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DIFFERENTIAL GEOMETRIC INVARIANTS FOR HOMOLOGY THREE SPHERES

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LECTURE 1.

1. Introduction. In these three lectures we will study differential geometric properties of smooth 3-manifolds that have the homology of S^3, i.e., homology 3-spheres. Poincaré originally conjectured that the only homology 3-sphere is S^3; however he quickly realized that homology 3-spheres exist in abundance. We list some examples.

1. Since the binary icosohedral group I is a subgroup of $SU(2)$, it acts on S^3 with quotient the famous Poincaré homology 3-sphere P^3 with $\pi_1(P^3) = I$ a finite group of order 120. In fact, it is known that the only finite non-trivial group that can occur as the fundamental group of a homology 3-sphere is I , and it is still unknown if P^3 is the only homology 3-sphere Σ with fundamental group I.

2. The Brieskorn homology 3-sphere $\Sigma(p,q,r) = \{z_1^p + z_2^q + z_3^r = 0\} \cap S^5$ is a homology 3-sphere whenever p, q, r are pairwise relatively prime [B]. In fact $\Sigma(2,3,5) = P^3$, and if $1/p + 1/q + 1/r < 1$ then $\pi_1(\Sigma(p,q,r))$ is infinite.

3. A rather ubiquitous collection of homology 3-spheres are the Seifert fibered homology 3-spheres $\Sigma(a_1, \ldots, a_n)$ (see [**NR**]). These 3-manifolds Σ possess an S^1-action with orbit space S^2. If $\Sigma \neq S^3$,

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then the S^1-action has no fixed points and has finitely many exceptional orbits (multiple fibers) of order $a_1, \ldots, a_n$. Σ is a homology sphere exactly when the orders of the exceptional fibers are pairwise relatively prime. If $n \geq 3$, then $\Sigma \neq S^3$ and the orders classify $\Sigma = \Sigma(a_1, \ldots, a_n)$ up to diffeomorphism. If $n \leq 2$, then $\Sigma = S^3$. As our notation predicts, the Brieskorn sphere $\Sigma(p,q,r)$ is Seifert fibered with 3 exceptional orbits of orders p, q, and r. Also one can show that

$$\begin{aligned}&\pi_1(\Sigma(a_1,\ldots,a_n)) = < x_1,...,x_n, h | h \text{ central}, x_i^{a_i}\\ =&h^{-b_i}, i = 1,...,n, x_1 \cdots x_n = h^{b_0} > .\end{aligned}$$

Here $b_0, b_1, ..., b_n$ are chosen so that

$$a(-b_0 + \sum_{i=1}^{n} \frac{b_i}{a_i}) = 1 \tag{1.1}$$

where $a = a_1 \cdots a_n$. We shall say that Σ has Seifert invariants $\{b_0; (a_1, b_1), \ldots, (a_n, b_n)\}$. (These are, of course, not unique.)

4. Given a knot K in S^3 one can perform a $1/n$, $n \in \mathbf{Z}$, Dehn surgery on K to obtain a homology 3-sphere. The homology 3-spheres $\Sigma(p, q, pqn \pm 1)$ are obtained by $\pm 1/n$ surgery on the (p,q) torus knot; the homology 3-spheres $\Sigma(p,q,r,s)$ with $qr - ps = \pm 1$ are obtained by a ± 1 surgery on the connected sum of the (q,r) and $(-p,s)$ torus knots. It is not known if every irreducible homology 3-sphere can be obtained by a Dehn surgery on some knot in S^3. However, every homology 3-sphere can be obtained as an integral surgery on a link in S^3. Recently, Gordon and Luecke [GL] have shown that nontrivial Dehn surgery on a nontrivial knot never yields S^3. Furthermore, any homology 3-sphere obtained by Dehn surgery on a knot is irreducible.

5. Given a knot K in S^3 one can take the n-fold cyclic cover of S^3 branched over K, denoted K_n. This is a homology 3-sphere when $|\prod_{i=1}^{n} \Delta(\omega^i)| = 1$, where $\Delta(t)$ is the Alexander polynomial of K

normalized so that there are no negative powers of t and has non-zero constant coefficients and $\omega = e^{\frac{2\pi i}{n}}$. The homology 3-sphere $\Sigma(p,q,r)$ is the r-fold cover of the (p,q) torus knot and every $\Sigma(a_1,\ldots,a_n)$ is the 2-fold cover of S^3 branched over a rational knot (see [**BZ**]). Not every homology 3-sphere is a cyclic branched cover [**My**]. However, every 3-manifold is an irregular branched cover of the figure eight knot [**HLJ**].

The target of our study in these lectures will be the oriented homology cobordism properties of homology 3-spheres, where oriented homology 3-spheres Σ_1 and Σ_2 are *oriented homology cobordant* provided there is an oriented 4-manifold W with $\partial W = \Sigma_1 \coprod -\Sigma_2$. Equvalently, Σ_1 and Σ_2 are oriented homology cobordant provided $\Sigma_1 \sharp \Sigma_2$ bounds an acyclic 4-manifold. This leads to the study of the abelian group Θ_3^H which is the set of oriented homology three spheres modulo the equivalence relation of oriented homology cobordism. The group operation is connected sum $\sharp$. This is an abelian group with the additive inverse of Σ being $-\Sigma$. Until recently the only known fact concerning the group Θ_3^H is the Kervaire-Milnor-Rochlin homomorphism $\mu : \Theta_3^H \to \mathbf{Z}_2$. A homology 3-sphere Σ bounds a smooth simply-connected 4-manifold W^4 with trivial tangent bundle (i.e. W^4 is spin). The signature $\sigma(W^4)$ of W^4 is known to be divisible by 8, so let $\mu(\Sigma) = \sigma(W^4)/8 \pmod 2$. To show that $\mu(\Sigma)$ is independent of the choice of W^4 utilizes Rochlin's theorem which states that the signature of a closed spin (almost parallelizable) 4-manifold is divisible by 16. For if W' is another such 4-manifold, $\sigma(W \cup -W') = \sigma(W) - \sigma(W') \equiv 0 \pmod{16}$, so that $\sigma(W)/8 \equiv \sigma(W')/8 \pmod 2$. Similarily, $\mu(\Sigma)$ is an homology cobordism invariant.

A reasonable question to ask at this point is "Why study homology 3-spheres or Θ_3^H?" First, understanding homology 3-spheres and the 4-manifolds they bound is useful in constructing interesting 4-manifolds. A given homology 3-sphere Σ may bound two 4-manifolds W_1 and W_2

with intersection forms I_1 and I_2 respectively so that $M^4 = W_1 \cup_\Sigma -W_2$ may be a desirable closed 4-manifold with intersection form $I_1 \oplus I_2$. Conversely, if the intersection form of a closed 4-manifold M^4 decomposes as $I_1 \oplus I_2$, then there is a homology 3-sphere Σ in M^4 splitting M^4 into two 4-manifolds W_1 and W_2 with intersection forms I_1 and I_2 respectively [**FT**]. This has been useful in constructing exotic 4-manifolds and group actions. For example in [**FS1**] it is shown that $\Sigma(3,5,19)$ bounds a contractible manifold W^4 and that the double of $W^4 \cup_t W^4$ along the free involution $t : \Sigma \to \Sigma$ contained in the S^1-action on Σ is S^4 with a free involution τ (obtained by interchanging the copies of W^4) that is not in any sense smoothly equivalent to the antipodal map. Thus S^4/τ is a smooth homotopy RP^4 that is not s-cobordant to RP^4. Other constructions are given in [**FS2**].

Secondly, the structure of group Θ_3^H is closely related to the question of whether a topological n-manifold M^n , $n \geq 5$, is a polyhedron. In [**GS**] and [**Mat**] it is shown that M^n is a polyhedron iff an obstruction $\tau_M \in H^5(M^n; \ker(\mu : \Theta_3^H \to \mathbf{Z}_2))$ vanishes and if $\tau_M = 0$ there are $|H^5(M^n; \ker\mu)|$ triangulations up to concordance. Furthermore, $\tau_M = 0$ for all M iff there is a homology 3-sphere Σ with $\mu(\Sigma) = 1$ and such that $\Sigma \sharp \Sigma$ bounds a smooth acyclic 4-manifold. A reasonable conjecture at that time was that $\Theta_3^H = \mathbf{Z}_2$, so that $\ker\mu = 0$. To date the existence of a homology sphere with the above properties is unknown. However, at the end of this lecture we will utilize techniques from gauge theory to show that the group Θ_3^H is infinitely generated!

There is not much algebraic topology associated with homology 3-spheres. There is no interesting homology, and most homology 3-spheres Σ are $K(\pi_1(\Sigma), 1)'s$. Thus, we only have $\pi_1(\Sigma)$, a (super) perfect group. For an oriented integral homology 3-sphere Σ, A. Casson has introduced an integer invariant $\lambda(\Sigma)$ that is determined by the space $\mathcal{R}(\Sigma)$ of conjugacy classes of irreducible representations of $\pi_1(\Sigma)$ into $SO(3)$ (see [**AM**]). This invariant $\lambda(\Sigma)$ can be computed from a surgery or Heegard description of Σ and satisfies $\lambda(\Sigma) \equiv \mu(\Sigma)$

(mod 2). This powerful new invariant was used to settle an outstanding problem in 3-manifold topology; namely, showing that if Σ is a homotopy 3-sphere, then $\mu(\Sigma) = 0$. This successful approach to the study of homology 3-spheres certainly provides motivation for investigating differential geometric properties of homology 3-spheres. The connection with differential geometry is provided by the natural correspondence

$$\mathcal{R}(\Sigma) = \frac{\text{representations of } \pi_1(\Sigma) \text{ into } SO(3)}{\text{conjugation}}$$
$$\leftrightarrow \frac{\text{flat connections in } \Sigma \times \mathbb{R}^3}{\text{gauge equivalence}}$$

For a flat connection, the holonomy along a loop depends only on the homotopy class of the loop and defines an element of $\mathcal{R}(\Sigma)$. Conversely, consider the universal cover $\tilde{\Sigma}$ of Σ. Extend the operation of $\pi_1(\Sigma)$ on $\tilde{\Sigma}$ to $\tilde{\Sigma} \times \mathbb{R}^3$ by means of a representation $\pi_1(\Sigma) \to SO(3)$. The quotient $(\tilde{\Sigma} \times \mathbb{R}^3)/\pi_1(\Sigma)$ is a bundle over Σ which inherits a flat connection from the trivial connection on $\tilde{\Sigma} \times \mathbb{R}^3$.

C. Taubes [**T2**], utilizing gauge theoretic considerations, has reinterpreted Casson's invariant in terms of flat connections. Refining this approach, A. Floer [**F**] has recently defined another invariant of Σ, its "instanton homology", which takes the form of an abelian group $I_*(\Sigma)$ with a natural Z_8 grading that is an enhancement of $\lambda(\Sigma)$ in that $\lambda(\Sigma) = \frac{1}{2}\sum_{i=0}^{7}(-1)^i \mathrm{rank}_{\mathbb{R}} I_i(\Sigma)$. The definition of these instanton groups makes essential use of gauge theory on three- and four-manifolds and so it appears that they are generally difficult to compute. (We refer the reader to two excellent expository papers [**A**] and [**Br**] concerning these invariants and how they relate to recent invariants of Donaldson for 4-manifolds.) However in [**FS6**] it is shown that when Σ is Seifert fibered the techniques in [**FS3**] can be adapted to compute these instanton homology groups.

The goal of these lectures is to review some of the "classical" differential geometric invariants associated to flat connections and indicate how they have been used in the past and how gauge theoretic considerations provide new applications of these invariants. We will then show how these invariants combine to provide information about the instanton homology $I_*(\Sigma)$ of Σ. The instanton chain complex of Σ has groups graded by $\mathbb{Z}_8$ that are generated by the elements α of $\mathcal{R}(\Sigma)$. When α is isolated in $\mathcal{R}(\Sigma)$, its grading in the chain complex is determined by the spectral flow $SF(\theta, a_\alpha)$ of the operator $\star d_a$ where a runs along any path of connections from the trivial $SU(2)$ connection θ to the flat connection a_α induced by α and d_a is the covariant derivative determined by the connection a. We will provide a slight refinement of $I_*(\Sigma)$, $* \in \mathbb{Z}_8$, by defining $\tilde{I}_*(\Sigma)$, $* \in \mathbb{Z}$, with $\sum_{n \equiv j(8)} \tilde{I}_n = I_{n \text{ (mod 8)}}$ and show that for a given $n \in \mathbb{Z}$ there are Brieskorn homology 3-spheres with $\tilde{I}_n \neq 0$.

2. Chern-Simons Invariants. Let Σ be a homology 3-sphere. Since $H^2(\Sigma; \mathbb{Z}_2) = 0$, every principal $SO(3)$-bundle P over Σ is trivial, i.e. is isomorphic to $\Sigma \times SO(3)$. Given such a trivialization, one can identify the space of connections $\mathcal{C}$ of Sobelev type L^p_k with the space $L^p_k(\Omega^1(\Sigma) \otimes \mathfrak{so}_3)$ of 1-forms on Σ with values in the Lie algebra $\mathfrak{so}_3$ in such a way that the zero element of $\mathcal{C}$ corresponds to the product connection θ on $\Sigma \times SO(3)$. The gauge group of bundle isomorphisms of P can be identified with $\mathcal{G} = L^p_{k+1}(\Sigma, SO(3))$ acting on $\mathcal{C}$ by the nonlinear transformation law

$$g(a) = gag^{-1} + (dg)g^{-1}$$

We will assume that $k+1 > 3/p$ so that $\mathcal{G}$ consists of continuous maps. The quotients $\mathcal{B} = \mathcal{C}/\mathcal{G}$ can be considered as infinite dimensional manifolds except near those connections a for which the group

$$\mathcal{G}_a = \{g \in \mathcal{G} | g(a) = a\}$$

is non-trivial. Such connections are called *reducible*. The trivial connection θ is reducible by all constant maps $g : \Sigma \to SO(3)$.

The *irreducible* connections form an open dense set $\mathcal{B}^*$ in $\mathcal{B}$. The set of flat connections, i.e. the set of all a for which its curvature $F_a \in L^p_k(\Omega^2(\Sigma) \otimes \mathfrak{so}_3)$ satisfies $F_a = 0$, is invariant under $\mathcal{G}$. However, the group $\mathcal{G}$ is not connected; in fact $\pi_0(\mathcal{G}) = \mathbb{Z}$ given by the degree of $g : \Sigma \to SO(3)$.

Given any connection a, we can take a path $\gamma : I = [0,1] \to \mathcal{C}$ from a to the trivial connection θ. This path determines a connection A_γ in the trivial bundle over $\Sigma \times I$. Let

$$CS(a) = \frac{1}{4\pi^2} \int_{\Sigma \times I} Tr(F_{A_\gamma} \wedge F_{A_\gamma})$$

This function $CS : \mathcal{C} \to \mathbb{R}$ depends on the trivialization of P. If θ' is the trivial connection with respect to another trivialization, then let γ' be a path in $\mathcal{C}$ from a to θ'. We can glue the connections A_γ and $A_{\gamma'}$ together over $\Sigma \times \{0\}$ and along $\Sigma \times \{1\}$ via a gauge transformation to obtain a connection A in a principal $SO(3)$-bundle E over $\Sigma \times S^1$ and

$$\begin{aligned} &\frac{1}{4\pi^2} \int_{\Sigma \times I} Tr(F_{A_\gamma} \wedge F_{A_\gamma}) - \frac{1}{4\pi^2} \int_{\Sigma \times I} Tr(F_{A_{\gamma'}} \wedge F_{A_{\gamma'}}) \\ =&\frac{1}{4\pi^2} \int_{\Sigma \times S^1} Tr(F_A \wedge F_A) = p_1(E) \end{aligned}$$

where the last equality follows from Chern-Weil theory, with $p_1(E)$ the first Pontryagin class of E evaluated on the top class of $\Sigma \times S^1$. Since $w_2(E) = 0$, $p_1(E) \equiv 0 \pmod 4$. Up to sign, $CS : \mathcal{C} \to \mathbb{R}/4\mathbb{Z}$ is the Chern-Simons invariant of the connection a. A similar argument shows that CS descends to $CS : \mathcal{B} \to \mathbb{R}/4\mathbb{Z}$. This $\mathbb{R}/4\mathbb{Z}$ invariant can be regarded as a (mod 4) Pontrjagin charge of the connection A_γ, for $Tr(F_{A_\gamma} \wedge F_{A_\gamma})$ is the Chern-Weil integrand.

This Chern-Simons functional induces a functional $CS : \mathcal{R}(\Sigma) \to \mathbb{R}/4\mathbb{Z}$. Noting that $\mathcal{R}(\Sigma)$ is compact, define

$$\tau(\Sigma) = \min\{CS(\alpha) | \alpha \in \mathcal{R}(\Sigma)\} \in [0,4]$$

It appears that these Chern-Simons invariants have never been used in the study of homology 3-spheres. We will now see that coupled with the techniques of [**FS3**] these invariants are extremely useful.

3. Gauge theory for $\Sigma(a_1,\ldots,a_n)$. A Seifert fibered homology sphere $\Sigma = \Sigma(a_1,...,a_n)$ admits a natural S^1-action whose orbit space is S^2. Orient Σ as the link of an algebraic singularity. Equivalently, orient Σ as a Seifert fibration with Seifert invariants $\{b_0;(a_i,b_i), i = 1,\ldots,n\}$ given by (1.1). With this orientation Σ bounds the canonical resolution, a negative definite simply connected smooth 4-manifold. Let $W = W(a_1,...,a_n)$ denote the mapping cylinder of the orbit map. It is a 4-dimensional orbifold with boundary Σ and singularities cones on the lens spaces $L(a_i,b_i)$ (see [**FS3**]). Orient W so that its oriented boundary is $-\Sigma$. Then W has a positive definite intersection form. Let W_0 denote W with open cones around the singularities removed. Then

$$\begin{aligned}\pi_1(W_0) &= \pi_1(\Sigma)/ < h >= T(a_1,...,a_n)\\ &=< x_1,...,x_n | x_i^{a_i} = 1, i = 1,...,n,\ x_1\cdots x_n = 1 >\end{aligned}$$

When $n = 3$ this is the usual triangle group and in general it is a genus zero Fuchsian group. Since Σ is a homology sphere, there is a one-to-one correspondence between representations α of $\pi_1(\Sigma)$ into $SU(2)$ and representations, which we still call α, of $\pi_1(W_0)$ into $SO(3)$.

Given $\alpha \in \mathcal{R}(\Sigma)$, let $\mathbf{V}_\alpha$ denote the flat real 3-plane bundle over W_0 determined by α. When $\mathbf{V}_\alpha$ is restricted over $L(a_i,b_i) \subset \partial W_0$ it splits as $\mathbb{L}_{\alpha,i} \oplus \mathbb{R}$ where $\mathbb{R}$ is a trivial real line bundle and $\mathbb{L}_{\alpha,i}$ is a flat 2-plane bundle corresponding to the representation $\pi_1(L(a_i,b_i)) \to \mathbb{Z}_{a_i}$ of weight l_i, where $r(\alpha(x_i)) = \pi l_i/a_i$. Here, the preferred generator of $\pi_1(L(a_i,b_i))$ corresponds to the deck transformation

$$(z,w) \longmapsto (\zeta z, \zeta^{b_i} w)$$

of S^3 where $\zeta = e^{2\pi i/a_i}$. Thus $\mathbb{L}_{\alpha,i}$ is the quotient of $S^3 \times \mathbb{R}^2$ by the above action of $\mathbb{Z}_{a_i}$. The bundle $\mathbb{L}_{\alpha,i}$ extends over the cone $cL(a_i,b_i)$

as $(\mathbb{C}^2 \underset{\mathbb{Z}_{a_i}}{\times} \mathbb{R}^2) \oplus \mathbb{R}$, an $SO(3)$- V-bundle whose rotation number over the cone point is l_i (with respect to the preferred generator given above). Thus we obtain an $SO(3)$-V-vector bundle, which we also denote by $\mathbf{V}_\alpha$, over W. (See **[FS3]** for V-bundles.) In **[FS6]** we determine which (l_1, l_2, l_3) can arise for representations of $\pi_1(W_0)$, $n = 3$, thus determining $\mathcal{R}(\Sigma(p,q,r))$.

Given a representation $\alpha : \pi_1(\Sigma) \to SU(2)$, its Zariski tangent space in the space of all conjugacy classes of such representations is $H^1(\Sigma; \mathbf{V}_\alpha)$ (where $\mathbf{V}_\alpha$ also denotes $\mathbf{V}_\alpha$ restricted to Σ). This is the case since $\mathbf{V}_\alpha$ is the $\mathbb{R}^3$-bundle associated to the representation α via the adjoint action of $SU(2)$.

The quotient Σ/S^1 of the natural S^1-action on $\Sigma = \Sigma(a_1, ..., a_n)$ is the 2-sphere S^2 with an induced orbifold (V-manifold) structure. The orbifold fundamental group of Σ/S^1 is just $T(a_1, ..., a_n) = \pi_1(W_0)$.

In the following proposition we use the presentation for $\pi_1(\Sigma)$ given by (1.1).

PROPOSITION 1.2 **[FS6]**. *Let $\alpha : \pi_1(\Sigma) \to SU(2)$ be a representation with $\alpha(x_i) \neq \pm 1$ for $i = 1, ..., m$ and $\alpha(x_i) = \pm 1$ for $i = m+1, ..., n$. Then*

$$dim_{\mathbb{R}}\ H^1(\pi_1(\Sigma); \mathbf{V}_\alpha) = 2m - 6.$$

(Here $H^1(\pi_1(\Sigma); \mathbf{V}_\alpha)$ denotes group cohomology with coefficients in the adjoint representation associated to α.)

In **[FS6]** we show that

PROPOSITION 1.3 **[FS6]**. *If $\alpha : \pi_1(\Sigma) \to SU(2)$ is a representation with $\alpha(a_i) \neq 1$ for $i = 1, ..., m$, $\alpha(a_i) = 1$ for $i = m+1, ..., n$, then the connected component $\mathcal{R}_\alpha$ of α in the space $\mathcal{R}(\Sigma)$ is a closed manifold of dimension $2m - 6$.*

In particular, $\mathcal{R}(\Sigma(p,q,r))$ consists of isolated points.

An $SO(2)$ V-vector bundle L over W is classified by the Euler class $e \in H^2(W_0) \cong \mathbb{Z}$ of its restriction over $W_0 = W -$ (neighborhood of singular points). We shall denote by L_e the V-bundle corresponding to the class e times a generator in $H^2(W_0,\mathbf{Z})$. Let B be any connection on L_e which is trivial near ∂W. Then the relative Pontryagin number of L_e is

$$\frac{e^2}{a}=\frac{1}{4\pi^2}\int_W Tr(F_B\wedge F_B)=p_1(B)$$

where $a=a_1\cdots a_n$.

Let A denote the $SO(3)$-connection over $W\cup(\Sigma\times\mathbf{R})\cong W$ which is built from the above flat connection on W and a 1-parameter family $\{a_t\}$ of $SO(3)$-connections over Σ given by a path of connections between a_α and the trivial connection θ.

THEOREM 1.4 [**FS4**]. *Let the connection A be as above and suppose Σ has Seifert invariants $\{b_0;(a_1,b_1),\dots,(a_n,b_n)\}$ with b_0 even. (This can always be arranged.) If one of the a_i's is even, assume it is a_1, and arrange the Seifert invariants so that the b_i, $i\neq 1$, are even. If $e\equiv\sum_{i=1}^n l_i\frac{a}{a_i}$ (mod $2a$), then the $SO(2)$ V-bundle L_e satisfies*

(1) *L_e has the same rotation numbers (up to sign) as $\mathbf{V}_\alpha$ over the singular points of W,*
(2) *$w_2(L_e)=w_2(\mathbf{V}_\alpha)$, and*
(3) *$p_1(A)\equiv e^2/a$ (mod 4).*

Thus for each representation $\alpha\in\mathcal{R}(\Sigma(p,q,r))$ there is associated an Euler number e and $CS(\alpha)\equiv e^2/pqr \bmod 4\mathbf{Z}$. This gives the computation for $\tau(\Sigma(p,q,r))$. Note that $pqr\tau(\Sigma(p,q,r))\in\mathbf{Z}$.

In [**FS3**] we introduced an integer

(3.5)
$$\begin{aligned}R(e)&=R(a_1,\dots,a_n;e)\\&=\frac{2e^2}{a}-3+m+\sum_{i=1}^{m}\frac{2}{a_i}\sum_{k=1}^{a_i-1}\cot\left(\frac{\pi ak}{a_i^2}\right)\cot\left(\frac{\pi k}{a_i}\right)\sin^2\left(\frac{\pi ek}{a_i}\right)\end{aligned}$$

The role that this integer plays in the gauge theory of $\Sigma((a_1,\dots,a_n)$ is that $R(a_1,\dots,a_n;e)$ denotes the virtual dimension of the moduli space of self-dual connections in the $SO(3)$-V-bundle $L_e \oplus \mathbb{R}$ over W which are astmptotically trivial.

For example, for p and q relatively prime, $\tau(\Sigma(p,q,pqk-1)) = 1/(pq(pqk-1))$, for $k \geq 1$. One way to see this is to use the algorithm presented in [**FS6**] to find a representation with associated Euler number $e = 1$. A rather curious way to see this is to consider the orbifold W which is the mapping cylinder of the orbit map $\Sigma(p,q,pqk-1) \to S^2$ and consider the moduli space $\mathcal{M}$ of asymptotically trivial self-dual connections in the V-bundle $E = L_1 \oplus \mathbb{R}$ over W. Then dim$\mathcal{M}$ = R(1) =1 (cf. [**FS3**]), so that (perhaps after a compact perturbation) there is a component of $\mathcal{M}$ which is an arc with one endpoint corresponding to the reducible self-dual connection. The noncompactness of $\mathcal{M}$ indicates that a self-dual connection "pops off" the end (cf. §10 of [**T1**]). That is, there is a self dual connection C over $Y = \Sigma \times \mathbb{R}$ which is asymptotically trivial near $+\infty$ and is asymptotically a flat connection a_α near $-\infty$. For any asymptotically trivial connection A in E over W we have $\frac{1}{4\pi^2}\int_W Tr(F_A \wedge F_A) = e^2/(pq(pqk-1)) = 1/(pq(pqk-1))$.Then $CS(\alpha,\theta) = \frac{1}{4\pi^2}\int_Y Tr(F_C \wedge F_C) \leq 1/(pq(pqk-1))$, so that $CS(\alpha,\theta) = 1/(pq(pqk-1))$.

Let Θ_3^H denote the group of oriented homology 3-spheres modulo the equivalence relation of oriented homology cobordism. In [**FS3**] it is shown that if $R(p,q,r;1) \geq 1$, then $\Sigma(p,q,r)$ has infinite order in Θ_3^H.

THEOREM 1.5 [**FS6**]. *Let p and q be pairwise relatively prime integers. The collection of homology 3-spheres $\{\Sigma(p,q,pqk-1)|k \geq 1\}$ are linearly independent over $\mathbb{Z}$ in Θ_3^H.*

PROOF: Fix $k \geq 2$ and suppose that $\Sigma(p,q,pqk-1) = \sum_{j=1}^k n_j \Sigma(p, q, pqj-1)$ in Θ_3^H, where $n_j \in \mathbb{Z}$ and $n_k \leq 0$. Then there is a cobordism Y between $\Sigma(p,q,pqk-1)$ and the disjoint union $\coprod_{j=1}^k n_j \Sigma(p,q,pqj-1)$ with Y having the cohomology of a $(1 + \sum |n_j|)$-punctured 4-

sphere. Now cap off the $-n_k$ copies of $-\Sigma(p,q,pqk-1)$ by adjoining to Y the positive definite canonical resolutions Z bounded by $-\Sigma(p,q,pqk-1)$. Let X be the resulting positive definite 4-manifold. Let W denote the mapping cylinder of the orbit map $\Sigma(p,q,pqk-1) \to S^2$. Finally, let $\tilde{X} = W \cup_{\Sigma(p,q,pqk-1)} X$ and consider the $SO(3)$-V-bundle $E = L_e \oplus \mathbb{R}$ over the positive definite orbifold $\tilde{X}$, where $e \in H^2(\tilde{X};\mathbf{Z}) = H^2(W;\mathbf{Z}) \oplus (\oplus_{i=1}^{-n_k} H^2(Z;\mathbf{Z}))$ is a generator of $H^2(W;\mathbf{Z}) = \mathbf{Z}$. For any asymptotically trivial connection A in E over $\tilde{X}$ we have $\frac{1}{4\pi^2}\int_{\tilde{X}} Tr(F_A \wedge F_A) = 1/(pq(pqk-1))$. The moduli space $\mathcal{M}$ of asymptotically trivial self-dual connections in E has dimension $R(p,q,pqk-1;1) = 1$, so that (perhaps after a compact perturbation) there is a component of $\mathcal{M}$ which is an arc with one endpoint corresponding to the reducible self-dual connection (see [**FS3**]). The noncompactness of $\mathcal{M}$ indicates that a self-dual connection C "pops off" the end (cf. §10 of [**T1**]). That is, there is a self dual connection C over $Y = \pm\Sigma(p,q,pqj-1) \times \mathbb{R}$, for some j, which is asymptotically trivial near $+\infty$ and is asymptotically a flat connection a_α near $-\infty$. Also, there is a self-dual connection B over $\tilde{X}$ which is asymptotically some flat connections at the ends of $\tilde{X}$ so that $\frac{1}{4\pi^2}\int_{\tilde{X}} Tr(F_B \wedge F_B) \geq 0$. However, $1/(pq(pqk-1)) = \frac{1}{4\pi^2}\int_{\tilde{X}} Tr(F_A \wedge F_A) = \frac{1}{4\pi^2}\int_{Y} Tr(F_C \wedge F_C) + \frac{1}{4\pi^2}\int_{\tilde{X}} Tr(F_B \wedge F_B) \geq \frac{1}{pq(pqj-1)} + \frac{1}{4\pi^2}\int_{\tilde{X}} Tr(F_B \wedge F_B) > \frac{1}{pq(pqk-1)} + \frac{1}{4\pi^2}\int_{\tilde{X}} Tr(F_B \wedge F_B)$, so that $\frac{1}{4\pi^2}\int_{\tilde{X}} Tr(F_B \wedge F_B) < 0$, a contradiction. ∎

This theorem was originally proved by Furuta [**Fu**] using a similar technique.

Other non-cobordism relationships can be detected by the explicit computations of $\tau(\Sigma(p,q,r))$. For example, $\tau(\Sigma(2,3,7)) = 25/42$ and $\tau(-\Sigma(2,3,7)) = 4 - 121/42 = 47/42$. Thus, the proof of Theorem 1 shows that $\Sigma(2,3,5)$ is not a multiple of $\Sigma(2,3,7)$.

LECTURE 2.

1. η and ρ-invariants. Atiyah, Patodi, and Singer introduced in [**APS1-3**] a real-valued invariant for flat connections in a trivialized bundle over odd-dimensional manifolds. These invariants arose from their study of index theorems for manifolds with boundary. In this lecture we will discuss these invariants and show how they have been used in low dimensional topology.

Let Σ be a 3-manifold with a flat connection a in a trivialized $SO(3)$-bundle over Σ. Let Ω^p_{ad} denote the space $L^p_k(\Omega(\Sigma) \otimes \mathfrak{so}_3)$ and consider the self-adjoint elliptic operator

$$B_a = \star d_a - d_a \star : \Omega^0_{ad} \oplus \Omega^2_{ad} \to \Omega^0_{ad} \oplus \Omega^2_{ad}$$

where $d_a : \Omega^p_{ad} \to \Omega^{p+1}_{ad}$ is the covariant derivative determined by the connection a and $\star$ is the Hodge star operator, which depends upon a Riemannian metric on Σ. The eigenvalues λ of B_a are real and discrete. Atiyah-Patodi, and Singer [**APS1**] define the function

$$\eta_a(s) = \sum_{\lambda \neq 0} (\mathrm{sign}\lambda)|\lambda|^{-s}$$

We let $\eta(s)$ denote $\eta_\theta(s)$. Since the operator B_a involves the $\star$ operator, it depends upon a Riemannian metric on Σ and changes sign when the orientation is changed. In [**APS1-3**] it is shown that $\eta_a(s)$ has a finite value at $s = 0$. The importance of these η invariants is their role in the computation of the twisted signatures of 4-manifolds with boundary.

Let X be a 4-manifold with boundary Σ and let $\alpha : \pi_1(X) \to U(n)$ be a unitary representation of the fundamental group. This defines a flat vector bundle V_α over X, or equivalently a local coefficient system. There are cohomology groups $H^*(X; V_\alpha)$ and $H^*(X, \Sigma; V_\alpha)$ and these have a natural pairing into $\mathbb{C}$ given by cup product, the inner product on V_α and the evaluation of the top cycle of X mod

Y. This induces a non-degenerate form on $\hat{H}^*(X;V_\alpha)$, the image of the relative cohomology in the absolute cohomology. On $\hat{H}^2(X;V_\alpha)$ this form is Hermetian and the signature of this form is denoted by $\text{sign}_\alpha(X)$.

Assume that the metric on Σ is extended to a metric on X which is a product near Σ and that $\alpha|\Sigma = a$. It is shown in [**APS3**] that

$$\text{sign}_\alpha(X) = n\int_X L(p_1) - \eta_a(0)$$

where $L(p_1)$ is the Hirzebruch L_1-polynomial of X. Thus, these η invariants are to be viewed as signature defects. However, they depend upon a Riemannian metric on Σ. To resolve this dependency define the reduced η-function by

$$\rho_a(s) = \eta_a(s) - \eta(s)$$

where $eta(s) = eta_\theta(s)$ with θ the trivial $U(n)$ connection. An application of the above signature theorem to $\Sigma \times I$ shows that $\rho_a(0)$ is independent of the Riemannian metric on Σ and is a diffeomorphism invariant of Σ and a. We denote it by $\rho_a(\Sigma)$. Furthermore, if $\Sigma = \partial X$ with a extending to a flat unitary connection α over X, then

$$(\star) \qquad \rho_a(\Sigma) = n\text{sign}(X) - \text{sign}_\alpha(X)$$

These ρ_α-invariants were made important in low dimensional topology via the Casson-Gordon invariants for knots [**CG**]. In the realm of low dimensional topology these and related invariants were the only game in town in the 1970's and early 80's. We will now discuss these Casson-Gordon invariants and indicate how gauge theory can enter into their considerations.

2. Knots in S^3. Let K be a smooth knot in S^3. The knot K is *slice* if there is a smooth 2-disk $D \subset B^4$ with $K = \partial D$. Knots K_0, K_1 are *cobordant* if there is a smoothly embeded annulus in $S^3 \times I$ meeting

$S^3 \times \{t\}$ in K_t $(t = 0, 1)$. Addition of cobordism classes of oriented knots is given by connected sum, resulting in the knot cobordism group Θ_1^3.

The question of the hour is: When is a knot slice? There are necessary algebraic conditions. $K \subset S^3$ bounds an oriented surface $F \subset S^3$. Thicken F to an embedding $F \times I \subset S^3$. Given $x, y \in H_1(F)$, let $\alpha(x, y) =$ linking number of $x \times 0$ and $y \times 1$. This defines a bilinear form $\alpha : H_1(F) \times H_1(F) \to \mathbf{Z}$, such that $\alpha(x, y) - \alpha(y, x) =$ intersection number of x and y. The *Seifert form* for K is just the form α. The Seifert form is *null-cobordant* if it vanishes on a subgroup of $H_1(F)$ of dimension $\frac{1}{2}\dim H_1(F)$). J. Levine proved (in all dimensions) that if K is slice, then any Seifert form for K is null-cobordant. Such a knot is called *algebraically slice*. Furthermore, in higher (odd) dimensions the analogous condition is necessary and sufficient for K to be slice.

K is a *ribbon* knot if it bounds an immersed disc (ribbon) in S^3 each of whose singularities is two sheets intersecting in an arc which is interior to one of the sheets. Ribbon knots are slice, for push the interior of the ribbon into B^4 and then deform slightly a neighborhood of each arc. An old problem of Fox, which is still unresolved, is whether every slice knot is a ribbon knot. In **[CG]** there is presented an invariant for detecting when an algebraically slice knot is not ribbon and a modified version of this invariant detects when it is not slice. We discuss these ribbon invariants and indicate how gauge theory makes them slice invariants.

3. Casson-Gordon invariants. Let L be the double branched covering of the knot K in S^3. If K is slice, then the double covering of B^4 branched over the slicing disc is a 4-manifold W with $\tilde{H}(W; \mathbf{Q}) = 0$. Furthermore if the image of $H_1(L)$ in $H_1(W)$ has order m, then $|H_1(L)| = m^2$. Also, if the slicing disc is obtained by deforming a ribbon, then $\pi_1(L)$ surjects onto $\pi_1(W)$.

Let $\chi : H_1(L) \to U(1)$ be a representation with image the m-th roots of unity $\mathbf{C}_m$. The map χ is induced by a map $L \to K(\mathbf{C}_m, 1)$.

Since $\Omega^3 K(\mathbf{C}_m, 1)$ is finite, rL bounds a compact 4-manifold W over $K(\mathbf{C}_m, 1)$, for some $r > 0$. Thus the representation χ factors through $H_1(W)$ and induces a flat $U(1)$ bundle V_χ over W. Then let $\sigma(K, \chi) = \rho_\chi(L)/r$. This is independent of r.

These invariants defined by Casson-Gordon were originally applied to those knots K in S^3 whose double branched covering is a lens space $L = L(p,q)$. This contains the collection of 2-bridged knots. If K is ribbon, then $\pi_1(L) = \mathbf{Z}_{m^2}$ and $\pi_1(W) = \mathbf{Z}_m$. Using this W to compute $\sigma(K,\chi)$ one computes that $H_0(W; V_\chi) = 0$ if χ is non-trivial and since the m-fold covering of W is simply-connected, $H_1(W; V_\chi) = H_3(W; V_\chi) = 0$ (this is where the ribbon assumption is used). Also the Euler characteristic of W with V_χ coefficients is that of W, namely 1, so that $H_2(W; V_\chi)$ has dimension 1. Now by $\star$ $\rho_\chi(L) = \sigma_\chi(W)$, so that $\sigma(K,\chi) = \pm 1$. Calculations then show that there are algebraically slice 2- bridge knots K for which there is a χ with $\sigma(K,\chi) \neq \pm 1$, hence they are not ribbon. Casson-Gordon then proceed in **[CS]** to refine these invariants $\sigma(K,\chi)$, by considering infinite cyclic coverings, to define invariants of (K,χ) that show that these K are also not slice in the case that m is a prime power order.

At this point gauge theory can enter the picture to show that in fact if K is slice, then $\sigma(K,\chi) = \pm 1$ for any m. This was done in **[FS4]** as follows. Consider $X =$cone$(L) \cup_L W$, a pseudofree orbifold in the sense of **[FS3]**. Now the bundle E_χ over W extends as a flat V-bundle over X. The flat connection determined by E_χ is both self-dual and anti-self-dual. We now study the moduli space of self-dual and anti-self-dual connections in the $SO(3)$-V-bundle $F_\chi = E_\chi \oplus \mathbf{R}$. We denote these by $\mathcal{M}_+$ and $\mathcal{M}_-$, respectively. Now $\mathcal{M}_\pm$ is nonempty (since each contains the reducible flat connection determined by χ) and compact (since flat connections are representations into a compact group). The index theorem yields that the virtual dimension of $\mathcal{M}_\pm$ is $-2 \pm \sigma(K,\chi)$ and that this is an odd integer. If dim$\mathcal{M}_\pm > 0$, then a perturbation of the equations has a moduli space that is a compact manifold of dimension $\dim \mathcal{M}_{\pm}$ with an odd number of singularities of the form a cone on a complex projective space. It is then shown that an odd number of complex projective spaces cannot bound in $\mathcal{B}$. Thus $\dim \mathcal{M}_{\pm} < 0$, and $\sigma(K, \chi) = \pm 1$.

This program was extend by G. Matic in **[Ma]** and independently D. Rubermann in **[R]** to show that if L is a rational homology sphere with some finite cover a homology 3-sphere, then the same conclusion holds. The main new ingredient is the gauge theory for manifolds with ends developed by C. Taubes in **[T1]**. Rather than coning off the boundary, the idea is to add an open collar to W^4 and consider self-dual and anti-self-dual connections that are asymptotically flat. With the results of **[T1]** in place, the proof is formally the same.

Note that the only ρ_α-invariants that were used in the above set-up were those associated with representations $\alpha : \pi_1(L) \to U(2)$ that factored through a finite group. In general, there are many irreducible representations. What role do these invariants play in the study of Θ_1^3? We should keep this question in mind during the next lecture, where these irreducible representations play an essential role.

LECTURE 3.

1. An integer instanton invariant. Let Σ be a homology 3-sphere, $\alpha : \pi_1(\Sigma) \to SO(3)$ an irreducible (i.e. non-trivial) representation, and a_α the associated flat connection in the trivial bundle over Σ. Now the Chern-Simons invariant $CS(\alpha)$ of α is just the Pontrjagin charge (mod 4) of a connection on the trivial bundle over $\Sigma \times \mathbb{R}$ that is a_α near $-\infty$ and is the trivial connection θ near $= \infty$. Let $a_\alpha(\Sigma)$ denote that connection gauge equivalent to a_α with

$$\frac{1}{4\pi^2} \int_{\Sigma \times \mathbb{R}} \operatorname{Tr}(F_{a_\alpha(\Sigma)} \wedge F_{a_\alpha(\Sigma)}) = CS(\alpha) \in [0, 4]$$

We can associate an integer to the representation α as follows. Let $I(\alpha)$ denote the dimension of the moduli space $\mathcal{M}_{a_\alpha(\Sigma)}$ of self-dual connections on $\Sigma \times \mathbb{R}$ with Pontrjagin charge $CS(\alpha)$ that are asymptotically $a_\alpha(\Sigma)$ near $-\infty$ and θ near $+\infty$. The Atiyah-Patodi-Singer index theorem yields the virtual dimension of $\mathcal{M}_{a_\alpha(\Sigma)}$ as

$$(3.1) \qquad I(\alpha) = \int_{\Sigma\times\mathbb{R}} \hat{A}(\Sigma\times\mathbb{R})ch(V_-)ch(\mathfrak{g}) \\ - \frac{1}{2}(h_\theta + \eta_\theta(0)) + \frac{1}{2}(h_{a_\alpha(\Sigma)} + \eta_{a_\alpha(\Sigma)}(0))$$

where the forms $\hat{A}(\Sigma\times\mathbb{R})$ and $ch(V_-)$ are computed from the Riemannian connection on $\Sigma\times\mathbb{R}$ (choose a product metric, say) and $\mathfrak{g}$ is the $SO(3)$ bundle over $\Sigma\times\mathbb{R}$ with Pontrjagin charge $CS(\alpha)$. The term h_β is the sum of the dimensions of $H^i(\Sigma;\mathbf{V}_\beta)$, $i=0,1$. It is important here that α be irreducible. Recalling that $\rho_\beta = \eta_\beta(0) - \eta_\theta(0)$ (here θ is the trivial $SO(3)$-connection), and noting that $h_\theta = 3$ we have

$$(3.2) \quad I(\alpha) = \int_{\Sigma\times\mathbb{R}} \hat{A}(\Sigma\times\mathbb{R})ch(V_-)ch(\mathfrak{g}) - \frac{3}{2} + \frac{h_{a_\alpha(\Sigma)}}{2} + \frac{\rho_{a_\alpha(\Sigma)}}{2}$$

The integral term of (3.2) is

$$\int_{\Sigma\times\mathbb{R}} \hat{A}(\Sigma\times\mathbb{R})ch(V_-)ch(\mathfrak{g}) = 2\int_{\Sigma\times\mathbb{R}} p_1(\mathfrak{g}) + \frac{3}{2}\int_{\Sigma\times\mathbb{R}}(\mathcal{L}-\mathcal{E})$$

where $\mathcal{L}$ and $\mathcal{E}$ are the L-polynomial and the Euler form of $\Sigma\times\mathbb{R}$. Since $\mathfrak{g}$ is our $SO(3)$-bundle over $\Sigma\times\mathbb{R}$ with Pontrjagin charge $CS(\alpha)$, $\int_{\Sigma\times\mathbb{R}} p_1(\mathfrak{g}) = CS(\alpha)$. We then get as in [**APS1**, §4] that the integral term is

$$(3.3) \qquad 2CS(\alpha) - \frac{3}{2}[\chi(\Sigma\times\mathbb{R}) - \sigma(\Sigma\times\mathbb{R})] + \frac{3}{2}(\eta_\theta(0) - \eta_\theta(0))$$

Combining (3.2) and (3.3) we have

$$(3.4) \qquad I(\alpha) = 2CS(\alpha) - \frac{3}{2} + \frac{h_{a_\alpha(\Sigma)}}{2} + \frac{\rho_{a_\alpha(\Sigma)}}{2} \in \mathbf{Z}$$

This is well-defined as an integer, not just as an integer mod 8, since we have specified the Pontrjagin charge of the instanton moduli space.

We now have an interesting relationship between the Chern-Simons invariants introduced in Lecture 1 and the ρ_α-invariants introduced in Lecture 2, namely

$$4CS(\alpha) \equiv \rho_\alpha + h_{a_\alpha(\Sigma)} + 3 \mod 2\mathbf{Z}$$

Like the Chern-Simons invariant, this integer invariant $I(\alpha)$ is useful in detecting when homology 3-spheres are not homology cobordant. We will illustrate this after some computations are discussed.

2. Examples. In **[FS6]** we computed $I(\alpha)$ for $\alpha \in \mathcal{R}(\Sigma(p,q,r))$. We now indicate the key ideas behind the computation.

Recall the integer

(3.5)

$$\begin{aligned} R(e) &= R(a_1, \dots, a_n; e) \\ &= \frac{2e^2}{a} - 3 + m + \sum_{i=1}^{m} \frac{2}{a_i} \sum_{k=1}^{a_i - 1} \cot(\frac{\pi a k}{a_i^2}) \cot(\frac{\pi k}{a_i}) \sin^2(\frac{\pi e k}{a_i}) \end{aligned}$$

the virtual dimension of the moduli space of self-dual connections in the $SO(3)$-V-bundle $L_e \oplus \mathbf{R}$ over W which are asymptotically trivial. Combining (3.5) with the choices of e that arise from representations of $\pi_1(\Sigma(p,q,r))$ (see Lecture 1) and Theorem 1.4 yields the computation of $I(\alpha)$; for $CS(\alpha) = \frac{e^2}{pqr} - 4k \in [0,4]$. Then $I(\alpha) = R(p,q,r;e) - 8k$.

THEOREM 3.6 **[FS4]**. *Suppose $R(a_1, \dots, a_n) \geq 1$. If $\Sigma(a_1, \dots, a_n)$ is homology cobordant to a homology 3-sphere Σ, then*

(1) $\tau(\Sigma) \leq \tau(\Sigma(a_1, \dots, a_n)) = \frac{1}{a_1 \cdots a_n}$, *and*

(2) $1 \leq I(\alpha) \leq R(a_1, \dots, a_n)$ *for some representation $\alpha \in \mathcal{R}(\Sigma)$ with $CS(\alpha) \leq \frac{1}{a_1 \cdots a_n}$.*

Furthermore, $\Sigma(a_1, \ldots, a_n)$ is not homology cobordant via a simply-connected homology cobordism to any other Σ.

PROOF: Following the proof of Thoerem 1.4 we obtain a reducible (asymptotically trivial) self-dual connection on the bundle $E = L_e \oplus \mathbb{R}$ over the union $\tilde{X}$ of the mapping cylinder W of $\Sigma(a_1, \ldots, a_n)$ and the homology cobordism. Any asymptotically trivial connection A on E satisfies $\frac{1}{4\pi^2} \int_{\tilde{X}} Tr(F_A \wedge F_A) = \frac{1}{a_1 \cdots a_n}$. Again let $\mathcal{M}$ be the moduli space of asymptotically trivial self-dual connections on E. A sequence in $\mathcal{M}$ which has no convergent subsequence must either pop off an instanton at one of the cone points or a self-dual connection over $\Sigma(a_1', \ldots, a_n') \times \mathbb{R}$ as in (1.4). Since the smallest Pontrjagin number of a bundle on the suspension of a lens space $L(a_i, b_i)$ which admits a self-dual (V-) connection is $\frac{4}{a_i} > \frac{1}{a_1 \cdots a_n}$, no instanton can pop off at a cone point. It follows that no sequence in $\mathcal{M}$ can converge to a reducible self-dual connection (on some V-bundle) which has Pontrjagin charge less than $\frac{1}{a_1 \cdots a_n}$. Since $\dim \mathcal{M} = R(a_1, \ldots, a_n)$ is odd, we may use the technique of **[FS3]** for cutting down the moduli space to find a 1-dimensional submanifold $\mathcal{N}$ of $\mathcal{M}$ which is noncompact and has one endpoint corresponding to the reducible self-dual connection.

Now apply the proof of Theorem 1.4 to $\mathcal{N}$ to verify claim (1) and to show the existence of a self-dual connection on $\Sigma \times \mathbb{R}$ which is asymptotically trivial near $+\infty$ and is asymptotically some flat connection a_α near $-\infty$, $\alpha \in \mathcal{R}(\Sigma)$. This implies that $I(\alpha) \geq 1$ because of translational invariance in the $\mathbb{R}$-factor (c.f. **[F]**). The other inequality also follows from Theorem 1.4.

The last statement follows as in Proposition 1.7 of **[T1]**. Let U be a simply-connected homology cobordism from $\Sigma(a_1, \ldots, a_n)$ to Σ, and let V be the simply-connected homology cobordism from $\Sigma(a_1, \ldots, a_n)$ to itself obtained by doubling U along Σ. We obtain a reducible (asymptotically trivial) self-dual connection on the bundle $E = L_e \oplus \mathbb{R}$ over the union $\tilde{X}$ of the mapping cylinder W of $\Sigma(a_1, \ldots, a_n)$ with infinitely many copies of V adjoined. Again let $\mathcal{M}$

be the moduli space of asymptotically trivial self-dual connections on E. Now, since $\tilde{X}$ has a simply-connected end, $\mathcal{M}$ is compact [**T1**]. As above, since $\dim \mathcal{M} = R(a_1, \ldots, a_n) > 0$, we can cut down to obtain a compact moduli space with one end point, a contradiction. ∎

For example, $\tau(2,7,15) = \tau(2,3,35)$ and $R(2,3,35) = 1$. However, for the unique α in $\mathcal{R}(\Sigma(2,7,15))$ with $CS(\alpha) = \tau(\Sigma(2,7,15))$, $I(\alpha) = -5$, so that $\Sigma(2,7,15)$ is not homology cobordant to $\Sigma(2,3,35)$.

3. Extended Instanton homology. Let Σ be an arbitrary homology 3-sphere. Suppose that for each $\alpha \in \mathcal{R}(\Sigma)$, $H^1(\Sigma; V_\alpha) = 0$. Such representations are called *regular*. The Brieskorn spheres satisfy this condition. Define a chain complex

$$\cdots \xrightarrow{\partial} IC_j(\Sigma) \xrightarrow{\partial} IC_{j-1}(\Sigma) \xrightarrow{\partial} \cdots$$

as follows. Each $IC_j(\Sigma)$ is a free abelian group generated by those $\alpha \in \mathcal{R}(\Sigma)$ with $I(\alpha) = j$. Let $\alpha \in IC_j(\Sigma)$ be a representation. The $\partial\alpha \in IC_{j-1}(\Sigma)$ is given by

$$\partial\alpha = \sum_{\substack{\beta \in \mathcal{R}(\Sigma) \\ I(\beta) = j-1}} [\alpha; \beta]\beta$$

where $[\alpha; \beta] \in \mathbf{Z}$ is the number of components (with orientation) of the 1-dimensional moduli space over $\Sigma \times \mathbf{R}$ that are asymptotically α at $-\infty$ and β at $+\infty$. It is shown in [**F**] that $\partial\partial = 0$. We let $\tilde{I}_*(\Sigma)$ denote the homology of this chain complex. The instanton homology $I_*(\Sigma)$ as defined by Floer in [**F**] is graded by $\mathbf{Z}_8$ where the grading is $I(\alpha) \mod 8$, so that $\sum_{n \equiv j(8)} \tilde{I}_n = I_{n \mod 8}$. If there are representations that are not regular, one must perturb the flatness equations to arrive at nearby (not necessarily flat) connections that are regular (see [**F**]). Be aware that in [**F**] the instanton grading is $-3 - I(\alpha)(\mod 8)$. We refer the reader to two excellent survey articles ([**A**] and [**Br**]) concerning instanton homology and its relationship to Donaldson polynomials.

4. Examples. In [**FS6**] we listed some examples of $I_*(\Sigma(p,q,r))$. The groups I_i are free over $\mathbf{Z}$ and vanish for odd i, so we denote the instanton homology $I_*(\Sigma(p,q,r))$ of $\Sigma(p,q,r)$ as an ordered 4-tuple (f_0, f_1, f_2, f_3) where f_i is the rank of $I_{2i+1}(\Sigma(p,q,r))$.

$$I_*(\Sigma(2,3,6k\pm 1)) = \begin{cases} (\frac{k\mp 1}{2}, \frac{k\pm 1}{2}, \frac{k\mp 1}{2}, \frac{k\pm 1}{2}) & \text{for } k \text{ odd} \\ (\frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k}{2}) & \text{for } k \text{ even} \end{cases}$$

$$I_*(\Sigma(2,5,10k\pm 1)) = \begin{cases} (\frac{3k\mp 1}{2}, \frac{3k\pm 1}{2}, \frac{3k\mp 1}{2}, \frac{3k\pm 1}{2}) & \text{for } k \text{ odd} \\ (\frac{3k}{2}, \frac{3k}{2}, \frac{3k}{2}, \frac{3k}{2}) & \text{for } k \text{ even} \end{cases}$$

$$I_*(\Sigma(2,5,10k\pm 3)) = \begin{cases} (\frac{3k\pm 1}{2}, \frac{3k\pm 1}{2}, \frac{3k\pm 1}{2}, \frac{3k\pm 1}{2}) & \text{for } k \text{ odd} \\ (\frac{3k\pm 2}{2}, \frac{3k}{2}, \frac{3k\pm 2}{2}, \frac{3k}{2}) & \text{for } k \text{ even} \end{cases}$$

$$I_*(\Sigma(2,7,14k\pm 1)) = \begin{cases} (3k\mp 1, 3k\pm 1, 3k\mp 1, 3k\pm 1) & \text{for } k \text{ odd} \\ (3k, 3k, 3k, 3k) & \text{for } k \text{ even} \end{cases}$$

$$I_*(\Sigma(2,7,14k\pm 3)) = (3k, 3k\pm 1, 3k, 3k\pm 1)$$

$$I_*(\Sigma(2,7,14k\pm 5)) = (3k\pm 1, 3k\pm 1, 3k\pm 1, 3k\pm 1)$$

$$I_*(\Sigma(3,4,12k\pm 1)) = \begin{cases} (\frac{5k\mp 1}{2}, \frac{5k\pm 1}{2}, \frac{5k\mp 1}{2}, \frac{5k\pm 1}{2}) & \text{for } k \text{ odd} \\ (\frac{5k}{2}, \frac{5k}{2}, \frac{5k}{2}, \frac{5k}{2}) & \text{for } k \text{ even} \end{cases}$$

$$I_*(\Sigma(3,4,12k-5)) = \begin{cases} (\frac{5k-1}{2}, \frac{5k-3}{2}, \frac{5k-1}{2}, \frac{5k-3}{2}) & \text{for } k \text{ odd} \\ (\frac{5k-2}{2}, \frac{5k-2}{2}, \frac{5k-2}{2}, \frac{5k-2}{2}) & \text{for } k \text{ even} \end{cases}$$

$$I_*(\Sigma(3,5,15k\pm 2)) = \begin{cases} (4k, 4k\pm 1, 4k, 4k\pm 1) & \text{for } k \text{ odd} \\ (4k\pm 1, 4k, 4k\pm 1, 4k\mp 3) & \text{for } k \text{ even} \end{cases}$$

We now list a few computations of $\tilde{I}_*(\Sigma(p,q,r))$. For the fist examples we will list these as n-vectors with the i-th entry denoting the dimension of $\tilde{I}_{2i-1}$.

$$\tilde{I}_*(\Sigma(2,3,5)) = (1,0,1,0)$$
$$\tilde{I}_*(\Sigma(2,3,11)) = (1,1,1,1)$$

$$\tilde{I}_*(\Sigma(2,3,17)) = (1,1,2,1,1)$$
$$\tilde{I}_*(\Sigma(2,3,23)) = (1,2,2,2,1)$$
$$\tilde{I}_*(\Sigma(2,3,29)) = (1,1,3,2,2,1)$$
$$\tilde{I}_*(\Sigma(2,3,35)) = (1,2,3,3,2,1)$$
$$\tilde{I}_*(\Sigma(2,3,41)) = (1,1,4,3,3,2)$$
$$\tilde{I}_*(\Sigma(2,3,47)) = (1,2,3,4,3,2,1)$$

For $\alpha \in \mathcal{R}(\Sigma(2,3,6k-1))$, $CS(\alpha) \equiv \frac{e^2}{36k-6} \mod 4$ where $e \equiv 1 \mod 6$. Choosing the largest e and computing $R(2,3,6k-1;e)$, it can be shown that for a given any positive integer N, there is a K with $\tilde{I}_{2N-1}(\Sigma(2,3,6K-1)) \neq 0$. Similarily, it can be shown that for any negative integer N, there is a K with $\tilde{I}_{2|N|-1}(\Sigma(K,K+1,K(K+1)+1)) \neq 0$.

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