Topology Vol. 26, No. 3, pp. 385–393, 1987
Printed in Great Britain.

0040-9383/87 $03.00 + .00

RATIONAL HOMOLOGY COBORDISMS OF SPHERICAL SPACE FORMS

RONALD FINTUSHEL* and RONALD STERN†

(Received 18 November 1985)

§1. INTRODUCTION

THE RECENT work of Donaldson [5–7] and of the authors [8–10] has pointed out that the study of the differential geometric and analytic nature of a smooth 4-manifold in the guise of the study of Yang–Mills connections yields new and surprising results. This is surely no accident. In this paper we will begin to place under the umbrella of Yang–Mills theory many of the earlier results in 4-manifolds which utilize invariants arising from the G-signature theorem and we will generalize these results.

Our main result concerns rational homology cobordisms of spherical space forms. Let W^4 be a compact smooth 4-manifold with boundary components $\partial_1, \ldots, \partial_p$ which are spherical space forms and suppose that W^4 has the rational homology of a p-punctured sphere. To any character $\chi\colon H_1(W^4;\mathbb{Z})\to U(1)$ we associate three integers $\sigma(W^4,\chi)$, $\rho(W^4,\chi)$ and $\mu(W^4,\chi)$ as follows. Each boundary component ∂_j of W^4 is a quotient of S^3 by a finite group G_j acting orthogonally on S^3 and which extends to an orthogonal action on D^4 fixing the origin. For $g\in G_j$, let $r_j(g)$ and $s_j(g)$ denote the rotation angles of the action of g on D^4. Let $\chi_j=\chi j^*\colon H_1(\partial_j;\mathbb{Z})\to U(1)$ where j is the inclusion of ∂_j into W^4. The character χ determines a flat $SO(2)$ bundle L_χ over W^4 which when restricted to a boundary component ∂_j is the $SO(2)$ bundle

$$S^3\times S^1/G_j\to S^3/G_j$$

where G_j acts on S^3 as above and acts on S^1 via the representation $\chi_j\colon H_1(\partial_j;\mathbb{Z})\to U(1)$. For $g\in G_j$, let $t_j(g)$ denote the rotation angle of this action of g on S^1. Let $e_\chi\in H^2(W^4;\mathbb{Z})$ denote the Euler class of L_χ. Define

$$\sigma(W^4,\chi)=\sum_{j=1}^{p}(2/|G_j|)\sum_{id\neq g\in G_j}\cot[r_j(g)/2]\cot[s_j(g)/2]\sin^2[t_j(g)/2]$$

$$\rho(W^4,\chi)=\sum_{j=1}^{p}(2/|G_j|)\sum_{id\neq g\in G_j}\sin^2[t_j(g)/2]$$

$$\mu(W^4,\chi)=\#\,[\{e\in H^2(W^4;\mathbb{Z})|j^*(e)=\pm j^*(e_\chi)\in H^2(\partial_j;\mathbb{Z}),\ j=1,\ldots,p\}/\{e\sim -e\}].$$

THEOREM 1.1. *Let W^4 be a compact smooth 4-manifold with boundary components $\partial_1,\ldots,\partial_p$ which are spherical space forms, and suppose that W^4 has the rational homology of a p-punctured sphere. For any character $\chi\colon H_1(W^4;\mathbb{Z})\to U(1)$, let $\chi_j\colon H_1(\partial_j;\mathbb{Z})\to U(1)$ be its*

*Partially supported by National Science Foundation Grant DMS8501789.
†Partially supported by National Science Foundation Grant DMS8402214.

restriction to ∂_j. Also suppose that for some j, $H^2(W^4, \partial_j; \mathbb{Z})$ has no 2-torsion and the image of χ_j does not have order two. Then

(i) $\sigma(W^4; \chi) \equiv \rho(W^4, \chi) \bmod 2$, *and*
(ii) *if* $|\sigma(W^4, \chi)| \geqslant 4 - \rho(W^4, \chi)$, *then* $\mu(W^4, \chi) \equiv 0 \bmod 2$.

In §6 we will show that when $p=1$ and ∂_1 is a lens space, $\sigma(W^4, \chi)$ is the Casson–Gordon invariant [CG] for detecting when a 2-bridge knot is not ribbon. Theorem 1.1 shows that this invariant also detects when such a knot is not slice. This strengthening of the Casson–Gordon invariant in the case when $H_1(W^4; \mathbb{Z}_2)=0$ was also obtained by Burnette [3], using our work [9]. Also, in section 6 we will show that Theorem 2.1 implies two odd order lens spaces are homology cobordant if and only if they are diffeomorphic. This strengthens results of Gilmer and Livingston [11].

Here is an outline of the proof of Theorem 1.1 which is carried out in §2–5. Let X be the rational homology manifold obtained by coning off each boundary component of W^4. This is a V-manifold in the sense of [13, 14]. The character χ determines a flat $SO(3)$ vector bundle over W^4 which extends to a flat $SO(3)$ V-bundle E over X. As pointed out by Lawson [15], our work [9] can be recast in the framework of V-manifolds and V-vector bundles. (In fact, this is the way we first proved our main theorem in [9].) So in §2 we recall the relevant material from [9] and [15] to study the self-dual connections in E. In §3 we compute the formal dimension of the moduli space of self-dual connections in E, which turns out to be the odd integer $R(W^4, \chi) = -3 + \rho(W^4, \chi) + \sigma(W^4, \chi)$. We also compute the formal dimension of the moduli space of anti-self-dual V-connections in E to be the odd integer $S(W^4, \chi) = -3 + \rho(W^4, \chi) - \sigma(W^4, \chi)$. In particular we show that it is compact and discuss the reducible flat connections that correspond to connections with S^1 isotropy. We then show in §5 that if either $R(W^4, \chi)$ or $S(W^4, \chi)$ is positive, then $\mu(W^4, \chi) \equiv 0 \bmod 2$, which directly implies our theorem.

The proof of Theorem 1.1 is elementary in the sense that it does not require any of the hard analysis in [9] arising from the compactness results of Uhlenbeck [16, 17]. The moduli space of flat connections is compact for rather trivial reasons. The key observation is that a flat connection is both self-dual and anti-self-dual and the dimension of the appropriate moduli space of these connections can be computed, and compared, using the index theorem of Kawasaki [14].

§2. THE V-MANIFOLD SET-UP

Let W^4 be a compact smooth 4-manifold with boundary components $\partial_1, \ldots, \partial_p$ which are spherical space forms. Thus each ∂_j is the quotient of S^3 by the free orthogonal action of a finite group G_j. Let $X = W^4 \cup [\cup \text{cone}(\partial_j)]$. This rational homology manifold is a V-manifold in the sense of [13, 14]. The local uniformizing system consists of a manifold cover for W^4 union an open exterior collar on ∂X together with one copy of intD^4 for each boundary component ∂_j with G_j acting orthogonally, extending the action of G_j on S^3 radially. For $g \in G_j$, let $r_j(g)$ and $s_j(g)$ denote the rotation angles of the action of g on D^4.

A character $\chi: H_1(W^4; \mathbb{Z}) \to U(1)$ determines a flat $SO(2)$ vector bundle L' over W^4, i.e. L' is the vector bundle $(\tilde{W} \times \mathbb{C})/\pi_1(W)$ where $\tilde{W}$ is the universal cover of W with its $\pi_1(W)$ action and $\pi_1(W)$ acts on $\mathbb{C}$ via $\pi_1(W) \to H_1(W) \xrightarrow{\chi} U(1)$. The $SO(2)$ bundle $L'_j = L' | \partial_j$ is

$$S^3 \times \mathbb{R}^2/G_j \to S^3/G_j$$

where G_j acts on S^3 as above and acts on $\mathbb{R}^2$ via the representation $\chi_j = \chi j^*$: $H_1(\hat{c}_j; \mathbb{Z}) \to U(1)$, where j is the inclusion of $\hat{c}_j$ into W^4. For $g \in G_j$, let $t_j(g)$ denote the rotation angle of this action of g on $\mathbb{R}^2$.

We now construct an $SO(2)$ V-bundle L over X with fiber $\mathbb{R}^2$. This will be the $SO(2)$ vector bundle L' over $W^4 \cup$ collar, and, for each $j = 1, \ldots, p$, the trivial bundle $D^4 \times \mathbb{R}^2$ over D^4 on which the group G_j acts on D^4 with rotation angles $r_j(g)$ and $s_j(g)$ and with rotation angles $t_j(g)$ on $\mathbb{R}^2$. We use the covering $(D^4 \backslash (1/2)D^4) \times \mathbb{R}^2 \to L'|$collar to piece these together. Stabilize L to obtain an $SO(3)$ V-bundle E over X. We now discuss the relevant material from [9] and [15] concerning self-dual connections in the V-vector bundle E.

A general closed V-manifold X is the orbit space of a G-manifold M, M a closed manifold and G a compact Lie group acting on M with only finite isotropy subgroups and of trivial principal orbit type (see [14, p. 144]). (For example, take M to be the total space of the V-bundle of orthogonal tangent frames on X and take G to be the corresponding orthogonal group [14].) Also, a V-bundle over X is just a G-equivariant bundle F over M. Thus, the differential geometric and analytic nature of X and E is just the G-equivariant geometry and analysis of M and F. In particular, we assume that E and X have Riemannian metrics in the sense of V-manifolds, i.e. G-equivariant metrics on M and F. Let $\Omega^k(E) = \Gamma(\Lambda^k T^* X \otimes E)$ be the k-forms on X with values in E in the sense of V-manifolds, i.e. G-equivariant forms on M with values in F. Then a connection ∇ in E is a linear map $d^\nabla: \Omega^0(E) \to \Omega^1(E)$ which satisfies a Leibniz rule. Let $\mathscr{C}$ denote the space of all Riemannian connections, and let $\mathscr{G}$ denote the gauge group. Let $\mathscr{B} = \mathscr{C}/\mathscr{G}$. Let $\mathscr{A}$ denote the space of self-dual connections and form the moduli space $\mathscr{M} = \mathscr{A}/\mathscr{G}$. As usual, one works with Sobolev spaces, so that for a fixed connection $\nabla_0 \in \mathscr{C}$, we have $\mathscr{C}_k = \{\nabla_0 + A \mid A \in L_k^2(\Omega^1(\mathscr{G}_E))\}$ and for $\nabla_0 \in \mathscr{A}$, $\mathscr{A}_k = \{\nabla_0 + A \mid A \in L_k^2(\Omega^1(\mathscr{G}_E)),\ d_-^\nabla(A) + [A, A]_- = 0\}$. The gauge group of $\mathscr{C}_k$ is $\mathscr{G}_{k+1} = L_{k+1}^2(\Omega^0(\mathrm{Aut}_{SO(3)}(E)))$. One defines reducible and irreducible connections as usual and obtains $\pi: \mathscr{C}_3^* \to \mathscr{B}_3^*$ is a principal bundle and $\mathscr{B}_3^*$ is a smooth Hausdorff Hilbert manifold with local charts

$$\mathscr{O}_{\nabla, \varepsilon} = \{\nabla + A \mid A \in L_3^2(\Omega^1(\mathscr{G}_E)), \delta^\nabla A = 0, \quad \|A\|_{L_3^2} < \varepsilon\}$$

for ε sufficiently small, where δ^∇ is the adjoint of d^∇. Here * denotes the appropriate space of irreducible connections. If ∇ is reducible and non-trivial, then the stabilizer subgroup of $\mathscr{G}$ at ∇ is denoted Γ_∇ and $\pi: \mathscr{O}_{\nabla,\varepsilon}/\Gamma_\nabla \to \mathscr{B}$ is a homeomorphism onto a neighborhood of $\pi(\nabla)$.

For $\nabla \in \mathscr{A}$, there is the fundamental elliptic complex

$$0 \to \Omega^0(\mathscr{G}_E) \xrightarrow{d^\nabla} \Omega^1(\mathscr{G}_E) \xrightarrow{d^\nabla_-} \Omega^2(\mathscr{G}_E)_- \to 0$$

with cohomology groups H_∇^0, H_∇^1 and $H_{\nabla,-}^2$ which may be identified with spaces of harmonic forms. Denote the index of this complex by $I(\nabla)$. In the next section we will show that $I(\nabla) = R(W^4, \chi)$. Using the Kuranishi technique, given $\nabla \in \mathscr{A}$ there is a neighborhood $\mathscr{U}$ of $0 \in H_\nabla^1$, $\mathscr{U} \subset \mathscr{O}_{\nabla,\varepsilon}$, and a differentiable map $\Phi: \mathscr{U} \to H_{\nabla,-}^2$ with $\Phi(0) = 0$ which is Γ_∇ equivariant when ∇ is reducible, and such that $\mathscr{M} \cap \mathscr{O}_{\nabla,\varepsilon} \cong \Phi^{-1}(0)$ if ∇ is irreducible and $\mathscr{M} \cap (\mathscr{O}_{\nabla,\varepsilon}/\Gamma_\nabla) \cong \Phi^{-1}(0)/\Gamma_\nabla$ if ∇ is reducible.

Since our bundle E is flat, any $\nabla \in \mathscr{A}$ is also anti-self-dual. We can then perform the above analysis for anti-self-dual connections, replacing $-$ by $+$ where appropriate. In particular, we have for $\nabla \in \mathscr{A}$ the elliptic complex

$$0 \to \Omega^0(\mathscr{G}_E) \to \Omega^1(\mathscr{G}_E) \to \Omega^2(\mathscr{G}_E)_+ \to 0$$

whose index we denote by $J(\nabla)$. In the next section we will show that $J(\nabla) = S(W^4, \chi)$.

§3. INDEX COMPUTATIONS

Let the V-manifold X and the $SO(3)$ V-bundle E be as in §2. Let $\nabla \in \mathscr{A}$ be the flat connection in L given by the character χ. We now compute the index $I(\nabla)$ of the elliptic complex

$$0 \to \Omega^0(\mathscr{G}_E) \to \Omega^1(\mathscr{G}_E) \to \Omega^2(\mathscr{G}_E)_- \to 0.$$

This computation will be performed by using a version of the Atiyah–Singer index theorem for V-manifolds given in [14] and is formally the same as the computation in [9] (and [15]).

The index of the operator $d_-^\nabla \oplus \delta^\nabla : \Omega^1(\mathscr{G}_E) \to \Omega^0(\mathscr{G}_E) \oplus \Omega^2(\mathscr{G}_E)_-$, where δ^∇ is the adjoint of d^∇, equals $I(\nabla)$ and is the same as that of the Dirac operator D: $\Gamma(V_+ \otimes V_- \otimes \mathscr{G}_E) \to \Gamma(V_- \otimes V_- \otimes \mathscr{G}_E)$, where V_- and V_+ are the complex spinor V-bundles of $\pm 1/2$-spinors on X (see [15, 9]). By the index theorem of [14], the index of D is

$$\mathrm{ch}(\mathscr{G}_E \otimes \mathbb{C})\,\mathrm{ch}(V_-)\,\hat{A}(X)[X]$$
$$+ \sum_{j=1}^{p} |G_j|^{-1} \sum_{id \neq g \in G_j} [\{\mathrm{ch}_g(V_+ - V_-)\,\mathrm{ch}_g(V_-)\,N^g \otimes \mathbb{C})\,\mathrm{ch}_g(\mathscr{G}_E)\}/\mathrm{ch}_g(\Lambda_{-1})]\ [\text{cone point}]$$

as in [15] or [9].

The first term in the index of D is

$$3/2[\sigma(X) - \chi(X)] - \sum_{j=1}^{p} [3/(2|G_j|)] \sum_{id \neq g \in G_j} [\sigma_g(X) - \chi_g(X)]$$

where for $g \in G_j$, $\sigma_g(X) = -\cot[r_j(g)/2]\cot[s_j(g)/2]$ and $\chi_g(X) = 1$.

The second term in the index of D is

$$-\sum_{j=1}^{p} (2|G_j|)^{-1} \sum_{id \neq g \in G_j} \{1 + \cot[r_j(g)/2]\cot[s_j(g)/2]\}\{3 - 4\sin^2[t_j(g)/2]\}$$

as in [9]. Adding these together we get

PROPOSITION 3.1. $I(\nabla) = \dim H^1_\nabla - \dim H^0_\nabla - \dim H^2_{\nabla,-} = R(W^4, \chi)$.

As our flat connection ∇ is also anti-self-dual, we can compute the index $J(\nabla)$ of the elliptic complex

$$0 \to \Omega^0(\mathscr{G}_E) \to \Omega^1(\mathscr{G}_E) \to \Omega^2(\mathscr{G}_E)_+ \to 0.$$

As above the index of the operator $d_+^\nabla \oplus \delta^\nabla : \Omega^1(\mathscr{G}_E) \to \Omega^0(\mathscr{G}_E) \oplus \Omega^2(\mathscr{G}_E)_+$ equals $J(\nabla_0)$ and is the same as that of the Dirac operator D': $\Gamma(V_- \otimes V_+ \otimes \mathscr{G}_E) \to \Gamma(V_+ \otimes V_+ \otimes \mathscr{G}_E)$. Again, by the index theorem of [14], the index of D' is

$$\mathrm{ch}(\mathscr{G}_E \otimes \mathbb{C})\,\mathrm{ch}(V_+)\,\hat{A}(X)[X]$$
$$+ \sum_{j=1}^{p} |G_j|^{-1} \sum_{id \neq g \in G_j} [\{\mathrm{ch}_g(V_- - V_+)\,\mathrm{ch}_g(V_+)\,(N^g \otimes \mathbb{C})\,\mathrm{ch}_g(\mathscr{G}_E)\}/\mathrm{ch}_g(\Lambda_{-1})]\ [\text{cone point}].$$

The first term in the index of D' is

$$-3/2[\sigma(X) + \chi(X)] + \sum_{j=1}^{p} [3/(2|G_j|)] \sum_{id \neq g \in G_j} [\sigma_g(X) + \chi_g(X)].$$

The second term in the index of D is

$$\sum_{j=1}^{p} (2|G_j|)^{-1} \sum_{id \neq g \in G_j} \{-1 + \cot[r_j(g)/2]\cot[s_j(g)/2]\}\{3 - 4\sin^2[t_j(g)/2]\}$$

as in [10]. Adding these together we get

PROPOSITION 3.2. $J(\nabla)=\dim H^1_\nabla-\dim H^0_\nabla-\dim H^2_{\nabla,+}=S(W^4,\chi)$.

Remark 3.3. The flat vector bundle L_χ also defines a local coefficient system and there are cohomology groups $H^*(W^4, L_\chi)$ and $H^*(W^4, \partial W^4; L_\chi)$. These have a natural pairing into $\mathbb{C}$ given by the cup-product, the inner product on L_χ and evaluation of the top cycle of W^4 mod ∂W^4. This induces a non-degenerate form on $H^*(W^4; L_\chi)$, the image of the relative cohomology in the absolute cohomology (all coefficients in L_χ). On $H^2(W^4; L_\chi)$ this form is Hermitian. The signature of Hermitian form is $\sigma(W^4;\chi)$ (see [2]).

§4. THE MODULI SPACE OF FLAT CONNECTIONS

As a general closed V-manifold X is the orbit space of a G-manifold M, M a closed manifold and G a compact Lie group acting on M with only finite isotropy subgroups and of trivial principal orbit type, and a V-bundle over X is just a G-bundle over M, many theorems for manifolds carry over to V-manifolds. For our purposes we need the following lemmas.

LEMMA 4.1. *Let X be a 4-dimensional closed V-manifold and E a flat $SO(3)$ bundle over X. A connection ∇ in E is flat if and only if it is self-dual if and only if it is anti-self-dual.*

Proof. If X were a closed manifold and E a vector bundle, the Chern–Weil theory shows that

$$0=p_1(E)=(1/4\pi^2)\int_X \|R^\nabla_+\|^2-\|R^\nabla_-\|^2$$

from which the lemma follows. As pointed out in [14, pp. 144–147], such a formula also holds for V-manifolds. ∎

It is well known that the space of gauge equivalence classes of flat $SO(3)$ connections on a bundle F (of fixed topological type) over a compact manifold M is canonically identified with an open and closed subset of the set of conjugacy classes of representations $\pi_1(M)$ in $SO(3)$ with the compact open topology [12, p. 210]. Since $SO(3)$ and G are compact and the condition that a connection be G invariant is a closed condition, we have

LEMMA 4.2. *The moduli space of flat connections in E is compact.*

Combining 4.1 and 4.2 we have

LEMMA 4.3. *Let X be a 4-dimensional closed V-manifold and E a flat $SO(3)$ V-bundle over X. Then the moduli space $\mathcal{M}$ of self-dual connections and the moduli space $\mathcal{N}$ of anti-self-dual connections in E are compact (and equal).*

Recall that our vector V-bundle E is the Whitney sum of the vector V-bundle L_χ and the trivial $\mathbb{R}^1$-bundle ε over X and that $e_\chi\in H^2(W^4;\mathbb{Z})$ is the Euler class of $L_\chi|W^4$. We call $L_\chi\oplus\varepsilon$ a *reduction* of E. Given another flat $\mathbb{R}^2$ V-bundle over X, let $e_L\in H^2(W^4;\mathbb{Z})$ denote the Euler class of $L|W^4$.

PROPOSITION 4.4. *$L_\chi\oplus\varepsilon$ is equivalent (as a V-bundle) to $L\oplus\varepsilon$ if and only if*

(i) $e_\chi\equiv e_L$ mod 2 *and*
(ii) $j^*(e_L)=\pm j^*(e_\chi)\in H^2(\partial_j;\mathbb{Z})$ *for each* $j=1,\ldots,p$.

Proof. The proof is the same as that of Proposition 2.5 of [10]. Suppose that $L_\chi \oplus \varepsilon$ and $L \oplus \varepsilon$ are equivalent. Since $L_\chi \oplus \varepsilon$ and $L \oplus \varepsilon$ are isomorphic over W^4, (i) holds. Over each cone(∂_j) we may identify $L \oplus \varepsilon|\text{cone}(\partial_j)$ with $D^4 \times \mathbb{R}^2 \times \mathbb{R}$ where $g \in G_j$ acts on $D^4 \times \mathbb{R}^2$ via the action with rotation angles $r_j(g)$ and $s_j(g)$ on D^4 and with rotation angle $t_j'(g)$ on $\mathbb{R}^2$. Our given equivalence of $L_\chi \oplus \varepsilon$ with $L \oplus \varepsilon$ gives an equivalence of representations of G_j on $0 \times \mathbb{R}^3$; hence $t_j(g) = \pm t_j'(g)$ and (ii) holds.

Conversely, assume (i) and (ii) hold. Since (ii) holds, $L_\chi \oplus \varepsilon$ and $L \oplus \varepsilon$ are isomorphic over the closure of $X - W^4$. Over ∂W^4 we thus obtain an equivalence of $L_\chi \oplus \varepsilon$ with $L \oplus \varepsilon$ and we need to know that this extends to an equivalence over all of X. There are two obstructions to doing this, and they lie in $H^2(W^4, \partial W^4; \mathbb{Z}_2)$ and $H^4(W^4, \partial W^4; \mathbb{Z})$. Since (i) holds, the first obstruction vanishes; and since the bundles are flat, the second obstruction vanishes. ■

For the rest of the section recall that we are assuming that W^4 has the rational homology of a punctured 4-sphere.

PROPOSITION 4.5. *Suppose that some $H^2(W^4, \partial_j; \mathbb{Z})$ has no 2-torsion. Then, up to orientation, the number of reductions of E is $\mu(W^4, \chi)$.*

Proof. We have the exact sequence

$$0 \to H^2(W^4, \partial_j) \to H^2(W^4) \xrightarrow{j^*} H^2(\partial_j) \to H^3(W^4, \partial_j) \to H^3(W^4).$$

As $H^2(W^4, \partial_j; \mathbb{Z})$ has no 2-torsion, it is odd torsion so every element in the kernel of j^* is twice another element. Also $e \in H^2(W^4)$ and $-e$ determine the same reduction up to orientation. The result now follows from Proposition 4.4. ■

PROPOSITION 4.6. *Suppose that for some j, $H^2(W^4, \partial_j; \mathbb{Z})$ has no 2-torsion and the image of χ_j is not of order two. Then E is not isomorphic as a V-bundle to $L \oplus \eta$ where L is an $O(2)$ V-bundle and η is a non-orientable $O(1)$ V-bundle over X.*

Proof. Consider the exact sequence

$$0 \to H^1(W^4, \partial_j; \mathbb{Z}_2) \to H^1(W^4, \mathbb{Z}_2) \xrightarrow{j^*} H^1(\partial_j; \mathbb{Z}_2) \to H^2(W^4, \partial_j; \mathbb{Z}_2) \to .$$

Since $H^2(W^4, \partial_j; \mathbb{Z})$ is odd torsion, it follows from the Universal Coefficient Theorem that $H^1(W^4, \partial_j; \mathbb{Z}_2) = 0$, so j^* is a monomorphism. Thus a non-orientable $O(2)$-bundle over W^4 restricts to a non-orientable bundle over ∂_j. Hence L and η are still non-orientable when restricted to ∂_j. Since G_j has finite order, the representation into $O(2)$ which induces L is dihedral, hence the image in $SO(3)$ of the representation inducing $L \oplus \eta$ cannot be cyclic unless its image has order 2. But if E is equivalent to $L \oplus \eta$, then their restrictions to the cone on ∂_j must be induced by conjugate representations, up to orientation, of G_j into $SO(3)$, and the image of χ_j is cyclic since it lies in $SO(2)$. Hence E cannot be equivalent to $L \oplus \eta$. ■

§5. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 is now exactly the proof of our Theorem 2.1 in [9]. The character $\chi: H_1(W^4; \mathbb{Z}) \to U(1)$ determines the $SO(3)$ V-bundle $E = L_\chi \oplus \varepsilon$ over the V-manifold $X = W^4 \cup (\cup \text{cone}(\partial_j))$. Let $\mathscr{B}$ be the moduli space of all connections in E over X and let $\mathscr{M}$

denote the moduli space of self-dual connections. Also let $\mathcal{N}$ denote the moduli space of anti-self-dual connections in E. By Lemma 4.3 $\mathcal{M}=\mathcal{N}$ is compact.

For a flat connection ∇, section 2 points out that there is a neighborhood $\mathscr{U}$ of $0 \in H^1_\nabla$ and a differentiable map $\Phi: \mathscr{U} \to H^2_{\nabla,-}$ with $\Phi(0)=0$ which is Γ_∇ equivariant when ∇ is reducible, and such that $\mathcal{M} \cap \mathcal{O}_{\nabla,\varepsilon} \cong \Phi^{-1}(0)$ if ∇ is irreducible and $\mathcal{M} \cap (\mathcal{O}_{\nabla,\varepsilon}/\Gamma_\nabla) \cong \Phi^{-1}(0)/\Gamma_\nabla$ if ∇ is reducible. By Proposition 4.6 our hypothesis guarantees that each reducible connection has $\Gamma_\nabla \cong S^1$. Thus,

$$\dim H^1_\nabla - \dim H^2_{\nabla,-} = R(W^4,\chi) \qquad \text{if } \nabla \text{ is irreducible,}$$

$$\dim H^1_\nabla - \dim H^2_\nabla = R(W^4,\chi)+1 \qquad \text{if } \nabla \text{ is reducible.}$$

If $R(W^4,\chi)>0$ and if $H^2_{\nabla,-}=0$ for each flat ∇, then $\mathcal{M}$ would be a compact smooth manifold of dimension $R(W^4,\chi)$ with, by Propositions 4.5 and 4.6, $\mu(W^4,\chi)$ singular points such that each has a neighborhood which is the cone on a complex projective space. If $(1/2)\,[R(W^4,\chi)-1]$ is even, this implies that $\mu(W^4,\chi)$ is even since an odd number of $\mathbb{CP}(2k)$s cannot bound a smooth manifold. The argument of [9, §9] shows that $\mu(W^4,\chi)$ must be even in any case. Similarly, if $S(W^4,\chi)>0$, then $\mu(W^4,\chi)$ must be even.

Now recall that $R(W^4,\chi)=-3+\rho(W^4,\chi)+\sigma(W^4,\chi)$ and that

$$S(W^4,\chi)=-3+\rho(W^4,\chi)-\sigma(W^4,\chi)$$

are both odd integers since there is at least one reducible flat connection (see [9, Cor. 6.3]). Thus (i) follows. Also $R(W^4,\chi)>0$ or $S(W^4,\chi)>0$ if and only if $|\sigma(W^4;\chi)| \geqslant 4-\rho(W^4,\chi)$. Thus (ii) holds true. ∎

§6. APPLICATIONS

In this section we will give the applications of Theorem 1.1 mentioned in the Introduction.

Lens spaces that bound rational homology balls

Let $L(m,q)$ be a three-dimensional lens space. If $L(m,q)$ bounds a rational homology 4-ball W^4, then $m=k^2$, where k is the order of the image of $H_1(L(m,q))$ in $H_1(W^4)$ (see Lemma 3 of [4]).

THEOREM 6.1. *Suppose that $L(k^2,q)$ bounds a rational homology 4-ball W^4 such that $H^2(W^4,\hat{c})$ has no 2-torsion. Then for any $1 \leqslant r \leqslant k-1$*

$$(2/k^2)\sum_{s=1}^{k^2-1} \cot(\pi s/k^2)\cot(\pi qs/k^2)\sin^2(\pi rs/k)=\pm 1.$$

Proof. For $k=2$ this formula can be checked by hand. Now suppose $k \neq 2$. Recall that $L(k^2,q)=S^3/\mathbb{Z}_{k^2}$, where $\mathbb{Z}_{k^2}$ acts on S^3 by the formula $\zeta(z,w)=(\zeta z,\zeta^q w)$. Let j denote the inclusion of $L(k^2,q)$ into W^4; the image of the induced j_* is well known to be $\mathbb{Z}_k$ in $H_1(W^4)$. Thus j_* induces a character $\chi\colon H_1(L(k^2,q)) \to \mathbb{Z}_k \subset U(1)$. By [4, p. 12] there is a character $\chi'\colon H_1(W^4) \to U(1)$ with image the cyclic group of order k^a, for some a, with $\chi=\chi' j_*$. The character χ' determines a flat $SO(2)$ bundle over W^4 which when restricted to $L(k^2,q)$ is the $SO(2)$ bundle $(S^3 \times S^1)/\mathbb{Z}_{k^2}$. Here $g \in \mathbb{Z}_{k^2}$ acts on S^3 with rotation angles $2\pi s/k^2$ and $2\pi sq/k^2$, for some s, and q, both prime to k, and acts on S^1 with rotation angle $2\pi rs/k$, for some r. By appropriately choosing the representation of $\mathbb{Z}_k$ into $U(1)$, we can realize any $0 \leqslant r \leqslant k-1$. We now apply Theorem 1.1 to the character $r\chi'$.

$$\rho(W^4, r\chi') = (2/k^2)\sum_{s=1}^{k^2-1}\sin^2(\pi rs/k) = (1/k^2)\sum_{s=1}^{k^2-1}[1-\cos(2\pi rs/k)]$$

$$= (1/k^2)\left[(k^2-1) - \sum_{s=1}^{k^2-1}\cos(2\pi rs/k)\right] = (1/k^2)[(k^2-1)+1] = 1.$$

Now let $e = e_{r\chi'}$ and note that in $H^2(L(k^2, q))$, j^*e has order $k > 2$. Recall that $\mu(W, r\chi') = \#\{e' \in H^2(W) \mid j^*e' = \pm j^*e\}/\{-e' \sim e'\}$. So if $0 \neq f \in H^2(W, \partial) = \ker j^*$, then the elements $\{\pm e \pm f\}$ account for 2 in the count of $\mu(W, r\chi')$. (Note $e \neq -e$.) Also if $\pm f \neq \pm g \in \ker j^*$, then $\pm e \pm f \neq \pm e \pm g$ (since if $e + f = -e + g$ then $2e \in \ker j^*$, but j^*e has order greater than 2). Thus $\mu(W, r\chi') = 2(|\ker j^*| - 1) + 1$ is odd. Thus Theorem 1.1(ii) implies that

$$|\sigma(W^4, r\chi)| = \left|(2/k^2)\sum_{s=1}^{k^2-1}\cot(\pi s/k^2)\cot(\pi qs/k^2)\sin^2(\pi rs/k)\right| \leqslant 2.$$

But, since by Theorem 1.1(i) $\sigma(W^4, r\chi)$ is odd, the result follows. ■

Remark 6.2. Theorem 6.1 under the additional hypothesis that k be a prime power was first obtained by Casson and Gordon (Theorem 2 of [4]).

Homology cobordisms of lens spaces

Two 3-manifolds M and N are *homology cobordant* provided there is a smooth 4-manifold W with boundary components M and N and with the inclusion of either boundary component inducing an isomorphism on integral homology.

THEOREM 6.3. *Two three-dimensional lens spaces of odd order are homology cobordant if and only if they are diffeomorphic.*

Proof. Under the additional hypothesis that the orders of the lens spaces be a prime power, this was first proved by Gilmer and Livingston [11]. Let us first recall some aspects of their proof.

Let W^4 be a 4-manifold with boundary components $L(m, a)$ and $L(m, b)$ and with the inclusion of each boundary component inducing an isomorphism on integral homology. Using the fact that $H_2(W) = 0$, it is easy to see that the $Q/\mathbb{Z}$ linking form on ∂W vanishes when restricted to the kernel K of i_*: $H_1(\partial W) \to H_1(W)$. From this one sees that $K = \mathbb{Z}_m \subset H_1(L(m, a)) \oplus H_1(L(m, b))$ is generated by an element $(c, 1)$ with $b = c^2 a \pmod m$.

For any $L(m, n)$ m odd and divisor d of m, let χ_d: $H^1(L(m, n)) \to \mathbb{Z}_d$ be the character with $\chi_d(g) = 1$, where g is the generator of H_1 $(L(m, n))$ with self-linking $n/m \in Q/\mathbb{Z}$. Then for any divisor d of m and any $0 < r < d$ the character $r(\chi_d + (-c)\chi_d)$ vanishes on K and so extends to a character χ: $H_1(W) \to \mathbb{Z}_d$.

Now apply Theorem 1.1. Note that $\rho(W^4, \chi) = 2$. Since $H^2(W^4, \partial) = 0$, the proof of Theorem 6.1 shows that $\mu(W^4, \chi) = 1$. Thus $\sigma(W^4, \chi) = 0$. So for every divisor d of m and every $0 < r < d$, we have

$$\sum_{s=1}^{m-1}\cot(\pi s/m)\cot(\pi as/m)\sin^2(\pi rs/d) = \sum_{s=1}^{m-1}\cot(\pi s/m)\cot(\pi bs/m)\sin^2(\pi rcs/d).$$

As pointed out in Theorem 3.4 of [11] this implies that the corresponding α-invariants of the lens spaces $L(m, a)$ and $L(m, b)$ are equal so that the α-invariant classification of Atiyah and Bott [1] shows that these lens spaces are diffeomorphic. ■

REFERENCES

1. M. F. ATIYAH and R. BOTT: A Lefschetz fixed point formula for elliptic complexes, II. Applications. *Ann. Math.* **88** (1968), 451–491.
2. M. F. ATIYAH, V. K. PATODI and I. M. SINGER: Spectral asymmetry and Riemannian geometry. II, *Math. Proc. Camb. Phil. Soc.* **78** (1975), 405–432.
3. K. BURNETTE: Thesis. University of Utah (1986).
4. A. CASSON and C. McA. GORDON: Cobordisms of classical knots. Mimeographed Notes, Orsay (1975).
5. S. DONALDSON: An application of gauge theory to four dimensional topology. *J. Differential Geom.* **18** (1983), 279–315.
6. S. DONALDSON: Irrationality and the *h*-cobordism conjecture. Preprint.
7. S. DONALDSON: Connections, cohomology and the intersection forms of 4-manifolds. *J. Diff. Geom.* **24** (1986), 275–341.
8. R. FINTUSHEL and R. STERN: $SO(3)$-connections and the topology of 4-manifolds. *J. Diff. Geom.* **20** (1984), 523–539.
9. R. FINTUSHEL and R. STERN: Pseudofree orbifolds. *Ann. Math.* **122** (1985), 335–364.
10. R. FINTUSHEL and R. STERN: $O(2)$-actions on the 5-sphere. *Invent. Math.* **87** (1987), 457–476.
11. P. GILMER and C. LIVINGSTON: On embedding 3-manifolds in 4-space. *Topology* **22** (1983), 241–252.
12. W. GOLDMAN: The symplectic nature of the fundamental group of surfaces. *Adv. Math.* **54** (1984), 200–225.
13. T. KAWASAKI: The signature theorem for V-manifolds. *Topology* **17** (1978), 75–83.
14. T. KAWASAKI: The index of elliptic operators over V-manifolds. *Nagoya Math. J.* **84** (1981), 135–157.
15. T. LAWSON: Normal bundles for an embedded RP^2 in a positive definite 4-manifold. *J. Differential Geom.* **22** (1985), 215–231.
16. K. UHLENBECK: Connections with L^p bounds on curvature. *Commun. Math. Phys.* **83** (1982), 31–42.
17. K. UHLENBECK: Removable singularities in Yang–Mills fields. *Commun. Math. Phys.* **83** (1982), 11–30.

Tulane University
New Orleans, LA 70118, *U.S.A.*

University of Utah
Salt Lake City, UT 84112, *U.S.A.*