-forms x with $\Delta x = 0$.

At this point, we can summarize our discussion with the following diagram

$$\begin{tikzcd}
H^2(M;\mathbb{Z}) \arrow[r, leftrightarrow, "\text{bundle theory}"] \arrow[d, hook] & \{\text{Isomorphism classes of } SO(2) \text{ bundles over } M\} & \\
H^2(M;\mathbb{R}) \arrow[r, leftrightarrow, "\text{DeRham theorem}"] & H^2_{DR}(M) \arrow[r, leftrightarrow, "\text{Hodge theory}"] & \{\text{harmonic 2-forms}\}.
\end{tikzcd}$$

We would like to make this into a "commutative" diagram by associating to an $SO(2)$ bundle over M a DeRham 2-form and to make this association unique. This is the realm of differential geometry through the study of connections and their curvatures!

Let E be a vector bundle over M. A *connection* ∇ on E is merely a rule which allows one to take derivatives of sections of E in the direction of tangent vectors of M. So, given a section $\sigma \in \Gamma(E)$ and a tangent vector field X on M, then $\nabla_X \sigma$ is the derivative of σ in the direction of X and it must satisfy a Leibniz rule. In other words, a *connection* on E is a linear differential operator $\nabla : \Gamma(E) \to \Gamma(T^*(M) \otimes E)$ such that

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$$

where $f: M \to R$. If E is endowed with a fiberwise metric $\langle , \rangle$ we then require ∇ to be Riemannian, that is

$$d\langle \sigma_1, \sigma_2\rangle = \langle \nabla\sigma_1, \sigma_2\rangle + \langle \sigma_1, \nabla\sigma_2\rangle$$

It will be convenient for us to set the notation

$$\Omega^k(F) = \Gamma(\Lambda^k T^* M \otimes F)$$

for any bundle F over M. So, for example, $\nabla : \Omega^0(E) \to \Omega^1(E)$. A connection ∇ has a natural extension

$$d^\nabla : \Omega^k(E) \to \Omega^{k+1}(E)$$

defined by $d^\nabla(\alpha \otimes \sigma) = d\alpha \otimes \sigma + (-1)^k \alpha \wedge \nabla\sigma$, where $\alpha \in \Omega^k$ and $\sigma \in \Omega^0(E)$.

The *curvature* R^∇ of a connection ∇ on E is a 2-form with values in $\mathfrak{g}_E$

where $\mathfrak{g}_E \subset \mathrm{Hom}(E, E)$ is the subbundle of endomorphisms which are skew-symmetric on each fiber. In other words $R^\nabla \in \Omega^2(\mathfrak{g}_E)$. It is defined by

$$R^\nabla_{X,Y} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Also, we have $R^\nabla = d^\nabla \circ \nabla$.

In the case at hand, E is an $SO(2)$ bundle, so that $\mathfrak{g}_E$ is the trivial real line bundle over M. Thus if ∇ is a connection in an $SO(2)$ bundle L over M, then its curvature $R^\nabla \in \Omega^2(\mathfrak{g}_E) = \Omega^2$ is a real 2-form! It is a consequence of the Bianchi identities that R^∇ is a closed 2-form, i.e., $dR^\nabla = 0$. Furthermore, it is a fundamental result of Chern that the real Euler class $e(L)_\mathbb{R} \in H^2_{DR}(M)$ is represented by $\frac{1}{2\pi} R^\nabla$. We could now complete our diagram by associating to each $SO(2)$ bundle L over M the curvature 2-form of a connection on L. Any two such connections determine the same DeRham cohomology class.

But let's be greedy. There are many connections on L and many representatives for the real Euler class $e(L)$ of L. Given any closed 2-form α representing $e(L)$ there is a connection ∇_α whose curvature is $2\pi\alpha$ (for fix any connection ∇ and note $\frac{1}{2\pi} R^\nabla = \alpha + dw$ for some $w \in \Omega^1$. Set $\nabla_\alpha = \nabla - 2\pi w$, and then $\frac{1}{2\pi} R^{\nabla_\alpha} = \frac{1}{2\pi} R^\nabla - dw = \alpha$). In particular there is a connection ∇_θ whose curvature is the unique harmonic 2-form θ representing $e(L)$. But there are many such connections ∇_θ! To see this, let ∇ be a connection in a vector bundle E over M and let $G_E \subset \mathrm{Hom}(E, E)$ be the bundle of orthogonal endomorphisms of E. If $g \in \Gamma(G_E)$, then $\nabla^g = g \cdot \nabla \cdot g^{-1}$ is a new connection whose curvature $R^{\nabla^g} = g \cdot R^\nabla \cdot g^{-1}$. ∇^g is said to be *gauge equivalent* to ∇ and $\mathcal{G}_E = \Gamma(G_E)$ is called the *gauge group*. In our case of an $SO(2)$ bundle L over M, $R^{\nabla^g} = g \cdot R^\nabla \cdot g^{-1} = R^\nabla$. In fact, $R^{\nabla'} = R^\nabla$ if and only if there exists a $g \in \mathcal{G}_E$ with $\nabla' = \nabla^g$! (If $R^{\nabla'} = R^\nabla$, then $\nabla' = \nabla + w$ where $dw = 0$. Since $H_1(M;\mathbb{R}) = 0$, $w = ds$ for some $s \in \Omega^0$. So $\nabla' = \nabla + ds = e^{-s}(e^s ds + e^s \nabla) = e^{-s} \nabla e^s$; hence ∇ and ∇' are gauge equivalent.)

We now associate to each $SO(2)$ bundle L over M the unique gauge equivalence class of connections whose curvatures are harmonic. This completes our commutative diagram.

$$
\begin{array}{ccccc}
H^2(M;\mathbb{Z}) & \xleftrightarrow{\text{bundle theory}} & \left\{\begin{array}{l}\text{Isomorphism classes}\\ \text{of } SO(2) \text{ bundles}\\ \text{over } M\end{array}\right\} & \xleftrightarrow{\text{differential geometry}} & \left\{\begin{array}{l}\text{gauge equivalence}\\ \text{classes of}\\ \text{connections with}\\ \text{harmonic curvature}\end{array}\right\} \\
\big\downarrow & & & & \big\downarrow \text{curvature}/2\pi \\
H^2(M;\mathbb{R}) & \xleftrightarrow{\text{DeRham theory}} & H^2_{DR}(M) & \xleftrightarrow{\text{Hodge theory}} & \{\text{harmonic 2-forms}\}
\end{array} \tag{2.1}
$$

Now that we have a cosmopolitan description of the free abelian group $H^2(M;\mathbb{Z})$ which utilizes material from (what should be) a standard graduate curriculum, so what?

3 Why Yang-Mills?

As is pointed out in the first section, in order to gain some understanding of 4-manifolds, the intersection form should come into play. This did not happen in the second section. As an attempt to introduce the intersection form into our scheme, a topologist might consider stable isomorphism classes of $SO(2)$ bundles over M rather than just isomorphism classes. That is, put an equivalence relation $\sim$ on $SO(2)$ bundles by declaring that $L \sim L'$ if and only if $L \oplus \varepsilon$ and $L' \oplus \varepsilon$ are isomorphic as $SO(3)$ bundles, where ε is a trivial $\mathbb{R}^1$ bundle over M.

To see if we have accomplished anything, what equivalence relation have we induced on $H^2(M;\mathbb{Z})$? By the classification of $SO(3)$ bundles over a 4-complex, due to Dold and Whitney [DW], $L \sim L'$ if and only if $e(L)_{(2)} = e(L')_{(2)}$ and $(e(L))^2 = (e(L'))^2$, where, for $a \in H^2(M;\mathbb{Z})$, $a_{(2)} \in H(M;\mathbb{Z}_2)$ is the mod 2 reduction of a, and a^2 is $(a \vee a)$ evaluated on the fundamental class of M. So by introducing the equivalence relation $\sim$ on $H^2(M;\mathbb{Z})$ given by $a \sim b$ if and only if $a_{(2)} = b_{(2)}$ and $a^2 = b^2$, we have the 1-1 correspondence

$$H(M;\mathbb{Z})/\sim \;\leftrightarrow \left\{\begin{array}{l}\text{isomorphism classes of}\\ SO(2) \text{ bundles over } M\end{array}\right\}/\sim.$$

Let's now attempt to complete the picture.

The novel (at least for a topologist) viewpoint in the previous section was the study of connections on L. So we now study connections on $E = L \oplus \varepsilon$; in particular, we should study those connections whose curvature forms are harmonic. But what does this mean, since $R^\nabla \in \Omega^2(\mathfrak{g}_E)$ and $\mathfrak{g}_E$ is no longer trivial (in fact, for $SO(3)$ bundles E, $\mathfrak{g}_E \simeq E$). As mentioned above, we have the sequence

$$\Omega^0(E) \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{d^\nabla} \Omega^2(E) \xrightarrow{d^\nabla} \cdots.$$

However, $R^\nabla \in \Omega^2(\mathfrak{g}_E)$, so we should look for a sequence involving forms with values in $\mathfrak{g}_E$. Given a connection ∇ in E, it induces a connection ∇ in $\mathfrak{g}_E$ given by $\nabla(\theta) = [\nabla, \theta]$ where $\theta \in \Omega^0(\mathfrak{g}_E)$, i.e., $\nabla(\theta)(\sigma) = \nabla(\theta(\sigma)) - \theta(\nabla\sigma)$ for any section σ of E. We then have the sequence

$$\Omega^0(\mathfrak{g}_E) \xrightarrow{\nabla = d^\nabla} \Omega^1(\mathfrak{g}_E) \xrightarrow{d^\nabla} \Omega^2(\mathfrak{g}_E) \xrightarrow{d^\nabla} \cdots$$

and, as in the real case, the Bianchi identities translate to the fact that $d^\nabla R^\nabla = 0$. Again, each d^∇ has a formal adjoint δ^∇, and we can form the Laplacian $\Delta^\nabla = d^\nabla \delta^\nabla$

$+\delta^\nabla d^\nabla$. We then wish to study those connections ∇ in $E = L \oplus \varepsilon$ whose curvatures are harmonic, i.e., $\Delta^\nabla R^\nabla = 0$. If M is compact, this translates into two equations, $d^\nabla R^\nabla = 0$ and $\delta^\nabla R^\nabla = 0$, which by the Bianchi identities reduces to $\delta^\nabla R^\nabla = 0$. This is nothing more than the Yang-Mills equation!! A *Yang-Mills connection* is a connection whose curvature is harmonic.

As we saw in the previous section, there is the action of the gauge group on the space of connections which takes a Yang-Mills connection to a Yang-Mills connection. We are now led to the study of gauge equivalence classes of Yang-Mills connections on $E = L \oplus \varepsilon$, i.e., the moduli space $\mathcal{M}$ of Yang-Mills connections on E.

Note that $\mathcal{M} \neq \emptyset$, since $E = L \oplus \varepsilon$ has Yang-Mills connections arising from the unique gauge equivalence class of Yang-Mills connections on L direct summed with the trivial connection on E. Such connections are called *reducible* connections. The number of gauge equivalence classes of reducible Yang-Mills connections is, then, just the number m of distinct (up to orientation) splitting of E as $L' \oplus \varepsilon$ for some $SO(2)$ bundle L'. This number, as we saw above, is half the number of solutions to the equations

(i) $a^2 = (e(L))^2$

(ii) $a_{(2)} = e(L)_{(2)}$,

for $a \in H^2(M; \mathbb{Z})$. (ii) says that $a = e(L) + 2b$ for some $b \in H^2(M; \mathbb{Z})$, so m is half the number of solutions to the equation

(iii) $(e(L) + 2b)^2 = (e(L))^2$

which is equivalent to the equation

(iii)′ $b \cdot (e(L) + b) = 0$.

Perhaps by studying M, the irreducible Yang-Mills connections will provide a cobordism between the reducible solutions which are completely determined by the intersection form on M.

4 Why Self-Dual Connections?

In order to complete the lower row of (2.1), we would like to relate harmonic forms with cohomology. Unfortunately, the sequence

$$(4.1)\quad \Omega^0(\mathfrak{g}_E) \xrightarrow{d^\nabla} \Omega^1(\mathfrak{g}_E) \to \Omega^2(\mathfrak{g}_E) \to \cdots$$

is not a complex, since $d^\nabla \ d^\nabla = R^\nabla$, which may not vanish. Don't despair!

Differential geometers have long been aware that dimension four has a property that distinguishes itself from other dimensions. The rotation group $SO(n)$ is a simple Lie group for all $n \neq 4$ and $SO(4)$ double covers $SO(3) \times SO(3)$, so that the Lie algebra $so(4)$ of $SO(4)$ is isomorphic to $so(3) \times so(3)$. Thus, since the six dimensional space $\Lambda^2(\mathbb{R}^4)$ of 2-forms on the inner product space $\mathbb{R}^4$ is isomorphic to $so(4)$, $\Lambda^2(\mathbb{R}^4)$ decomposes as the sum of 3-dimensional spaces $\Lambda^2_+ + \Lambda^2_-$. An alternate description of this decomposition is given in terms of the Hodge star operator $*: \Lambda^2(\mathbb{R}^4) \to \Lambda^2(\mathbb{R}^4)$. If $(e_1, \ldots, e_4)$ is an oriented basis for $\mathbb{R}^4$, then $*(e_i \vee e_j) = e_k \vee e_l$, where (i,j,k,l) is an even permutation of $(1,2,3,4)$. As $(*)^2 = 1$, $\Lambda^2(\mathbb{R}^4)$ decomposes as the ± 1 eigenspaces $\Lambda^2_\pm$ of $*$. Thus, if M admits a Riemannian metric, $\Lambda^2(T^*M) = \Lambda^2_+(M) \oplus \Lambda^2_-(M)$, and this decomposition is an invariant of the conformal class of the metric on M. An element of $\Lambda^2_+(M)(\Lambda^2_-(M))$ is called a *self dual* (*anti-self dual*) 2-form.

Since $\Omega^2(\mathfrak{g}_E) = \Gamma(\Lambda^2(T^*(M)) \otimes \mathfrak{g}_E)$, $*$ extends to an operator $*: \Omega^2(\mathfrak{g}_E) \to \Omega^2(\mathfrak{g}_E)$ given by $* \otimes \mathrm{id}$. Thus $\Omega^2(\mathfrak{g}_E) \cong \Omega^2_+(\mathfrak{g}_E) \oplus \Omega^2_-(\mathfrak{g}_E)$. But $R^\nabla \in \Omega^2(\mathfrak{g}_E)$, so $R^\nabla = R^\nabla_+ + R^\nabla_-$. This is a *very* special property of 4-dimensional geometry—the curvature decomposes into its self dual and anti-self dual components.

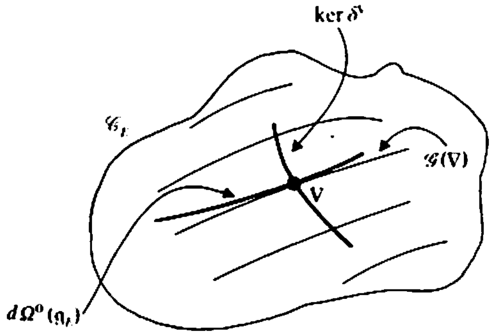
The adjoint $\delta^\nabla: \Omega^p(\mathfrak{g}_E) \to \Omega^{p-1}(\mathfrak{g}_E)$ can be given by $\delta^\nabla = (-1)^{p+1} * d^\nabla *$. If $R^\nabla_- \equiv 0$, then $\delta^\nabla R^\nabla = -*d^\nabla R^\nabla = 0$. Thus *self dual (anti-self dual) connections*, i.e., connections ∇ for which R^∇_- (R^∇_+) vanishes, are Yang-Mills connections.

We now obtain a complex from (4.1) as follows. Suppose ∇ is self dual connection. Then the sequence

$$(4.2)\quad \Omega^0(\mathfrak{g}_E) \xrightarrow{d^\nabla} \Omega^1(\mathfrak{g}_E) \xrightarrow{d^\nabla_-} \Omega^2_-(\mathfrak{g}_E) \to 0$$

with d^∇_- the orthogonal projection of d^∇ onto $\Omega^2_-(\mathfrak{g}^E)$, is a complex since $d^\nabla_- \cdot d^\nabla(\sigma) = [R^\nabla_-, \sigma] = 0$ for $\sigma \in \Omega^0(\mathfrak{g}_E)$. So, by considering self dual connections (which are Yang-Mills connections), we can extract from (4.1) a complex, hence consider its cohomology groups H^0_∇, H^1_∇ and H^2_∇.

The complex (4.1) and its cohomology groups contain much information: First, if ∇ and ∇' are Riemannian connections in E, $\nabla - \nabla' \in \Omega^1(\mathfrak{g}_E)$, so that, as an affine space, the space $\mathscr{C}_E$ of Riemannian connections on E is isomorphic to $\Omega^1(\mathfrak{g}_E)$. Furthermore, if $\nabla' = \nabla + A$ for some $A \in \Omega^1(\mathfrak{g}_E)$, $R^{\nabla'} = R^\nabla + d^\nabla A + [A, A]$, where $[A, A]_{X,Y} \equiv [A_X, A_Y]$. Second, the tangent space to the orbit of the gauge group $\mathscr{G} = \Gamma(G_E)$ at ∇, considered as a subspace of $\Omega^1(\mathfrak{g}_E) \cong T_\nabla C_E$, is the image $d^\nabla(\Omega^0(\mathfrak{g}_E))$. To see this, we can view $\Omega^0(\mathfrak{g}_E) = \Gamma(\mathfrak{g}_E)$ as the infinitesimal gauge transformations. So given an element $\sigma \in \Omega^0(\mathfrak{g}_E)$, consider the corresponding curve $g_t = \exp(t\sigma)$ in G_E and note that $(d/dt)\nabla^{g_t}|_{t=0} = [\nabla, \sigma] = \nabla(\sigma) = d^\nabla(\sigma)$. Thus $\ker \delta^\nabla$ can be thought of as the tangent space of $\mathscr{C}/\mathscr{G}$ at $[\nabla]$.



Third, if ∇ is self dual and $A \in \Omega^1(\mathfrak{g}_E)$, $\nabla + A$ is self dual if and only if $0 = R_-^{\nabla+A} = R_-^{\nabla} + d_-^{\nabla} A + [A, A]_- = d_-^{\nabla} A + [A, A]_-$. The linear part of this equation is $d_-^{\nabla} A = 0$. So, if we only consider linear information, a neighborhood of $[\nabla]$ in the moduli space $\mathscr{S}$ of gauge equivalence classes of self dual connections on M should be $\{A \in \Omega^1(\mathfrak{g}_E) | \delta^{\nabla} A = 0 \text{ and } d^{\nabla} A = 0\}$, that is, by Hodge theory, a neighborhood of 0 in H^1_{∇}.

What about the reducible Yang-Mills connections in $E = L \oplus \varepsilon$. It certainly is not the case that every harmonic 2-form is self dual. However, since the intersection pairing is positive (negative) definite on the self dual (anti-self dual) 2-forms, if the intersection form on M is positive definite, *every* harmonic 2-form is self dual. Thus under the assumption that the intersection form on M is positive definite (and $H^1(M:\mathbb{R}) = 0$, a fact we used in ε 2), we have that if m is half the number of solutions to

$$(e(L) + b) \cdot b = 0 \text{ for } b \in H^2(M:\mathbb{Z})$$

then contains m reducible connections.

It is from this point of view that in [*FS*1] we show, for instance, that $E_8 \oplus \Phi, \Phi$ any positive definite symmetric unimodular form, cannot occur as the intersection form on any closed smooth 4-manifold M with $H_1(M:\mathbb{Z})$ containing no 2-torsion. The outline of the proof is simple. Suppose such an M exists. We can assume $H_1(M:\mathbb{R}) = 0$, for surger out the free part of $H_1(M:\mathbb{Z})$ and note that the intersection form is unaffected. There exists an element $x \in H^2(M:\mathbb{Z})$ with $x^2 = 2$. Let L be the $SO(2)$ bundle over M with $e(L) = x$ and consider the $SO(3)$ bundle $E \cong L \oplus \varepsilon$. The work of K. Uhlenbeck ([U 1], [U 2]) can be used to show that $\mathscr{S}$ is *compact*. Then, using the work of Atiyah-Hitchin-Singer [AHS], we show that $\mathscr{S}$ is a manifold of dimension $2p_1(\mathfrak{g}_E) - 3 = 2p_1(E) - 3 = 2x^2 - 3 = 1$. $\mathscr{S}$ is then a disjoint union of circles and intervals whose

endpoints correspond to the reducible self dual connections. But the solutions to $b \cdot (x + b) = 0$ is just the order of the torsion subgroup of $H^2(M:\mathbb{Z})$ (for $b \cdot (x + b) = 0$ if and only if $(x + 2b)^2 = x^2$ and $(x + 2b)^2 = (x + b + b)^2 = (x + b)^2 + (b)^2$. But then $b = \pm x$ or b is torsion since x is minimal, i.e., x cannot be written as $c + d$ with $c^2 < x^2$ or $d^2 < x^2$). Thus, by the universal coefficient theorem $m = |\text{tor } H^2(M:\mathbb{Z})| = |\text{tor } H_1(M:\mathbb{Z})|$. If $H_1(M:\mathbb{Z})$ has no 2-torsion, m is then odd. But intervals have an even number of end points, a contradiction!

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