INSTANTON HOMOLOGY OF SEIFERT FIBRED HOMOLOGY THREE SPHERES

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[Received 4 October 1988—Revised 5 May 1989]

1. *Introduction*

For an oriented integral homology 3-sphere Σ, A. Casson has introduced an integer invariant $\lambda(\Sigma)$ that is defined by using the space $\mathcal{R}(\Sigma)$ of conjugacy classes of irreducible representations of $\pi_1(\Sigma)$ into SU(2) (see [**1**]). This invariant $\lambda(\Sigma)$ can be computed from a surgery or Heegard description of Σ and satisfies $\lambda(\Sigma) \equiv \mu(\Sigma) \pmod 2$, where $\mu(\Sigma)$ is the Kervaire–Milnor–Rochlin invariant of Σ. This powerful new invariant was used to settle an outstanding problem in 3-manifold topology; namely, showing that if Σ is a homotopy 3-sphere, then $\mu(\Sigma) = 0$. C. Taubes [**17**], utilizing gauge-theoretic considerations, has reinterpreted Casson's invariant in terms of flat connections. Refining this approach, A. Floer [**13**] has recently defined another invariant of Σ, its 'instanton homology', which takes the form of an abelian group $I_*(\Sigma)$ with a natural $\mathbb{Z}_8$-grading that is an enhancement of $\lambda(\Sigma)$ in that

$$\lambda(\Sigma) = \tfrac{1}{2} \sum_{i=0}^{7} (-1)^i \operatorname{rank}_{\mathbb{R}} I_i(\Sigma).$$

The definition of these instanton groups makes essential use of gauge theory on three- and four-manifolds and so it appears that they are generally difficult to compute. However, we shall show in this paper that when Σ is Seifert fibred our techniques in [**10**] can be adapted to compute these instanton homology groups.

In § 2 we give a method for computing all the conjugacy classes of representations of the fundamental group of a Seifert fibred homology sphere Σ into SU(2). In the special case that Σ has three exceptional orbits (that is, Σ is a Brieskorn sphere) then $\mathcal{R}(\Sigma)$ consists of a finite number of points. In general, if Σ has n exceptional orbits ($n \geqslant 3$) then the components of $\mathcal{R}(\Sigma)$ are even-dimensional manifolds with top dimension $2n - 6$.

Briefly, instanton homology is defined as follows. (We refer the reader to two excellent expository papers [**2**] and [**9**] concerning these invariants and how they relate to recent invariants of Donaldson for 4-manifolds.) Let $\mathcal{B}_\Sigma$ be the Banach manifold of L_2^4-gauge equivalence classes of L_1^4-SU(2) connections over Σ. Then we may view $\mathcal{R}(\Sigma)$ as contained in $\mathcal{B}_\Sigma$. If all non-trivial representations of $\pi_1(\Sigma)$ into SU(2) are non-degenerate, i.e. if $H^1(\Sigma, ad(\alpha)) = 0$ (see § 2), then the instanton homology of Σ is obtained by using the chain complex $R_*(\Sigma)$ freely generated by the non-trivial representations in $\mathcal{R}(\Sigma)$. For such a non-degenerate $\alpha \in \mathcal{R}(\Sigma)$, its grading in $R_*(\Sigma)$ is defined to be the spectral flow $\mathrm{SF}(\theta, \alpha)$ of the operator $\star d_a$ where a runs along any path of connections from the trivial SU(2) connection θ to

The work of the first author was partially supported by NSF Grant DMS 8501789-02, and that of the second author by NSF Grant DMS 8703413.

A.M.S. (1980) subject classification: 57N10.

Proc. London Math. Soc. (3) 61 (1990) 109–137.

the flat connection a_α induced by α and where d_a is the covariant derivative determined by the connection a. The spectral flow is defined to be the net change in the number of eigenvalues from negative to positive as a moves along the path (cf. [**5**, § 7] or [**13**, § 2b]). This number is well-defined (mod 8) in $\mathcal{B}_\Sigma$; so the chain complex $R_*(\Sigma)$ is graded by $\mathbb{Z}_8$.

The differential of this chain complex is defined using instanton moduli spaces over $\Sigma \times \mathbb{R}$; however, this will not concern us since one of the key results of this paper is that for a non-degenerate irreducible representation $\alpha \in \mathcal{R}(\Sigma)$, the spectral flow $\mathrm{SF}(\theta, \alpha)$ is even. Thus when all non-trivial representations of a Seifert fibred homology sphere are non-degenerate, the instanton homology $I_*(\Sigma)$ is $R_*(\Sigma)$. This will be true for Brieskorn homology spheres (Proposition 2.5).

In § 3 we compute the instanton index $\mathrm{SF}(\theta, \alpha)$ for representations of Seifert-fibred homology spheres and prove that it is always even, so that the differential of the instanton complex is always trivial for a Brieskorn sphere. This then gives the instanton homology groups for a Brieskorn sphere. We conclude this section with an explicit computation for $\Sigma = \Sigma(3, 4, 5)$. When Σ has $n > 3$ exceptional orbits, the problems in computing the instanton homology groups become more difficult due to non-degeneracy of the representations. Utilizing a particular perturbation of the curvature equation suggested by a 'normally non-degenerate Morse theory', we indicate how the Euler characteristic of $\mathcal{R}(\Sigma)$ enters into the instanton homology computations. We also show how the components of $\mathcal{R}(\Sigma)$ may be viewed as configuration spaces of linkages in the 3-sphere. The section closes with an explicit computation of $I_*(\Sigma(2, 3, 5, 7))$.

In § 5 we indicate some applications of the ideas presented in §§ 2–4. If Σ is *any* homology three sphere, define $\tau(\Sigma)$ to be the minimum of all the Chern–Simons invariants of non-trivial flat connections in the trivial SO(3)-bundle over Σ. We investigate the homology cobordism properties of this τ-invariant and show that the homology cobordism group Θ_3^H of oriented homology 3-spheres is infinitely generated. This was first proved by M. Furuta [**14**].

Finally, in the appendix, we list some computations of instanton homology groups for Brieskorn spheres.

2. *Representations of Seifert fibred homology spheres*

We shall describe in this section a technique for determining the components of $\mathcal{R}(\Sigma)$ for a Seifert fibred homology sphere Σ. Equivalently, we shall describe the representations into SO(3) of the orbifold fundamental group of the mapping cylinder of the orbit map $\Sigma \to S^2$. At the end of this section we give an alternative description of these representations in the special case where Σ has three exceptional fibres and relate the number of these representations to the signature of the Milnor fibre of Σ.

Let $\Sigma = \Sigma(a_1, \ldots, a_n)$ be a Seifert fibred homology sphere. It follows that $a_1, \ldots, a_n$ are pairwise relatively prime. To find the representations of $\pi_1(\Sigma)$ in SU(2) we shall apply a technique of S. Boyer [**8**]. We have the presentation

$$\pi_1(\Sigma) = \langle x_1, \ldots, x_n, h \mid h \text{ central}, x_i^{a_i} = h^{-b_i}, i = 1, \ldots, n, x_1 \ldots x_n = h^{-b_0} \rangle.$$

Here $b_0, b_1, \ldots, b_n$ are chosen so that

$$a\left(-b_0 + \sum_{i=1}^{n} \frac{b_i}{a_i}\right) = 1, \tag{2.0}$$

where $a = a_1 \dots a_n$. We shall say that Σ has Seifert invariants $\{b_0; (a_1, b_1), \dots, (a_n, b_n)\}$. (These are, of course, not unique.)

We are interested in irreducible (i.e. non-abelian) representations in SU(2). Since Σ is a homology sphere, this is equivalent to considering non-trivial representations.

LEMMA 2.1. *If α is an irreducible representation of $\pi_1(\Sigma)$ in* SU(2), *then* $\alpha(h) = \pm 1$.

Proof. If $\alpha(h) \neq \pm 1$, then conjugate in SU(2) to make

$$\alpha(h) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for some $\lambda \in S^1 \subset \mathbb{C}$. For any $g \in \pi_1(\Sigma)$, $\alpha(g)$ commutes with $\alpha(h)$. So $\alpha(g)$ preserves the λ-eigenspace of $\alpha(h)$. This means that for each $g \in \pi_1(\Sigma)$, $\alpha(g)$ lies in the same circle in SU(2) as $\alpha(h)$; hence α is reducible.

Let us now try to find irreducible representations with $\alpha(h) = H = \pm 1$. So $\alpha(x_i)^{a_i} = H^{-b_i} = \varepsilon_i = \pm 1$, for $i = 1, \dots, n$. Identify SU(2) with the unit quaternions S^3. Then for $A \in \mathrm{SU}(2)$, $\mathrm{trace}(A) = 2\,\mathrm{Re}(A)$ and $r\colon S^3 \to [0, \pi]$ given by $r(A) = \arccos(\mathrm{Re}(A))$ is a complete conjugacy invariant. We have $r(\alpha(x_i)) = l_i(\pi/a_i)$ where l_i is even if $\varepsilon_i = +1$ and is odd if $\varepsilon_i = -1$.

LEMMA 2.2. *If $\alpha\colon \pi_1(\Sigma(a_1, \dots, a_n)) \to \mathrm{SU}(2)$ is an irreducible representation, then at most $n-3$ of the $\alpha(x_i)$ are* ± 1.

Proof. If $\alpha(x_i) = \pm 1$ for $i = 3, \dots, n$, then, since $\alpha(x_1) \dots \alpha(x_n) = \alpha(h)^{b_0}$, $\alpha(x_1)\alpha(x_2) = \pm 1$. But $\alpha(x_1)^{a_1} = \pm 1$ and $\alpha(x_2)^{a_2} = \pm 1$. Since a_1 and a_2 are relatively prime, $\alpha(x_1) = \pm 1$, $\alpha(x_2) = \pm 1$. Hence $\alpha(\pi_1(\Sigma)) \subset \mathbb{Z}_2$, a contradiction.

After conjugation, we may assume that $X_1 = \alpha(x_1) = e^{ir(\alpha(x_1))} \in S^1 \subset S^3$. The possible choices for X_1 correspond to integers l_1 satisfying $0 \leq l_1 \leq a_1$ and l_1 is even if $\varepsilon_1 = +1$ and is odd if $\varepsilon_1 = -1$. Similarly, for each $i = 2, \dots, n$, choose $Y_i \in S^1 \subset S^3$ so that $Y_i = e^{ir(\alpha(x_i))}$. The choices for Y_i are described above with 'i' replacing '1' (with the proviso of Lemma 2.2). Let $S_i = S(i, l_i)$ denote the conjugacy class of Y_i in SU(2). Then $S_i = r^{-1}r(Y_i) \cong S^2$ (if $Y_i \neq \pm 1$). We must find $X_i = \alpha(x_i) \in S_i$, for $i = 2, \dots, n$, such that $X_1 \dots X_n = H^{-b_0}$. In other words, we must find $X_i \in S_i$, for $i = 2, \dots, n-1$, such that $(X_1 \dots X_{n-1})^{a_n} = H^{-b_0 a_n + b_n}$ $(= \pm 1)$.

Let $\varphi\colon S_2 \times \dots \times S_{n-1} \to [0, \pi]$ be the map

$$\varphi(Z_2, \dots, Z_{n-1}) = r(X_1 \cdot Z_2 \cdot \dots \cdot Z_{n-1}).$$

We must find X_i $(i = 2, \dots, n-1)$ such that $\varphi(X_2, \dots, X_{n-1}) = l'_n(\pi/a_n)$ where l'_n is even if $H^{-b_0 a_n + b_n} = +1$ and is odd otherwise.

In S^3 the product $S_2 \dots S_n$ is an $S^2 \times I$, B^3, or all of S^3, depending upon whether the interval $r(S_2 \dots S_{n-1})$ lies in $(0, \pi)$, contains one endpoint or both. Then $X_1 \cdot S_2 \dots S_{n-1}$ is the image under a rigid motion of S^3 taking 1 to X_1, and hence the image of φ in $[0, \pi]$ is easily determined. For definiteness, suppose that $H^{-b_0 a_n + b_n} = 1$. Then for each even l'_n such that $0 \leq l'_n \leq a_n$ and $l'_n(\pi/a_n) \in \mathrm{Im}\,\varphi$, we can find $(X_2, \dots, X_{n-1}) \in S_2 \times \dots \times S_{n-1}$ such that $r(X_1 X_2 \dots X_{n-1}) = l'_n(\pi/a_n)$. A

representation α is then determined by $\alpha(h) = H$, $\alpha(x_i) = X_i$ $(i = 1, \ldots, n-1)$, and its conjugacy class is given by the simultaneous conjugation of $X_2, \ldots, X_{n-1}$ by $S^1 \subset S^3$.

As a simple example consider $\Sigma(4, 3, 5)$. We have

$$\pi_1(\Sigma(4, 3, 5)) = \langle x, y, z, h \mid h \text{ central}, x^4 = h^1, y^3 = h^4, z^5 = h^2, xyz = h^2 \rangle.$$

Then $\alpha(h) = H = \pm 1$. Suppose $H = -1$. Now $\alpha(x)^4 = X^4 = -1$, and we set $X = e^{l\pi i/4}$, $l = 1, 3$. Then $Y = \alpha(y)$ is a 3rd root of 1, so its conjugacy class S_Y has $r(S_Y) = \frac{2}{3}\pi$. Because $\alpha(z) = \alpha((xy)^{-1}h^{-2}) = (XY)^{-1}$ and $\alpha(z)^5 = 1$, the equation $(XY)^5 = 1$ must be satisfied. Thus $r(XY) = \frac{2}{5}\pi$ or $\frac{4}{5}\pi$. Now $X \cdot S_Y$ is a 2-sphere centred at X and $(X \cdot S_Y) \cap r^{-1}(\frac{2}{5}\pi)$ is either empty or a circle (and similarly for $\frac{4}{5}\pi$). If $l = 1$ then $r(X \cdot S_Y) = [\frac{2}{3}\pi - \frac{1}{4}\pi, \frac{2}{3}\pi + \frac{1}{4}\pi]$ which contains $\frac{4}{5}\pi$ but not $\frac{2}{5}\pi$. Thus $H = -1$, $X = e^{\pi i/4}$, and $Y \in (X^{-1} \cdot r^{-1}(\frac{4}{5}\pi)) \cap S_Y$ determines a conjugacy class of representations, and for this class $r(Z) = \frac{4}{5}\pi$.

If $l = 3$, then $r(X \cdot S_Y) = [\frac{1}{12}\pi, \frac{7}{12}\pi]$ which contains $\frac{2}{5}\pi$ but not $\frac{4}{5}\pi$; so we obtain another representation, this time with $r(Z) = \frac{2}{5}\pi$. Similarly when $H = +1$ we obtain two more conjugacy classes of representations.

As in our example, it follows directly that:

PROPOSITION 2.3. *The representation space $\mathcal{R}(\Sigma(p, q, r))$ is finite.*

A Seifert fibred homology sphere $\Sigma = \Sigma(a_1, \ldots, a_n)$ admits a natural S^1-action whose orbit space is S^2. Orient Σ as the link of an algebraic singularity. Equivalently, orient Σ as a Seifert fibration with Seifert invariants $\{b_0; (a_i, b_i), i = 1, \ldots, n\}$ given by (2.0). With this orientation Σ bounds the canonical resolution, a negative definite simply connected smooth 4-manifold. Let $W = W(a_1, \ldots, a_n)$ denote the mapping cylinder of the orbit map. It is a 4-dimensional orbifold with boundary Σ and singularities cones on the lens spaces $L(a_i, b_i)$ (see **[10]**). Orient W so that its oriented boundary is $-\Sigma$. Then W has a positive definite intersection form. Let W_0 denote W with open cones around the singularities removed. Then

$$\begin{aligned}\pi_1(W_0) = \pi_1(\Sigma)/\langle h \rangle &= T(a_1, \ldots, a_n) \\ &= \langle x_1, \ldots, x_n \mid x_i^{a_i} = 1, i = 1, \ldots, n, x_1 \ldots x_n = 1 \rangle.\end{aligned}$$

When $n = 3$, this is the usual triangle group and in general it is a genus-zero Fuchsian group. Since Σ is a homology sphere, there is a one-to-one correspondence between representations α of $\pi_1(\Sigma)$ into SU(2) and representations, which we still call α, of $\pi_1(W_0)$ into SO(3).

Let $\mathbb{V}_\alpha$ denote the flat real 3-plane bundle over W_0 determined by α. When $\mathbb{V}_\alpha$ is restricted over $L(a_i, b_i) \subset \partial W_0$, it splits as $\mathbb{L}_{\alpha,i} \oplus \mathbb{R}$ where $\mathbb{R}$ is a trivial real line bundle and $\mathbb{L}_{\alpha,i}$ is a flat real 2-plane bundle corresponding to the representation $\pi_1(L(a_i, b_i)) \to \mathbb{Z}_{a_i}$ of weight l_i, where $r(\alpha(x_i)) = \pi l_i/a_i$. Here, the preferred generator of $\pi_1(L(a_i, b_i))$ corresponds to the deck transformation

$$(z, w) \mapsto (\zeta z, \zeta^{b_i} w)$$

of S^3 where $\zeta = e^{2\pi i/a_i}$. Thus $\mathbb{L}_{\alpha,i}$ is the quotient of $S^3 \times \mathbb{R}^2$ by the above action of $\mathbb{Z}_{a_i}$. The bundle $\mathbb{L}_{\alpha,i}$ extends over the cone $cL(a_i, b_i)$ as $(\mathbb{C}^2 \times_{\mathbb{Z}_{a_i}} \mathbb{R}^2) \oplus \mathbb{R}$, an SO(3)-$V$-bundle whose rotation number over the cone point is l_i (with respect to the preferred generator given above). Thus we obtain an SO(3)-V-vector bundle, which we also denote by $\mathbb{V}_\alpha$, over W. (See **[11]** for V-bundles.)

An SO(3)-V-bundle has a well-defined second Stiefel–Whitney class w_2 which is the obstruction to lifting it to an SU(2)-V-bundle. This means that $w_2(\mathbb{V}_\alpha) = 0$ if $H = \alpha(h) = +1$ and $w_2(\mathbb{V}_\alpha) \neq 0$ if $H = -1$.

PROPOSITION 2.4. *Let $\Sigma = \Sigma(a_1, \ldots, a_n)$ with a_1 even, and let α: $\pi_1(\Sigma) \to \mathrm{SU}(2)$ be a representation with rotation number l_1 corresponding to a_1. Then $w_2(\mathbb{V}_\alpha) = 0$ if and only if l_1 is even.*

Proof. The equality $-b_0 a + \sum_{i=1}^n ab_i/a_i = 1$ implies that b_1 is odd, so $X^{a_1} = H$. Thus l_1 is even if and only if $H = +1$.

Given a representation α: $\pi_1(\Sigma) \to \mathrm{SU}(2)$, its Zariski tangent space in the space of all conjugacy classes of such representations is $H^1(\Sigma\,; \mathbb{V}_\alpha)$ (where $\mathbb{V}_\alpha$ also denotes $\mathbb{V}_\alpha$ restricted to Σ). This is the case since $\mathbb{V}_\alpha$ is the $\mathbb{R}^3$-bundle associated to the representation α via the adjoint action of SU(2).

The quotient Σ/S^1 of the natural S^1-action on $\Sigma = \Sigma(a_1, \ldots, a_n)$ is the 2-sphere S^2 with an induced orbifold (V-manifold) structure. The orbifold fundamental group of Σ/S^1 is just $T(a_1, \ldots, a_n) = \pi_1(W_0)$.

In the following proposition we use the presentation for $\pi_1(\Sigma)$ given at the beginning of this section.

PROPOSITION 2.5. *Let α: $\pi_1(\Sigma) \to \mathrm{SU}(2)$ be a representation with $\alpha(x_i) \neq \pm 1$ for $i = 1, \ldots, m$ and $\alpha(x_i) = \pm 1$ for $i = m+1, \ldots, n$. Then*

$$\dim_{\mathbb{R}} H^1(\pi_1(\Sigma)\,; \mathbb{V}_\alpha) = 2m - 6.$$

(Here $H^1(\pi_1(\Sigma)\,; \mathbb{V}_\alpha)$ denotes group cohomology with coefficients in the adjoint representation associated to α.)

Proof. Let α also denote the corresponding SO(3) representation. The 1-cocycles of $\pi_1(\Sigma)$ with coefficients in $\mathbb{V}_\alpha$ are crossed homomorphisms z: $\pi_1(\Sigma) \to \mathfrak{su}(2) \cong \mathbb{R}^3$. Let $z_i = z(x_i)$; then $z_i = 0$ for $i = m+1, \ldots, n$ and $z_1, \ldots, z_m$ satisfy the relations

$$(*) \qquad (I + \alpha(x_i) + \alpha(x_i)^2 + \ldots + \alpha(x_i)^{a_i - 1})z_i = 0,$$

$$(**) \qquad z_1 + \alpha(x_1)z_2 + \alpha(x_1x_2)z_3 + \ldots + \alpha(x_1 \ldots x_{m-1})z_m = 0.$$

For $i = 1, \ldots, m$, each $\alpha(x_i) \in \mathrm{SO}(3)$ has an axis $\mathbb{R}_i$ and perpendicular plane of rotation

$$\mathbb{C}_i = \ker(I + \alpha(x_i) + \alpha(x_i)^2 + \ldots + \alpha(x_i)^{a_i - 1}).$$

To satisfy $(*)$ we need to choose $z_i \in \mathbb{C}_i$, for $i = 1, \ldots, m$. This gives $2m$ degrees of freedom in the choice of the crossed homomorphism z, but these are still subject to $(**)$.

Consider the linear map L: $\mathbb{C}_1 \oplus \ldots \oplus \mathbb{C}_m \to \mathbb{R}^3$ given by

$$L(z_1, \ldots, z_m) = z_1 + \alpha(x_1)z_2 + \alpha(x_1x_2)z_3 + \ldots + \alpha(x_1 \ldots x_{m-1})z_m.$$

Since α is irreducible, at least two of the axes, say $\mathbb{R}_1$ and $\mathbb{R}_2$, are distinct. The axis $\mathbb{R}_1$ is fixed by $\alpha(x_1)$, so $\mathbb{R}_1$ and $\alpha(x_1)\mathbb{R}_2$ are distinct. Hence their

perpendicular planes $\mathbb{C}_1$ and $\alpha(x_1)\mathbb{C}_2$ span $\mathbb{R}^3$. Thus L is surjective and the relation $(**)$ constrains three of the degrees of freedom in the choice of z. The space of cocycles has dimension $2m-3$.

The coboundaries are the principal crossed homomorphisms; that is, those of the form

$$z(x) = v - \alpha(x)v$$

for fixed $v \in \mathbb{R}^3$. This is 3-dimensional; so $\dim_{\mathbb{R}} H^1(\pi_1(\Sigma)\,;\mathbb{V}_\alpha) = 2m-6$.

LEMMA 2.6. *There are isomorphisms*

$$H^1(\pi_1(\Sigma)\,;\mathbb{V}_\alpha) \cong H^1(\pi_1(W_0)\,;\mathbb{V}_\alpha) \cong H^1(\Sigma\,;\mathbb{V}_\alpha) \cong H^1(W_0\,;\mathbb{V}_\alpha)$$

and $H^2(W_0\,;\mathbb{V}_\alpha) \to H^2(\Sigma\,;\mathbb{V}_\alpha)$ *is injective.*

Proof. That $H^1(\pi_1(\Sigma)\,;\mathbb{V}_\alpha) \cong H^1(\Sigma\,;\mathbb{V}_\alpha)$ and $H^1(\pi_1(W_0)\,;\mathbb{V}_\alpha) \cong H^1(W_0\,;\mathbb{V}_\alpha)$ is a standard fact. That $H^1(\pi_1(W_0)\,;\mathbb{V}_\alpha) \cong H^1(\pi_1(\Sigma)\,;\mathbb{V}_\alpha)$ follows from the proof of Proposition 2.5.

To verify the last injection, let E_i $(i = 1, \ldots, m)$ denote those exceptional fibres in Σ which correspond to the generators x_i of $\pi_1(\Sigma)$ for which $\alpha(x_i) \neq \pm 1$. Then let Σ_e be the closure of Σ with the union S of solid tori neighbourhoods of these m exceptional fibres E_i removed. Finally, let W_e be the mapping cylinder of the fibration $\Sigma_e \to S^2 - \{m \text{ points}\}$. Then W_0 has the homotopy type of $W_1 = \Sigma \cup_{\Sigma_e} W_e$. Applying the Mayer–Vietoris sequence to the spaces $W_1 = W_e \cup_{\partial S} S$ and $\Sigma = \Sigma_e \cup_{\partial S} S$, we have the commutative diagram of cohomology groups (with coefficients in $\mathbb{V}_\alpha$) with exact rows

$$\begin{array}{ccccccc}
H^1(W_e) \oplus H^1(S) & \longrightarrow & H^1(\partial S) & \longrightarrow & H^2(W_1) & \longrightarrow & H^2(W_e) \oplus H^2(S) \\
\downarrow f & & \downarrow i & & \downarrow j & & \downarrow g \\
H^1(\Sigma_e) \oplus H^1(S) & \longrightarrow & H^1(\partial S) & \longrightarrow & H^2(\Sigma) & \longrightarrow & H^2(\Sigma_e) \oplus H^2(S)
\end{array}$$

Since W_e deformation retracts to $S^2 - \{m \text{ points}\}$, $H^2(W_e) = 0$ and clearly $H^2(S) = 0$. Also, since $m \geq 3$, α restricted to $S^2 - \{m \text{ points}\}$ is irreducible; so $H^0(W_e\,;\mathbb{V}_\alpha) = H^0(S^2 - \{m \text{ points}\}) = 0$, and a Gysin sequence shows that $H^1(W_e\,;\mathbb{V}_\alpha) \cong H^1(\Sigma_e\,;\mathbb{V}_\alpha)$ and f is an isomorphism. Clearly i is an isomorphism, and a diagram chase shows that j is injective.

For Seifert homology spheres (Brieskorn complete intersections) $\Sigma(a_1, \ldots, a_n)$ $(n \geq 4)$ we have:

PROPOSITION 2.7. *If* α: $\pi_1(\Sigma) \to \mathrm{SU}(2)$ *is a representation with* $\alpha(a_i) \neq \pm 1$ *for* $i = 1, \ldots, m$, $\alpha(a_i) = \pm 1$ *for* $i = m+1, \ldots, n$, *then the connected component* $\mathcal{R}_\alpha$ *of* α *in the space* $\mathcal{R}(\Sigma)$ *is a closed manifold of dimension* $2m-6$.

Proof. We have seen above that each α: $\pi_1(\Sigma) \to \mathrm{SU}(2)$ gives rise to a representation α: $\pi_1(W_0) \to \mathrm{SO}(3)$ and in turn to a flat $\mathrm{SO}(3)$-V-vector bundle $\mathbb{V}_\alpha$ over W. Thus we have its restriction $\mathbb{V}_\alpha$ over $S^2 = \Sigma/S^1$, a flat V-bundle. The orbifold Σ/S^1 has orbifold fundamental group $\Gamma = \pi_1(W_0)$, and fixing a basepoint

of Σ/S^1 off its singular set we get a holonomy map $\Gamma \to SO(3)$. This map is induced from

$$\begin{tikzcd} \pi_1(\Sigma/S^1 - \{\text{singular points}\}) \arrow[rr] \arrow[dr] & & SO(3) \arrow[dl] \\ & \Gamma & \end{tikzcd}$$

where the top map is the holonomy of the flat bundle $\mathbb{V}_\alpha$ restricted over the m-punctured sphere $\Sigma/S^1 - \{\text{singular points}\}$. Using the holonomy map $\Gamma \to SO(3)$ we get the standard identification of gauge-equivalence classes of flat SO(3)-V-vector bundles over the orbifold Σ/S^1 with conjugacy classes of representations $\Gamma \to SO(3)$, that is, with $\mathcal{R}(\Sigma)$.

Since $\mathcal{R}(\Sigma(p, q, r))$ is finite by Proposition 2.3, we may assume that $n \geq 4$. In this case it is well-known that the universal covering orbifold of Σ/S^1 is the hyperbolic plane $\mathbb{H}^2$, and $\Sigma/S^1 = \mathbb{H}^2/\Gamma$ with Γ acting as a discontinuous group of hyperbolic motions. Thus over Σ/S^1, $\mathbb{V}_\alpha$ can be identified with $\mathbb{H}^2 \times_\Gamma \mathbb{R}^3$, where α: $\Gamma \to SO(3)$ gives the action on $\mathbb{R}^3$.

On Σ/S^1 we have cohomology with local coefficients $H^*(\Sigma/S^1; \mathbb{V}_\alpha)$ which is simply the Γ-equivariant cohomology $H_*(\mathrm{Hom}_\Gamma(C_*(\mathbb{H}^2), \mathbb{R}^3))$. Since we are dealing with orthogonal representations, we get a pairing $\mathbb{V}_\alpha \otimes \mathbb{V}_\alpha \to \mathbb{R}$ induced from the inner product on the fibres. So we have a cup-product pairing

$$H^0(\Sigma/S^1; \mathbb{V}_\alpha) \otimes H^2(\Sigma/S^1; \mathbb{V}_\alpha) \to H^2(\Sigma/S^1; \mathbb{R}) = \mathbb{R},$$

which is non-degenerate since Poincaré duality holds on Σ/S^1. Since α is an irreducible representation, $H^0(\Sigma/S^1; \mathbb{V}_\alpha) = 0$; so the above pairing implies $H^2(\Sigma/S^1; \mathbb{V}_\alpha) = 0$.

Near the equivalence class of α, the space of gauge-equivalence classes of flat SO(3)-V-bundles over Σ/S^1 is given by the zero set of the Kuranishi map:

$$H^1(\Sigma/S^1; \mathbb{V}_\alpha) \to H^2(\Sigma/S^1; \mathbb{V}_\alpha).$$

But $H^2(\Sigma/S^1; \mathbb{V}_\alpha) = 0$ and $H^1(\Sigma/S^1; \mathbb{V}_\alpha) \cong H^1(\Gamma; \mathbb{V}_\alpha) = H^1(\pi_1(W_0); \mathbb{V}_\alpha)$ has dimension $2m - 6$ by Proposition 2.5.

We now give an alternative description of the representations of $\pi_1(\Sigma(p, q, r))$ into SU(2). By Lemma 2.1 and the fact that $\Sigma(p, q, r)$ is a $\mathbb{Z}_2$-homology sphere, there is a natural identification of representations of $\pi_1(\Sigma(p, q, r))$ into SU(2) with representations of $T(p, q, r)$ into SO(3) where

$$T(p, q, r) = \langle a, b, c \mid a^p = b^q = c^r = abc = 1 \rangle$$

is the orbifold fundamental group of the mapping cylinder of the orbit map $\Sigma \to S^2$.

Let α: $T(p, q, r) \to SO(3)$ be a homomorphism. Then $\alpha(a)$ is a rotation of S^2 with antipodal fixed points f_a and g_a. Let $S(a, b)$ be the great circle in S^2 containing f_a and f_b. Then $S(a, b)$, $S(b, c)$, and $S(a, c)$ divide S^2 into four antipodal pairs of spherical triangles. The angles of these four sets of triangles are of the form

$$(\pi k/p, \pi l/q, \pi m/r), \quad (\pi(p-k)/p, \pi(q-l)/q, \pi m/r),$$
$$(\pi(p-k)/p, \pi l/q, \pi(r-m)/r), \quad \text{and} \quad (\pi k/p, \pi(q-l)/q, \pi(r-m)/r).$$

Since the sum of the angles of each triangle must be larger than π, we have the inequalities

$$\frac{\pi k}{p}+\frac{\pi l}{q}+\frac{\pi m}{r}>\pi, \quad \frac{\pi k}{p}-\frac{\pi l}{q}+\frac{\pi m}{r}<\pi,$$

$$\frac{\pi k}{p}+\frac{\pi l}{q}-\frac{\pi m}{r}<\pi, \quad \frac{-\pi k}{p}+\frac{\pi l}{q}+\frac{\pi m}{r}<\pi.$$

We now assume that q and r are odd integers. To isolate only one of these eight spherical triangles it suffices to find a triple of integers (k, l, m) such that

(1) l and m have the same parity,

(2) $$\frac{l}{q}+\frac{m}{r}<1,$$

(3) $$\frac{k}{p}+\frac{l}{q}+\frac{m}{r}>1,$$

(4) $$\frac{k}{p}-\frac{l}{q}+\frac{m}{r}<1,$$

(5) $$\frac{k}{p}+\frac{l}{q}-\frac{m}{r}<1,$$

PROPOSITION 2.8. *There is a one-to-one correspondence between elements of* $\mathcal{R}(\Sigma(p, q, r))$ *and triples of integers* (k, l, m), *with* $0<k<p$, $0<l<q$, $0<m<r$, *satisfying* (1)–(5) *above.*

The Brieskorn homology sphere

$$\Sigma(p, q, r) \cong \{(z_1, z_2, z_3) \in \mathbb{C}^3 |\ z_1^p+z_2^q+z_3^r=0\} \cap S^5$$

as a link of a singularity bounds the Brieskorn manifold

$$B(p, q, r)=\{(z_1, z_2, z_3) \in \mathbb{C}^3 |\ z_1^p+z_2^q+z_3^r=\varepsilon\}.$$

Brieskorn [6] showed that the signature of $B(p, q, r)$ is given by $\sigma_+ - \sigma_-$ where

$$\sigma_+ = \#\left\{(k, l, m) |\ 0<\frac{k}{p}+\frac{l}{q}+\frac{m}{r}<1 \pmod 2\right\},$$

$$\sigma_- = \#\left\{(k, l, m) |\ -1<\frac{k}{p}+\frac{l}{q}+\frac{m}{r}<0 \pmod 2\right\}.$$

COROLLARY 2.9. $\mathcal{R}(\Sigma(p, q, r))$ *contains* $-\sigma(B(p, q, r))/4$ *elements.*

Proof. Let $|X|$ denote the number of elements in a finite set X. If

$$X_i=\left\{(k, l, m) |\ i<\frac{k}{p}+\frac{l}{q}+\frac{m}{r}<i+1\right\},$$

then

$$-\sigma(B(p, q, r)) = -|X_0| + |X_1| - |X_2|.$$

Let $S = \{(k, l, m) \mid 0 < k < p, 0 < l < q, 0 < m < r\}$. Then

$$|S| = |X_0| + |X_1| + |X_2| = (p-1)(q-1)(r-1).$$

Let $\mathcal{R}_i$ denote the set of elements of S that satisfy Conditions (1) through (i) above. The bijection of S given by $(k, l, m) \mapsto (p-k, q-l, r-m)$ maps X_0 to X_2 so that $|X_0| = |X_2|$ and hence

$$-\sigma(B(p, q, r)) = |X_1| - 2\,|X_0| = |S| - 4\,|X_0|.$$

We need to show that $|\mathcal{R}(\Sigma(p, q, r))| = \frac{1}{4}\,|S| - |X_0|$. The bijection of S given by $(k, l, m) \mapsto (k, l, r-m)$ maps $\mathcal{R}_1$ to $S - \mathcal{R}_1$ so that $|\mathcal{R}_1| = \frac{1}{2}\,|S|$. The bijection of $\mathcal{R}_1$ given by $(k, l, m) \mapsto (k, q-l, r-m)$ maps $\mathcal{R}_2$ to $\mathcal{R}_1 - \mathcal{R}_2$ so that $|\mathcal{R}_2| = \frac{1}{4}\,|S|$. Furthermore,

$$\mathcal{R}_2 - \mathcal{R}_3 = \{(k, l, m) \mid (k, l, m) \in X_0,\ l \text{ and } m \text{ have the same parity}\}.$$

The transformations $(k, l, m) \mapsto (p-k, q-l, m)$ (if $l/q < m/r$) or $(k, l, m) \mapsto (p-k, l, r-m)$ (if $l/q > m/r$) gives a bijection between $\mathcal{R}_3 - \mathcal{R}_5$ and $\{(k, l, m) \mid (k, l, m) \in X_0,\ l \text{ and } m \text{ have the opposite parity}\}$. Thus

$$\begin{aligned}\tfrac{1}{4}\,|S| = |\mathcal{R}_2| &= |\mathcal{R}_2 - \mathcal{R}_5| + |\mathcal{R}_5| = |\mathcal{R}_2 - \mathcal{R}_3| + |\mathcal{R}_3 - \mathcal{R}_5| + |\mathcal{R}_5| \\ &= |X_0| + |\mathcal{R}(\Sigma(p, q, r))|\end{aligned}$$

and the result follows.

We thank David Austin for help with the combinatorics in the proof of Corollary 2.9. This result has also been proved by J. Wahl.

It will follow from Proposition 3.10 that the Casson invariant of a Brieskorn sphere Σ (with our orientation) is $-\frac{1}{2}\,|\mathcal{R}(\Sigma)|$. Thus we have

THEOREM 2.10. *The Casson invariant*

$$\lambda(\Sigma(p, q, r)) = \tfrac{1}{8}\sigma(B(p, q, r)).$$

See also Theorem 4.3.

3. *Instanton homology of Brieskorn spheres*

Again, let $\Sigma = \Sigma(a_1, \ldots, a_n)$. A representation $a \in \mathcal{R}(\Sigma)$ is *non-degenerate* if $H^1(\Sigma; \mathbb{V}_\alpha)$ is 0. For the Brieskorn homology sphere $\Sigma(p, q, r)$, Lemma 2.2 and Proposition 2.5 imply that each irreducible representation is non-degenerate. Furthermore, by Proposition 2.3 there are finitely many conjugacy classes of such representations.

Let $\mathcal{B}_\Sigma$ be the Banach manifold of L_1^4-SU(2)-connections over Σ modulo L_2^4-gauge equivalence, and consider $\mathcal{R}(\Sigma)$ as a submanifold of $\mathcal{B}_\Sigma$ where $\alpha \in \mathcal{R}(\Sigma)$ corresponds to the flat connection $a_\alpha \in \mathcal{B}_\Sigma$. The instanton chain complex is the free chain complex generated by the irreducible (i.e. non-trivial) representations in $\mathcal{R}(\Sigma)$. Floer assigns to each such representation α a grading $\mu(\alpha) \in \mathbb{Z}_8$ as follows.

Fix a Riemannian metric on Σ. For any connection a in the trivial real 3-plane bundle over Σ let D_a denote the elliptic operator on the bundle $(\Omega^0 \oplus \Omega^1) \otimes \mathfrak{su}(2)$ defined by

$$D_a(\phi, \tau) = (d_a^* \tau, d_a \phi + \star d_a \tau).$$

Here, d_a is the covariant derivative determined by the connection a, d_a^* its adjoint, and $\star$ is the Hodge star operator. The grading $\mu(\alpha) \in \mathbb{Z}_8$ of a non-degenerate irreducible representation $\alpha \in \mathcal{R}(\Sigma)$ is the spectral flow $\mathrm{SF}(\theta, \alpha)$ of the family of operators D_{a_t} where a_t is a path of connections from the trivial SU(2) connection θ to the flat connection a_α determined by α. Intuitively, it is the net number of eigenvalues of D_{a_t} whose sign changes from negative to positive as a_t varies from θ to a_α [**5**]. It is well-defined (mod 8) on $\mathcal{B}_\Sigma$.

More rigorously, define the $L^p_{q;\delta}$-norm of a section ξ of some bundle over $\Sigma \times \mathbb{R}$ to be the L^p_q-norm of $e_\delta \cdot \xi$ where e_δ is a smooth positive function on $\Sigma \times \mathbb{R}$ such that $e_\delta(x, t) = e^{\delta|t|}$ for $|t| \geq 1$. For any SU(2) connection A (on the trivial bundle) over $\Sigma \times \mathbb{R}$ consider the self-duality operator

$$d_A^* \oplus d_A^-\colon L^4_{1;\delta}(\Omega^1(\Sigma \times \mathbb{R}) \otimes \mathfrak{su}(2)) \to L^4_{0;\delta}((\Omega^0(\Sigma \times \mathbb{R}) \oplus \Omega^2_-(\Sigma \times \mathbb{R})) \otimes \mathfrak{su}(2)).$$

For small enough positive δ, this operator will be Fredholm (cf. [**13**, **17**]). A family of connections $\{a_t\colon 0 \leq t \leq 1\}$ in $\mathcal{B}_\Sigma$ defines a connection A over $\Sigma \times \mathbb{R}$ via $d_{A(t)} = d_{a_t} + \partial/\partial t$. Then the self-duality operator is $d_A^* \oplus d_A^- = D_{a_t} + \partial/\partial t$.

If $a_0 = a_\alpha$ and $a_1 = a_\beta$ where α and β are non-degenerate irreducible representations, then the spectral flow $\mathrm{SF}(\alpha, \beta)$ is (modulo 8) the index of the Fredholm operator

$$d_A^* \oplus d_A^-\colon L^4_{1;\delta}(\Omega^1(\Sigma \times \mathbb{R}) \otimes \mathfrak{su}(2)) \to L^4_{0;\delta}((\Omega^0(\Sigma \times \mathbb{R}) \oplus \Omega^2_-(\Sigma \times \mathbb{R})) \otimes \mathfrak{su}(2)),$$

where A is extended to be asymptotically constant for $t \leq 0$ and $t \geq 1$ and where $\delta > 0$ is smaller than the smallest non-zero absolute value of an eigenvalue of D_{a_α} or D_{a_β} [**13**, 2b.2]. Let us denote this index by $\mathrm{Ind}_\delta(d_A^* \oplus d_A^-)(\alpha, \beta)$. In case either of the representations α or β is the trivial representation θ, there is a sign convention to worry about since D_θ has a 3-dimensional null space. Floer defines the spectral flow in this instance to be the net number of eigenvalues which change value from less than $-\delta$ at $t = 0$ to greater than δ at $t = 1$. This definition is independent (mod 8) of the chosen path. So, for example, $\mathrm{SF}(\theta, \theta) = -3 \pmod 8$. Then Floer defines the grading $\mu(\alpha) \equiv \mathrm{SF}(\theta, \alpha) \pmod 8$ for an irreducible non-degenerate α. (Note that there is a sign error in the statement of [**13**, 2b.2].)

One can also consider the calculation of the index of the self-duality operator over $\Sigma \times \mathbb{R}$ as a boundary value problem, as was done by Atiyah, Patodi, and Singer [**3**, **4**, **5**]. They impose global boundary conditions and compute the index

$$\begin{aligned}(3.1)\qquad \mathrm{Index}(d_A^* \oplus d_A^-)(\alpha, \beta) = &\int_{\Sigma \times \mathbb{R}} \hat{A}(\Sigma \times \mathbb{R}) \mathrm{ch}(V_-) \mathrm{ch}(\mathfrak{g}) \\ &- \tfrac{1}{2}(h_\beta + \eta_\beta(0)) + \tfrac{1}{2}(-h_\alpha + \eta_\alpha(0)),\end{aligned}$$

where the forms $\hat{A}(\Sigma \times \mathbb{R})$ and $\mathrm{ch}(V_-)$ are computed from the Riemannian connection on $\Sigma \times \mathbb{R}$ (choose a product metric, say) and $\mathfrak{g}$ is the SO(3) bundle over $\Sigma \times \mathbb{R}$ with connection A induced from $\{a_t\}$. The term h_β is the sum of the dimensions of $H^i(\Sigma; \mathbb{V}_\beta)$ $(i = 0, 1)$, and η_β is the η-invariant of the signature operator $\star d_{a_\beta} - d_{a_\beta} \star$ over Σ restricted to even forms [**3**, § 3].

LEMMA 3.2. *Let $\{a_t\colon 0 \leq t \leq 1\}$ be a 1-parameter family of connections in $\mathcal{B}_\Sigma$ connecting $a_0 = a_\alpha$ to $a_1 = \theta$ where α is irreducible and non-degenerate. Then the spectral flow of D_{a_t} is*

$$\mathrm{SF}(\alpha, \theta) = \mathrm{Index}(d_A^* \oplus d_A^-)(\alpha, \theta),$$

the Atiyah–Patodi–Singer index of the self-duality operator over $\Sigma \times [0, 1]$. The grading $\mu(\alpha)$ in the instanton chain complex is given by

$$\mu(\alpha) \equiv -\mathrm{Index}(d_A^* \oplus d_A^-)(\alpha, \theta) - 3 \pmod 8.$$

Proof. Consider first the case where $a_1 = a_\beta$ for β non-degenerate and irreducible in $\mathcal{R}(\Sigma)$. Suppose $\xi \in \ker(d_A^* \oplus d_A^-)$. Since $D_{A(t)} = D_{a_\beta}$ for $t \geq 1$, we can expand $\xi(x, t)$ for $t \geq 1$ in terms of an orthonomal basis $\{\varphi_\lambda\}$ of eigenforms of D_{a_β}; that is, $\xi(x, t) = \sum \xi_\lambda^+(t)\varphi_\lambda(x)$. Since $\partial\xi/\partial t + D_{a_\beta}\xi = 0$, we have $\xi_\lambda^+(t) = e^{-\lambda t}\xi_\lambda^+(1)$. Thus if $\xi \in L_{1;\delta}^4$ then $e^{\delta t} \in L_1^4$ for $t \to +\infty$; so for each λ in the sum we must have $\delta < \lambda$. Thus $\xi(x, t) = \sum_{\lambda > 0} e^{-\lambda t}\xi_\lambda^+(1)\varphi_\lambda(x)$. (Recall that $\delta > 0$ is smaller than the smallest non-zero absolute value of an eigenvalue of D_{a_α} or D_{a_β}.) Similarly $(d_A^* \oplus d_A^-)^* = D_{a_\beta} - \partial/\partial t$ for $t \geq 1$, if $\xi \in \ker(d_A^* \oplus d_A^-)^* \cap L_{0;\delta}^4$ then $\xi(x, t) = \sum_{\lambda < 0} e^{\lambda t}\xi_\lambda^+(1)\varphi_\lambda(x)$ for $t \geq 1$.

The same argument applies at the other end. If $\{\psi_\lambda\}$ is an orthonormal basis of eigenforms of D_{a_α}, we have

$$\xi(x, t) = \sum_{\lambda < 0} e^{-\lambda t}\xi_\lambda^+(0)\psi_\lambda(x)$$

if $\xi \in \ker(d_A^* \oplus d_A^-) \cap L_{1;\delta}^4$, and

$$\xi(x, t) = \sum_{\lambda > 0} e^{\lambda t}\xi_\lambda^+(0)\psi_\lambda(x)$$

if $\xi \in \ker(d_A^* \oplus d_A^-)^* \cap L_{0;\delta}^4$ for $t \leq 0$.

The fact that $\mathrm{SF}(\alpha, \beta)$ equals $\mathrm{Ind}_\delta(d_A^* \oplus d_A^-)(\alpha, \beta)$ means that the dimensions behave as if a change in the sign of an eigenvalue from negative to positive as t varies from 0 to 1 counts basis elements of $\ker(d_A^* \oplus d_A^-) \cap L_{1;\delta}^4$, and a change in the sign from positive to negative counts basis elements of $\ker(d_A^* \oplus d_A^-)^* \cap L_{0;\delta}^4$. (See Fig. 1.)

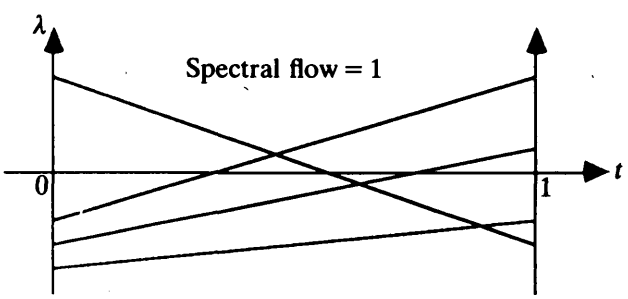

FIG. 1

Now consider the case where $a_0 = a_\alpha$ and $a_1 = \theta$, where α is non-degenerate and irreducible in $\mathcal{R}(\Sigma)$. Then for $t < 1$ and close to 1 there are three small eigenvalues of $D_{a(t)}$. Then according to Floer's convention for spectral flow, any eigenvalue λ which changes from $\lambda > \delta$ at $t = 0$ to $\lambda = 0$ at $t = 1$ is counted

negatively, whereas an eigenvalue which changes from $\lambda < \delta$ at $t = 0$ to $\lambda = 0$ at $t = 1$ is not counted. (See Fig. 2.)

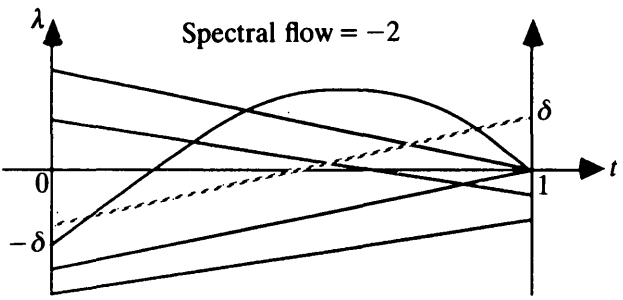

FIG. 2

The eigenvalues which flow from positive at $t = 0$ to 0 at $t = 1$ correspond, under the above-described relationship between index and spectral flow, to basis forms of $\ker(d_A^* \oplus d_A^-)^*$ which are not in $L^4_{0;\delta}$, but which are extended $L^4_{0;\delta}$ forms. That is, for large positive t, such forms are the sum $\xi(x, t) + \zeta(x)$ of an $L^4_{0;\delta}$ form ξ and an element ζ of the kernel of D_θ. Thus

$$\mathrm{SF}(\alpha, \theta) = \mathrm{Ind}_\delta(d_A^* \oplus d_A^-)(\alpha, \theta) - h_\infty(F),$$

where $h_\infty(F)$ is the dimension of the subspace of $\ker D_\theta$ consisting of limiting values of extended $L^4_{0;\delta}$ forms ξ in $(\Omega^0(\Sigma \times \mathbb{R}) \oplus \Omega^2_-(\Sigma \times \mathbb{R})) \otimes \mathfrak{su}(2)$ which satisfy $(d_A^* \oplus d_A^-)^*(\xi) = 0$. Thus according to Atiyah, Patodi, and Singer [**3**, 3.14],

$$\mathrm{SF}(\alpha, \theta) = \mathrm{Index}(d_A^* \oplus d_A^-)(\alpha, \theta).$$

Now Floer has shown that for α irreducible and non-degenerate,

$$-3 = \mathrm{SF}(\theta, \theta) = \mathrm{SF}(\theta, \alpha) + \mathrm{SF}(\alpha, \theta).$$

So $\mu(\alpha) \equiv \mathrm{SF}(\theta, \alpha) \equiv -\mathrm{SF}(\alpha, \theta) - 3 \equiv -\mathrm{Index}(d_A^* \oplus d_A^-)(\alpha, \theta) - 3 \pmod 8$.

Our goal in this section is to compute this instanton index $\mu(\alpha)$ for representations of Seifert-fibred homology spheres. This will give the instanton homology for Brieskorn spheres, since, as we shall see, the boundary operator of the instanton complex is always trivial for a Brieskorn sphere.

PROPOSITION 3.3. *Let $\Sigma = \Sigma(a_1, \ldots, a_n)$ be any Seifert-fibred homology sphere and let α: $\pi_1(\Sigma) \to \mathrm{SU}(2)$ be an irreducible representation. Let $\mathbb{V}_\alpha$ be the induced flat* SO(3)-*V-bundle over the orbifold W with flat connection A_α. (Here we view W as an open orbifold with end $\Sigma \times \mathbb{R}$ with product metric.) Then over W_0 the index of the self-duality operator is*

$$\mathrm{Ind}_\delta(d_{A_\alpha}^* + d_{A_\alpha}^-)(W_0) = 0.$$

Proof. By [**3**, Proposition 4.9] the (twisted) L^2_δ-cohomology of W_0 is the image of the relative cohomology $H^*(W_0, \Sigma \cup L; \mathbb{V}_\alpha)$ in the absolute cohomology $H^*(W; \mathbb{V}_\alpha)$. Here '$L$' denotes the disjoint union of the lens space boundary components of W_0. Because over each lens space $\mathbb{V}_\alpha$ lifts to a bundle over S^3, it

follows that $H^1(L\,;\mathbb{V}_\alpha)=0$. Thus Lemma 2.6 implies that for the complex

$$0\longrightarrow \Omega^0(\mathfrak{g})\xrightarrow{d_{A_\alpha}}\Omega^1(\mathfrak{g})\xrightarrow{d^-_{A_\alpha}}\Omega^2_-(\mathfrak{g})\longrightarrow 0$$

(here $\mathfrak{g}\cong\mathbb{V}_\alpha$) over W_0, the betti numbers of the L^2_δ-harmonic forms are $h^1=h^2=0$. Furthermore, $h^0=0$, because A_α is irreducible and so has no covariant constant sections. Thus the L^2_δ-index of $d^*_{A_\alpha}+d^-_{A_\alpha}$ is $h^1-h^0-h^2=0$. Because we are here discussing smooth harmonic forms, $\mathrm{Ind}_\delta(d^*_{A_\alpha}+d^-_{A_\alpha})(W_0)=0$ as well.

Again by [**3**, 3.14] we have

$$\mathrm{Index}(d^*_{A_\alpha}+d^-_{A_\alpha})(W_0)=\mathrm{Ind}_\delta(d^*_{A_\alpha}+d^-_{A_\alpha})(W_0)-h_\infty(F)_{W_0},$$

where $h_\infty(F)_{W_0}$ is the dimension of the subspace of $\ker D_{a_\alpha}$ consisting of limiting values of extended $L^4_{0;\delta}$ sections f of $(\Omega^0\oplus\Omega^2_-)(\mathfrak{g})$ satisfying $(d^*_{A_\alpha}+d^-_{A_\alpha})^*(f)=0$. Now any f satisfying $(d^*_{A_\alpha}+d^-_{A_\alpha})^*(f)=0$ lies in $H^0(W_0\,;\mathbb{V}_\alpha)\oplus H^2_-(W_0\,;\mathbb{V}_\alpha)$. By Propositions 2.5 and 2.6 we know that the betti number $h_1(W_0\,;\mathbb{V}_\alpha)=2m-6$. But also

$$\sum_{i=0}^{4}(-1)^i h_i(W_0\,;\mathbb{V}_\alpha)=3\chi(W_0)=6-3n.$$

Further, we know that $h_0(W_0\,;\mathbb{V}_\alpha)=0$; and also $h_4(W_0\,;\mathbb{V}_\alpha)=0$. Thus

$$6-3n=-(2m-6)+h_2(W_0\,;\mathbb{V}_\alpha)-h_3(W_0\,;\mathbb{V}_\alpha).$$

By duality, $h_3(W_0\,;\mathbb{V}_\alpha)=h_1(W_0,\Sigma\cup L\,;\mathbb{V}_\alpha)$, and the exact sequence for the pair $(W_0,\Sigma\cup L)$ along with Proposition 2.6 shows that $h_1(W_0,\Sigma\cup L\,;\mathbb{V}_\alpha)=3n-2m$. Thus $h_2(W_0\,;\mathbb{V}_\alpha)=0$ and so $H^2_-(W_0\,;\mathbb{V}_\alpha)=0$. This means that the term $h_\infty(F)_{W_0}$ of the Atiyah–Patodi–Singer formula vanishes. It then follows from Proposition 3.3 and [**3**] that

$$\begin{aligned}0=&\int_{W_0}\hat{A}(W_0)\mathrm{ch}(V_-)\mathrm{ch}(\mathfrak{g})-\tfrac{1}{2}(h_\alpha+\eta_\alpha(0))(\Sigma)\\&+\tfrac{1}{2}\sum_{i=1}^{n}(-h_\alpha+\eta_\alpha(0))(L(a_i,b_i)),\end{aligned}\tag{3.4}$$

where the connection A_α is used on $\mathfrak{g}$.

We next wish to compute the instanton grading $\mu(\alpha)$ where α is a non-degenerate, irreducible representation. According to Proposition 3.2 it will suffice to compute $\mathrm{Index}(d^*_{A_1}\oplus d^-_{A_1})(\alpha,\theta)$ (mod 8), for any SO(3) connection A_1 over $\Sigma\times[0,1]$ which comes from a path $\{a_t\}$ of connections on Σ with $a_0=a_\alpha$ and $a_1=\theta$. The flat connection a_α over Σ extends to a flat SO(3)-connection on the flat bundle $\mathbb{V}_\alpha$ over W. Let A denote the SO(3)-connection over $W\cup(\Sigma\times\mathbb{R})\cong W$ which is built from the flat connection on W and the 1-parameter family $\{a_t\}$ chosen above. Adding (3.4) and (3.1) with θ replacing β and noting that $h_\alpha=0$ because α is non-degenerate and irreducible we get

$$\begin{aligned}\mathrm{Index}(d^*_A\oplus d^-_A)(\alpha,\theta)=&\int_{W_0\cup\Sigma\times\mathbb{R}}\hat{A}(W_0\cup\Sigma\times\mathbb{R})\mathrm{ch}(V_-)\mathrm{ch}(\mathfrak{g})\\&-\tfrac{1}{2}(h_\theta+\eta_\theta(0)(\Sigma))+\tfrac{1}{2}\sum_{i=1}^{n}(-h_\alpha+\eta_\alpha(0))(L(a_i,b_i)).\end{aligned}\tag{3.5}$$

By [3] the right-hand side is just $\mathrm{Index}(d_A^* \oplus d_A^-)(W_0)$. Thus we have:

PROPOSITION 3.6. *Let A denote the* SO(3)*-connection over $W \cup (\Sigma \times \mathbb{R}) \cong W$ which is built from the flat connection on W and a* 1*-parameter family $\{a_t\}$ of* SO(3)*-connections over Σ given by a path of connections between a_α and the trivial connection θ. Consider the corresponding operator*

$$d_A^* + d_A^-\colon\ \Omega^1(\mathfrak{g}) \to \Omega^0(\mathfrak{g}) \oplus \Omega^2_-(\mathfrak{g})$$

restricted to W_0. Then $\mu(\alpha) \equiv -\mathrm{Index}(d_A^* + d_A^-)(W_0) - 3 \pmod 8$.

For any asymptotically flat SO(3)-connection A over an open 4-dimensional manifold or orbifold Y, define its 'Pontryagin charge' to be

$$p_1(A) = \frac{1}{8\pi^2} \int_Y \mathrm{Tr}(F_A \wedge F_A).$$

We are especially interested in the case where A is flat on a neighbourhood of the end of Y.

An SO(2)-V-vector bundle L over W is classified by the Euler class $e \in H^2(W_0) \cong \mathbb{Z}$ of its restriction over $W_0 = W -$ (neighbourhood of singular points). We shall denote by L_e the V-bundle corresponding to the class e times a generator in $H^2(W_0, \mathbb{Z})$. Let B be any connection on L_e which is trivial near ∂W. Then the relative Pontryagin number of L_e is

$$\frac{e^2}{a} = \frac{1}{8\pi^2} \int_W \mathrm{Tr}(F_B \wedge F_B) = p_1(B),$$

where $a = a_1 \ldots a_n$.

For the statement of the next theorem recall that the rotation numbers $l_1, \ldots, l_n$ of $\mathbb{V}_\alpha$ are given in § 2 by the equations $r(\alpha(x_i)) = l_i(\pi/a_i)$.

THEOREM 3.7. *Let the connection A be as in* (3.6). *Suppose Σ has Seifert invariants $\{b_0\,; (a_1, b_1), \ldots, (a_n, b_n)\}$ with b_0 even.* (*This can always be arranged.*) *If one of the a_i is even, assume it is a_1, and arrange the Seifert invariants so that the b_i, with $i \neq 1$, are even. If $e \equiv \sum_{i=1}^n l_i(a/a_i) \pmod{2a}$, then the* SO(2) *$V$-bundle L_e satisfies*

(1) *L_e has the same rotation numbers* (*up to sign*) *as $\mathbb{V}_\alpha$ over the singular points of W,*

(2) *$w_2(L_e) = w_2(\mathbb{V}_\alpha)$, and*

(3) *$p_1(A) \equiv e^2/a \pmod 4$.*

Proof. To discuss the rotation numbers at the singular points of W we need to identify $\pi_1(L(a_i, b_i))$ with $\mathbb{Z}_{a_i} \subset S^1$ acting on $\mathbb{C}^2$. The generator

$$g_i = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{b_i} \end{pmatrix}, \quad \zeta = e^{2\pi i/a_i},$$

corresponds to $x_i \in \pi_1(W_0)$. Let b_i^* satisfy $b_i b_i^* \equiv 1 \pmod{a_i}$. The formula

$$1 = -b_0 a + \sum_{i=1}^n b_i \frac{a}{a_i}$$

shows that $b_i^* \equiv a/a_i \pmod{a_i}$. In **[10]** we showed that for L_1, the rotation numbers which correspond to the generators $g_i^{b_i^*}$ are the numbers $1 \pmod{a_i}$. Taking the quotient of L_1 by the $\mathbb{Z}_e$-action contained in the S^1-action yields L_e. It follows that the rotation numbers of L_e corresponding to $g_i^{b_i^*}$ are $e \pmod{a_i}$.

The rotation numbers of $\mathbb{V}_\alpha$ corresponding to the g_i are $\pm l_i \pmod{a_i}$; so the rotation numbers corresponding to $g_i^{b_i^*}$ are the numbers $\pm b_i^* l_i \pmod{a_i}$. To satisfy Condition (1) of the theorem we must solve for e in the congruences $e \equiv \pm b_i^* l_i \pmod{a_i}$, for $i = 1, \ldots, n$. Since the a_i are pairwise relatively prime, we have the solution

$$e \equiv \sum_{i=1}^{n} l_i \frac{a}{a_i} \pmod{a}.$$

Thus our task is to determine $e \pmod{2a}$.

As for Condition (2), if a is even, then Lemma 2.4 implies that $w_2(L_e) = w_2(\mathbb{V}_\alpha)$ as long as (1) is satisfied. If a is odd and $w_2(L_e) \neq w_2(\mathbb{V}_\alpha)$, since $H^2(W_0; \mathbb{Z}_2) \cong \mathbb{Z}_2$, replace e by $e + a$. Then $w_2(L_e) = w_2(\mathbb{V}_\alpha)$. (We shall see below that the correct choice for e is the one given in the statement of the theorem.) To guarantee that $p_1(A) = e^2/a \pmod 4$ requires more care.

Let E_e denote the SO(3)-V-vector bundle $L_e \oplus \mathbb{R}$, the Whitney sum of L_e with a trivial bundle. Let A_e be a connection in E_e which reduces to a connection on L_e which is trivial near the end of W and is flat in a neighbourhood of each singular point of W. Then $p_1(A_e) = e^2/a$.

Truncate W by removing neighbourhoods of the singular points, leaving W_0, and let $\delta W_0 = \partial W_0 - \Sigma$. Let E denote the bundle over W_0 with connection A. Note that since A_e is flat near the singular points of W,

$$p_1(A_e) = \frac{1}{8\pi^2} \int_{W_0} \mathrm{Tr}(F_{A_e} \wedge F_{A_e}).$$

Let $Y_0 = W_0 \cup_{\delta W_0} W_0$.

Over Y_0 we can form the SO(3) vector bundle $E \cup E$ by gluing via the identity so that we obtain a connection '$A \cup A$' on this bundle. Since the orientations get reversed, $p_1(A \cup A) = 0$. Up to sign, E_e and $\mathbb{V}_\alpha$ have the same rotation numbers over the singular points; so we can also form the bundle $E \cup E_{e,0}$ over Y_0 (where $E_{e,0}$ denotes $E_e|_{W_0}$). We want to do this by gluing the bundles together over δW_0 via a connection-preserving bundle isomorphism. (The rotation number condition ensures that A_e and A are equivalent flat connections over δW_0.) Note that if f_0, f_1: $E_e|_{\delta W_0} \to E|_{\delta W_0}$ are connection-preserving bundle isomorphisms, then $f_0^{-1} f_1$ is a gauge transformation of $E_e|_{\delta W_0}$ which preserves $A_e|_{\delta W_0}$. Thus, connection-preserving isomorphisms are unique up to composition with the stabilizer of $A_e|_{\delta W_0}$ in the group of gauge transformations. Since this stabilizer is connected, any two such gluings give isomorphic bundles over Y_0. If we can glue so that $w_2(E \cup E_{e,0}) = w_2(E \cup E)$, then, since A and A_e are trivial near the ends of Y, we have (just as for a closed 4-manifold) $p_1(A \cup A_e) \equiv p_1(A \cup A) \equiv 0 \pmod 4$. But

$$p_1(A \cup A_e) = \frac{1}{8\pi^2} \int_{W_0} \mathrm{Tr}(F_A \wedge F_A) + \frac{1}{8\pi^2} \int_{-W_0} \mathrm{Tr}(F_{A_e} \wedge F_{A_e}) = p_1(A) - \frac{e^2}{a},$$

verifying (3).

If $a = a_1 \ldots a_n$ is odd, then each component $L(a_i, b_i)$ of δW_0 is a $\mathbb{Z}_2$-homology

sphere; so in this case, $H^2(Y_0;\mathbb{Z}_2)$ splits as $H^2(W_0;\mathbb{Z}_2) \oplus H^2(-W_0;\mathbb{Z}_2)$ and the w_2's split, that is,

$$w_2(E \cup E) = w_2(E) \oplus w_2(E), \quad w_2(E \cup E_{e,0}) = w_2(E) \oplus w_2(E_{e,0}).$$

But $w_2(E_{e,0}) = w_2(L_e)$ and $w_2(E) = w_2(\mathbb{V}_\alpha)$; so if e is chosen so that $w_2(L_e) = w_2(\mathbb{V}_\alpha)$, Condition (3) will be satisfied. Now if $w_2(\mathbb{V}_\alpha) = 0$, then $\alpha(h) = +1$; so l_i is even for each i and $e = \sum_{i=1}^n l_i(a/a_i)$ is even; that is, $w_2(L_e) = 0 \in H^2(W_0;\mathbb{Z}_2) = \mathbb{Z}_2$. If $w_2(\mathbb{V}_\alpha) \neq 0$, then $\alpha(h) = -1$; so for $i = 1, \ldots, n$, $l_i \equiv b_i \pmod 2$. Thus

$$e = \sum_{i=1}^n l_i \frac{a}{a_i} = \sum_{i=1}^n b_i \frac{a}{a_i} \pmod 2.$$

But b_0 is even; so the relation

$$-b_0 a + \sum_{i=1}^n b_i \frac{a}{a_i} = 1$$

implies that $e \equiv 1 \pmod 2$. Thus $w_2(L_e) = w_2(\mathbb{V}_\alpha)$.

If one of the a_i is even, assume it is a_1. Then $H^2(W_0;\mathbb{Z}_2) \to H^2(L(a_1, b_1);\mathbb{Z}_2) = \mathbb{Z}_2$ is an isomorphism; so $w_2(E)$ and $w_2(E_{e,0})$ are carried by $H^2(L(a_1, b_1);\mathbb{Z}_2)$ (cf. Proposition 2.4). A Mayer–Vietoris sequence shows that

$$H^2(Y_0;\mathbb{Z}_2) \cong H^1(L(a_1, b_1);\mathbb{Z}_2) \oplus H^2(L(a_1, b_1);\mathbb{Z}_2).$$

Our argument will use an explicit splitting which we shall determine later. We may write

$$w_2(E \cup E) = \omega(E \cup E) \oplus w_2(E) \quad \text{and} \quad w_2(E \cup E_{e,0}) = \omega(E \cup E_{e,0}) \oplus w_2(E).$$

(Note that $w_2(E) = w_2(E_{e,0})$.) To satisfy Condition (3) we must find e so that $\omega(E \cup E_{e,0}) = \omega(E \cup E)$.

Form the double cover $\tilde{\Sigma}$ of Σ branched over the exceptional fibre of order a_1. By hypothesis, the Seifert invariants of Σ are $\{b_0;(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}$ where the b_i, with $i \neq 1$, are all even. Then $\tilde{\Sigma}$ is Seifert fibred with invariants $\{\frac{1}{2}b_0;(2a_1, b_1), (a_2, \frac{1}{2}b_2), \ldots, (a_n, \frac{1}{2}b_n)\}$. Since

$$-\frac{b_0}{2}(2a) + b_1\left(\frac{2a}{2a_1}\right) + \sum_{i=2}^n \left(\frac{b_i}{2}\right)\left(\frac{2a}{a_i}\right) = -b_0 a + \sum_{i=1}^n b_i\left(\frac{a}{a_i}\right) = 1,$$

$\tilde{\Sigma}$ is a homology sphere, $\tilde{\Sigma} = \Sigma(2a_1, a_2, \ldots, a_n)$. Let $\tilde{W}$ be the mapping cylinder of $\tilde{\Sigma} \to S^2$; it is a branched cover π: $\tilde{W} \to W$ where the branch set is the union of the orbit 2-sphere together with the 2-disk in the mapping cylinder corresponding to the a_1-fibre in Σ. The representation α: $\pi_1(\Sigma) \to \mathrm{SU}(2)$ lifts to $\tilde{\alpha}$: $\pi_1(\tilde{\Sigma}) \to \mathrm{SU}(2)$.

On π_1 the covering map $\tilde{\Sigma} \to \Sigma$ induces a homomorphism given on the generators by $\tilde{h} \to h^2$, $\tilde{x}_i \to x_i$, for $i = 1, \ldots, n$. Clearly $r(\tilde{\alpha}(\tilde{x}_i)) = r(\alpha(x_i))$ for all i; so Proposition 2.4 implies that $\tilde{\alpha}(\tilde{h}) = 1 \in \mathrm{SU}(2)$. Then $\tilde{\alpha}$: $\pi_1(\tilde{W}_0) \to \mathrm{SU}(2)$ and gives rise to a flat SU(2)-V-vector bundle $\tilde{\mathbb{V}}'_\alpha$ over $\tilde{W}$. The SO(3)-V-vector bundle corresponding to $\tilde{\mathbb{V}}'_\alpha$ is the pullback $\tilde{\mathbb{V}}_\alpha = \pi^*\mathbb{V}_\alpha$. Let $\tilde{E} = \pi^*E$ with its connection $\tilde{A} = \pi^*A$. Near δW_0, $\tilde{A}$ agrees with the flat connection on $\tilde{\mathbb{V}}_\alpha$ which is in turn induced from a flat SU(2) connection $\tilde{A}'$ on $\tilde{\mathbb{V}}'_\alpha$. The SO(3) bundle $\tilde{E}$ lifts to an SU(2) bundle $\tilde{E}'$ with connection which equals $\tilde{A}'$ near $\delta\tilde{W}'_0$.

We wish to find an SO(2)-V-vector bundle $\tilde{L}_f$ over $\tilde{W}$ with connection $\tilde{A}_f$, flat near the singular points of $\tilde{W}$ and trivial near the end of $\tilde{W}$, so that we can use a flat SU(2)-connection-preserving bundle isomorphism to form the SU(2)-vector

bundle $\tilde{E} \cup (\tilde{L}_{f,0} \oplus \tilde{L}_{f,0}^{-1})$ over $\tilde{Y}_0$. This can be achieved provided that the SU(2)-V-bundle rotation numbers of $\tilde{\mathbb{V}}_\alpha$ match those of $\tilde{L}_f \oplus \tilde{L}_f^{-1}$.

Since $\tilde{\alpha}(h) = +1$ in SU(2), each $\tilde{\alpha}(\tilde{x}_i)$, for $i = 2, \ldots, n$, is an a_ith root of unity and $\tilde{\alpha}(\tilde{x}_i)$ is a $2a_1$th root of unity. For $i = 2, \ldots, n$, $r(\tilde{\alpha}(\tilde{x}_i)) = r(\alpha(x_i)) = l_i(\pi/a_i)$; so l_i is even. Also $r(\tilde{\alpha}(\tilde{x}_1)) = r(\alpha(x_1)) = 2l_1(\pi/2a_1)$. Thus, for the generators $\tilde{g}_i$, the SU(2) rotation numbers of $\tilde{\mathbb{V}}'_\alpha$ are $\pm l_1$ and $\pm\frac{1}{2}l_i$, for $i = 2, \ldots, n$. For generators $\tilde{g}_i^{\tilde{b}_i^*}$ they are $\pm\tilde{b}_1^* l_1$ and $\pm\frac{1}{2}b_i^* l_i$ $(i = 2, \ldots, n)$. We need to choose f so it is congruent to these rotation numbers modulo $2a_1$ or a_i, as the case may be. It suffices to choose $f = \sum_{i=1}^n l_i(a/a_i) \pmod{2a}$.

The SO(3)-vector bundle induced from $\tilde{E}' \cup (\tilde{L}_{f,0} \oplus \tilde{L}_{f,0}^{-1})$ is $\tilde{E} \cup \tilde{E}_{2f,0}$ where $\tilde{E}_{2f,0} = \tilde{L}_{2f} \oplus \mathbb{R}$. In

$$H^2(\tilde{Y}_0; \mathbb{Z}_2) = H^1(L(2a_1, b_1); \mathbb{Z}_2) \oplus H^2(L(2a_1, b_1); \mathbb{Z}_2)$$

we get

$$0 = w_2(\tilde{E} \cup \tilde{E}_{2f,0}) = \omega(\tilde{E} \cup \tilde{E}_{2f,0}) \oplus w_2(\tilde{E}).$$

Similarly $0 = w_2(\tilde{E} \cup \tilde{E}) = \omega(\tilde{E} \cup \tilde{E}) \oplus w_2(\tilde{E})$ since we can form the SU(2)-bundle $\tilde{E}' \cup \tilde{E}'$. We claim that the Euler class $e \in H^2(W_0; \mathbb{Z}) = \mathbb{Z}$ which we are seeking is $e = f$. Note that $e = f$ solves the defining congruences $e \equiv b_i^* l_i \pmod{a_i}$. Over W the SO(2)-V-vector bundle L_f has Pontryagin number f^2/a (computed as $p_1(B)$ for any connection B flat near the singular points and trivial near the end of W). Since we work with connections trivial near the ends, we may view W and $\tilde{W}$ as 'closed' orbifolds with fundamental classes. Then

$$p_1(\pi^* L_f)[\tilde{W}] = p_1(L_f)(\pi_*[\tilde{W}]) = 2p_1(L_f)[W] = 2f^2/a = (2f)^2/2a.$$

Thus $\pi^* L_f = \tilde{L}_{2f}$, and so $\pi^* E_f = \tilde{E}_{2f}$. Since connection-preserving bundle isomorphisms form a connected set, $\pi^*(E \cup E_{f,0}) = \tilde{E} \cup \tilde{E}_{2f,0}$ and $\pi^*(E \cup E) = \tilde{E} \cup \tilde{E}$. So $\pi^* w(E \cup E_{f,0}) = 0 = \pi^* \omega(E \cup E)$. However

$$\pi^*\colon H^1(L(a_1, b_1); \mathbb{Z}_2) \to H^1(L(2a_1, b_1); \mathbb{Z}_2)$$

is an isomorphism; so $\omega(E \cup E_{f,0}) = \omega(E \cup E)$, as required.

(Here we can see how to split $H^2(Y_0; \mathbb{Z}_2)$ so that the above argument is correct. First take an arbitrary splitting of $H^2(\tilde{Y}_0; \mathbb{Z}_2)$ given by a map $H^2(\tilde{Y}_0; \mathbb{Z}_2) \to H^1(L(a_1, b_1); \mathbb{Z}_2)$, and then use the diagram

$$\begin{array}{ccccc} H^1(L(2a_1, b_1); \mathbb{Z}_2) & \leftrightarrows & H^2(\tilde{Y}_0; \mathbb{Z}_2) & \longrightarrow & H^2(L(2a_1, b_1); \mathbb{Z}_2) \\ \updownarrow \approx & & \uparrow & & \uparrow \\ H^1(L(a_1, b_1); \mathbb{Z}_2) & \leftrightarrows & H^2(Y_0; \mathbb{Z}_2) & \longrightarrow & H^2(L(a_1, b_1); \mathbb{Z}_2) \end{array}$$

to get the splitting map $H^2(Y_0; \mathbb{Z}_2) \to H^1(L(a_1, b_1); \mathbb{Z}_2)$.)

Let E be an SO(3)-V-vector bundle over W which is isomorphic to a direct sum $L_e \oplus \mathbb{R}$. We then have the invariant of **[10]**,

$$\begin{aligned} (3.8) \qquad R(e) &= R(a_1, \ldots, a_n; e) \\ &= \frac{2e^2}{a} - 3 + m + \sum_{i=1}^{m} \frac{2}{a_i} \sum_{k=1}^{a_i - 1} \cot\left(\frac{\pi a k}{a_i^2}\right) \cot\left(\frac{\pi k}{a_i}\right) \sin^2\left(\frac{\pi e k}{a_i}\right), \end{aligned}$$

where $e \not\equiv 0 \pmod{a_i}$ for $i = 1, \ldots, m$ and $e \equiv 0 \pmod{a_i}$ for $i = m+1, \ldots, n$. (Recall, $a = a_1 \ldots a_n$ and $ab_i/a_i \equiv 1 \pmod{a_i}$.)

THEOREM 3.9. *Let e be the Euler class of the* SO(2)-V-*vector bundle* L_e *of* (3.7) *restricted to* W_0, *and let* α *be non-degenerate and irreducible. Then*

$$\mu(\alpha) \equiv -R(e) - 3 \pmod 8.$$

Proof. We expand the formula for $\mathrm{Index}(d_A^* \oplus d_A^-)(W_0)$ given in (3.5). The integral term is

$$\int_{W_0 \cup (\Sigma \times \mathbb{R})} \hat{A}(W_0 \cup \Sigma \times \mathbb{R}) \mathrm{ch}(V_-) \mathrm{ch}(\mathfrak{g}) = 2 \int_{W_0 \cup (\Sigma \times \mathbb{R})} p_1(\mathfrak{g}) + \frac{3}{2} \int_{W_0 \cup (\Sigma \times \mathbb{R})} (\mathscr{L} - \mathscr{E}),$$

where $\mathscr{L}$ and $\mathscr{E}$ are the L-polynomial and the Euler form of $W_0 \cup \Sigma \times \mathbb{R}$. Since $\mathfrak{g}$ is our SO(3)-bundle with connection A of Proposition 3.6,

$$\int_{W_0 \cup (\Sigma \times \mathbb{R})} p_1(\mathfrak{g}) = p_1(A),$$

the Pontryagin charge. We then get as in [**3**, § 4] that the integral term is

$$2p_1(A) - \tfrac{3}{2}[\chi(W_0) - \sigma(W_0)] + \tfrac{3}{2}\eta_\theta(0)(\partial W_0).$$

By Theorem 3.7, $p_1(A) \equiv e^2/a \pmod 4$. Also $\chi(W_0) - \sigma(W_0) = 1 - n$, so (3.5) yields

$$\begin{aligned}\mathrm{Index}(d_A^* \oplus d_A^-)(W_0) \equiv{}& 2e^2/a - \tfrac{3}{2}(1-n) - \tfrac{1}{2}(h_\theta + \varrho_\theta)(\Sigma) \\ &+ \frac{1}{2}\sum_{i=1}^{n} (-h_\alpha + \varrho_\alpha)(L(a_i, b_i)) \pmod 8,\end{aligned}$$

where $\varrho_\beta = \eta_\beta(0) - \dim(\beta)\eta_\theta(0)$ is the ϱ-invariant of [**4**]. But $h_\theta(\Sigma) = 3$, $\varrho_\theta(\Sigma) = 0$, and $h_\alpha(L(a_i, b_i)) = 1$ for $i = 1, \ldots, m$ and 3 for $i = m+1, \ldots, n$, where α restricts non-trivially to the first m and trivially to the last $(n-m)$ lens spaces. Since $\varrho_\alpha(L(a_i, b_i)) = \varrho_\theta(\hat{L}(a_i, b_i)) = 0$ for $i = m+1, \ldots, n$ we have

$$\mathrm{Index}(d_A^* \oplus d_A^-)(W_0) \equiv 2e^2/a - 3 + m + \frac{1}{2}\sum_{i=1}^{m} \varrho_\alpha(L(a_i, b_i)) \pmod 8.$$

Since L_e restricts over the lens spaces with the same rotation numbers as $\mathbb{V}_\alpha$, the ϱ-invariants are exactly those computed in [**10**], that is,

$$\tfrac{1}{2}\varrho_\alpha(L(a_i, b_i)) = \sum_{i=1}^{m} \frac{2}{a_i} \sum_{k=1}^{a_i - 1} \cot\left(\frac{\pi a k}{a_i^2}\right) \cot\left(\frac{\pi k}{a_i}\right) \sin^2\left(\frac{\pi e k}{a_i}\right).$$

Thus $\mathrm{SF}(\alpha, \theta) = \mathrm{Index}(d_A^* \oplus d_A^-)(W_0) \equiv R(e) \pmod 8$. The theorem now follows from Proposition 3.6.

PROPOSITION 3.10. *For a Seifert fibred homology sphere* Σ *and irreducible non-degenerate representation* α: $\pi_1(\Sigma) \to \mathrm{SU}(2)$, *the instanton grading* $\mu(\alpha)$ *is even.*

Proof. By [**10**, Corollary 6.3], $R(e)$ is always odd.

Propositions 3.10 and 2.5 imply that our calculations of the instanton homology for Brieskorn homology spheres are complete.

As an example, consider the Brieskorn sphere $\Sigma(4,3,5)$. In § 2 we computed the vectors (l_1, l_2, l_3) that arise as representations of $\pi_1(\Sigma)$ into SU(2), where we chose $(b_1, b_2, b_3) = (-1, -4, -2)$. They are $(1, 2, 4)$ and $(3, 2, 2)$ when $H = -1$,

and (2,2,2) and (2,2,4) when $H = +1$. The corresponding e's are given by 103, 109, 94, and 118; and the corresponding $\mathrm{SF}(\alpha, \theta) \equiv R(e) \pmod 8$ are 1, 3, 5 and 7, respectively. So the instanton homology groups $I_i(\Sigma(4,3,5))$ are

$$I_i(\Sigma(4,3,5)) = \begin{cases} \mathbb{Z} & \text{for } i = 0,2,4,6, \\ 0 & \text{otherwise.} \end{cases}$$

Other examples of computations of instanton homology will be listed in the appendix.

4. *Seifert fibrations with more than three exceptional fibres*

Let $\Sigma = \Sigma(a_1, \ldots, a_n)$ and let α: $\pi_1(\Sigma) \to \mathrm{SU}(2)$ be an irreducible representation which has $m \geqslant 4$ non-zero rotation numbers when extended over the cones on the lens spaces in the mapping cylinder W. By Proposition 2.5, α is degenerate and by Proposition 2.7, the connected component $\mathcal{R}_\alpha$ of $\mathcal{R}(\Sigma)$ which contains α is a compact manifold of dimension $2m - 6 \geqslant 2$. For $a \in \mathcal{B}_\Sigma$ with curvature F_α we have

$$d_a^*(\star F_a) = (\star d_a \star)(\star F_a) = \star d_a F_\alpha = 0,$$

by the Bianchi identity; so $\star F_a$ takes values in $\{\psi \in L_0^4(\Omega^1(\mathfrak{g}))|\ d_a^*\psi = 0\} = \mathcal{T}_a$, the fibre of a bundle over $\mathcal{B}_\Sigma$. Furthermore, there is a natural inclusion $T\mathcal{B}_\Sigma \to \mathcal{T}$ since $T\mathcal{B}_\Sigma = \{\psi \in L_1^4(\Omega^1(\mathfrak{g}))|\ d_a^*\psi = 0\}$. Thus flat SU(2) connections on Σ are zeros of the section $a \mapsto \star F_a$ on $\mathcal{B}_\Sigma$. To get a basis for the instanton chain complex in this situation we must perturb this section to isolate its zeros and make them non-degenerate.

Following Bott [7], we apply ideas from the Morse theory of non-degenerate critical manifolds. Since $\mathcal{R}_\alpha$ is a smooth closed submanifold of $\mathcal{B}_\Sigma$, it has a tubular neighbourhood N [15]. Furthermore, the tangent bundle $T\mathcal{B}_\Sigma|_{\mathcal{R}_\alpha}$ splits L^2-orthogonally as $T\mathcal{R}_\alpha \oplus \nu\mathcal{R}_\alpha$, and N can be identified with the total space of $\nu\mathcal{R}_\alpha$. Since $\dim \mathcal{R}_\alpha = 2m - 6 = \dim H^1(\Sigma; \mathbb{V}_\alpha)$, for any $a \in \mathcal{R}_\alpha$ the linearization $\star d_a$ of the section $\star F_a$ of the bundle $\mathcal{T}$ over $\mathcal{B}_\Sigma$ has kernel equal to $T_a\mathcal{R}_\alpha$ and is non-degenerate on $\nu\mathcal{R}_\alpha$.

Let g: $\mathcal{R}_\alpha \to \mathbb{R}$ be a non-degenerate Morse function. Fix a cutoff function β on $\mathcal{B}_\Sigma$ such that $\beta \equiv 1$ on a small sub-tubular neighbourhood of $\mathcal{R}_\alpha$ in N and such that $\beta \equiv 0$ outside of N. Let π: $N \to \mathcal{R}_\alpha$ be the normal bundle projection. Set $\varphi(a) = \beta(a)g(\pi(a))$; so φ: $\mathcal{B}_\Sigma \to \mathbb{R}$. Let Φ denote the L^2-gradient of φ, and let f be the section $f(a) = \star F_a + \Phi(a)$ of $\mathcal{T}$ over $\mathcal{B}_\Sigma$.

LEMMA 4.1. *For a sufficiently small tubular neighbourhood N of $\mathcal{R}_\alpha$ the zeros $a \in \mathcal{B}_\Sigma$ of the section f are the zeros of $\star F_a$ outside of N and inside N they all lie in $\mathcal{R}_\alpha$ and are critical points of g. Furthermore, all the zeros of f which all lie on $\mathcal{R}_\alpha$ are non-degenerate.*

Proof. Since the linearization $\star d_a$ of $\star F_a$ is non-degenerate normal to $\mathcal{R}_\alpha$ in $\mathcal{B}_\Sigma$, we can choose a small enough tubular neighbourhood N so that $\star d_a$ has a non-zero component in the direction of the fibres of N for each $a \in N$. Each $a \in \mathcal{R}_\alpha$ has a neighbourhood $U_a \subset \mathcal{R}_\alpha$ such that $\pi^{-1}(U_a)$ can be trivialized: $\pi^{-1}(U_a) = U_a \times (\nu\mathcal{R}_\alpha)_a$. Over $\pi^{-1}(U_a)$ the bundle $\mathcal{T}$ also splits in the L^2-norm as $\mathcal{T} = \mathcal{T}_{\mathcal{R}} \oplus \mathcal{T}_\nu$, into spaces parallel to $\mathcal{R}_\alpha$ and to the fibres of $\nu\mathcal{R}_\alpha$.

Note that for $a \in R_\alpha$ and $\xi \in T_a\mathcal{B}_\Sigma$, $\eta \in T_a\mathcal{R}_\alpha$, we have

$$\langle \star d_a\xi, \eta \rangle = \langle \xi, \star d_a\eta \rangle = 0$$

since $T_a\mathcal{R}_\alpha = \ker(\star d_a)$. Thus $\star d_a$ has image which is L^2-orthogonal to $T_a\mathcal{R}_\alpha$. If $(a, v) \in U_a \times (v\mathcal{R}_\alpha)_a = \pi^{-1}(U_\alpha)$ then $\star d_a = (0, \star d_a v + O(|v|^2)) \in \mathcal{T}_{\mathcal{R},a} \oplus \mathcal{T}_{v,a} = \mathcal{T}_{(a,v)}$. Thus $f(a, v) = (\beta(a, v)\nabla g(a), \star d_a v + O(|v|^2))$, which is non-zero for $v \neq 0$, that is, outside of $\mathcal{R}_\alpha$ in N, since N was chosen so that $\star d_a \neq 0$ (and can be made even smaller so that $O(|v|^2)$ is negligible.)

If $a \in \mathcal{R}_\alpha$ is a zero of f then the linearization of f at a is $Df_a = (H(g)_a, \star d_a)$ which is non-degenerate since $H(g)_a$ is non-degenerate tangent to $\mathcal{R}_\alpha$ and $\star d_a$ is non-degenerate normal to $\mathcal{R}_\alpha$.

Using this perturbation of $a \mapsto \star F_a$ at each positive-dimensional component of $\mathcal{R}(\Sigma)$, we obtain a compact perturbation of $\star F_a$ whose zeros are all non-degenerate. These zeros form a basis for the instanton chain complex, and the gradings are computed using the spectral flow of the linearization of the perturbed operator. Fix a connected component $\mathcal{R}_\alpha$ of $\mathcal{R}(\Sigma)$. It follows from § 2 that the rotation numbers $\{l_i\}$ of the representations are constant along $\mathcal{R}_\alpha$. So the integer e of Theorem 3.7 is constant on $\mathcal{R}_\alpha$.

THEOREM 4.2. *Let $\mathcal{R}_\alpha$ be a connected component of $\mathcal{R}(\Sigma)$ and let g: $\mathcal{R}_\alpha \to \mathbb{R}$ be a Morse function. Then the critical points of g are basis elements of the instanton chain complex. Such a critical point b has grading $\mu(b) = -R(e) - 3 + \mu_g(b)$ where $\mu_g(b)$ is the Morse index of b relative to g and e is associated to α by Theorem* 3.7.

Proof. Floer's perturbations are not of the form which we have described; so we must argue that the described compact perturbations actually yield instanton homology. First, on $\Sigma \times \mathbb{R}$ we connect one of Floer's perturbations on $\Sigma \times (-\infty, 0]$ with our perturbation on $\Sigma \times [1, \infty)$ via an interpolation. There is then an induced perturbed self-duality operator on $\Sigma \times \mathbb{R}$. Floer's proof of [**13**, 2c.2] shows that one can further perturb this operator on $\Sigma \times (0, 1)$ to be regular; i.e. so that its zero set is a disjoint union of finite-dimensional manifolds. Then Floer's argument [**13,** Theorems 2,3] for topological invariance holds in this case as well, showing that the induced homology theories agree.

Floer's proof proceeds by showing that for his set Π of perturbations, the section $\bar{F}$: $\Pi \times \mathcal{B}(\Sigma \times [0, 1]) \to \mathcal{T}(\Sigma \times [0, 1])$ given by $\bar{F}(\pi, A) = F_-(a) + \pi(A)$, has surjective linearization at any (π, A) in the zero set of $\bar{F}$. He then applies the Sard–Smale Theorem. The key step in the proof is to show that he has enough perturbations so that the linearization $D\bar{F}_{(\pi,A)}(\rho, B)$ maps onto the cokernel of $\bar{F}$ at (π, A). He achieves this by producing for each ξ in the cokernel a ρ_ξ such that $D\bar{F}_{(\pi,A)}(\rho, B)(0) = \xi(0)$. He then calls on Arondzajn's theorem to the effect that the restriction of the zero set of $\bar{F}$ to $t = 0$ is injective, and that this is also true for $D\bar{F}$.

We can simply enlarge Π to a set of perturbations $\mathcal{P}$ of the form $\eta(t) = (1 - f(t))\pi(t) + f(t)\Phi$, for $0 \leq t \leq 1$, where $\pi \in \Pi$, Φ is one of our perturbations above, and f: $[0, 1] \to \mathbb{R}$ is any smooth function such that $f(t) = 0$ for t less than some positive ε. Now all of Floer's arguments will work.

Next, as above, first perturb $a \mapsto \star F_a$ to get $a \mapsto f(a) = \star F_a + \Phi(a)$ as in Lemma

4.1 so that all zeros are non-degenerate. Then $\mu(b)$ is the spectral flow $\mathrm{SF}'(\theta, b)$ of $Df = \star d_a + \nabla\Phi(a)$. As in Lemma 3.2 we obtain

$$\mathrm{SF}'(b, \theta) = \mathrm{Index}(d_A^* \oplus d_A^- + \pi)(b, \theta)$$

where π is a compact operator. Formula (3.1) gives

$$\mathrm{Index}(d_A^* \oplus d_A^-)(b, \theta) = \int_{\Sigma\times\mathbb{R}} \mathcal{I}(x)\, dx - \tfrac{1}{2}(h_\theta + \eta_\theta(0)) + \tfrac{1}{2}(-h_b + \eta_b(0)),$$

where the integrand is described in [**3**, Theorem 3.10] (it is called $\alpha_0(x)$ there) and has in our case the simple form given in (3.1). In the calculation, the operator $d_A^* + d_A^-$ used to compute $\mathcal{I}(x)$ has been changed by the addition of a compact operator, but the formula in [**3**] shows that this leaves $\mathcal{I}(x)$ unchanged. So we have

$$\mathrm{Index}(d_A^* \oplus d_A^- + \pi)(b, \theta) = \int_{\Sigma\times\mathbb{R}} \mathcal{I}(x)\, dx - \tfrac{1}{2}(h_\theta + \eta_\theta(0)) + \tfrac{1}{2}(-h_b' + \eta_b'(0)),$$

where $h_b' = 0 = h_b - \dim \mathcal{R}_\alpha$ and $\eta_b'(0) = \eta_b(0) + (\dim \mathcal{R}_\alpha - 2\mu_g(b))$. Hence,

$$\mathrm{SF}'(b, \theta) = \mathrm{Index}(d_A^* \oplus d_A^- + \pi)(b, \theta) = \mathrm{Index}(d_A^* \oplus d_A^-)(b, \theta) + \dim \mathcal{R}_\alpha - \mu_g(b).$$

From (3.4) and (3.1) we have

$$\mathrm{Index}(d_A^* \oplus d_A^-)(b, \theta) = \mathrm{Index}(d_A^* \oplus d_A^-)(W_0) - h_b = R(e) - h_b.$$

So $\mathrm{SF}'(b, \theta) = R(e) - \mu_g(b)$, and

$$\mu(b) = \mathrm{SF}'(\theta, b) = -\mathrm{SF}'(b, \theta) - 3 = -R(e) + \mu_g(b) - 3.$$

This together with the fact that Casson's invariant $\lambda(\Sigma)$ is half the Euler characteristic of the instanton chain complex, implies:

THEOREM 4.3. *For a Seifert-fibred homology three-sphere Σ, Casson's invariant $\lambda(\Sigma)$ satisfies $\lambda(\Sigma) = \frac{1}{2}\chi(\mathcal{R}(\Sigma))$.*

Proof. Casson's invariant is half the Euler characteristic of the instanton chain complex. This is the sum over all components $\mathcal{R}_\alpha$ of $\mathcal{R}(\Sigma)$ and critical points b of a Morse function g_α of $\mathcal{R}_\alpha$ of -1 to the power $(-3 - R(e_b) + \mu_g(b))$, that is, of $(-1)^{\mu_g(b)}$ since $\dim \mathcal{R}_\alpha = 2m - 6$ is even and $R(e_b)$ is odd. Thus the Euler characteristic of the instanton chain complex is $\chi(\mathcal{R}(\Sigma))$.

Before working an example we wish to comment further on the techniques in § 2 for computing conjugacy classes of representations. Given $\Sigma(a_1, \ldots, a_n)$, w_2 corresponding to $w = 0, 1$, and possible rotation numbers $(l_1, \ldots, l_n)$, we wish to find out whether there is a representation α: $\pi_1(\Sigma) \to \mathrm{SU}(2)$ corresponding to the given data. If such a representation exists, we would like to describe the topological type of the connected component $\mathcal{R}_\alpha$ of α in $\mathcal{R}(\Sigma)$.

Fix $X_1 = e^{il_1\pi/a_1}$ and for $i = 2, \ldots, n$ let S_i be the 2-sphere $r^{-1}(l_i(\pi/a_i))$. We wish to find $X_i \in S_i$, for $i = 2, \ldots, n$, such that $X_1 \ldots X_n = (-1)^{wb_0} \in \mathrm{SU}(2)$. This means that in $\mathrm{SU}(2) \cong S^3$ we must find radii r_i $(i = 2, \ldots, n)$ of the 2-spheres $X_1 \ldots X_{i-1} S_i$ which form a linkage spanning from X_1 to $(-1)^{wb_0}$. (See Fig. 3.)

If such a linkage can be formed, then we can find a representation α as desired. Then $\mathcal{R}_\alpha$ is the connected component of the corresponding configuration in the space of all configurations of the given linkage modulo rotations leaving S^1 invariant.

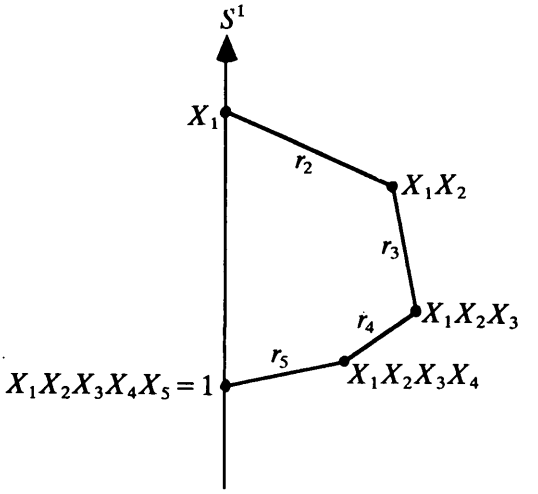

FIG. 3

For example, suppose $n=4$. Let $\Sigma=\Sigma(a_1, a_2, a_3, a_4)$ and α: $\pi_1(\Sigma)\to \mathrm{SU}(2)$ with $\alpha(x_i)\neq \pm 1$ for any i; so Proposition 2.5 predicts that $\mathcal{R}_\alpha$ is 2-dimensional. Let the rotation numbers of α be $(l_1, \ldots, l_4)$ and for definiteness, suppose $H^{wb_0}=1$. Also suppose that we have ordered the a_i so that $l_i(\pi/a_i)>l_j(\pi/a_j)$ for $i>j$. This means that the length $|r_i|$ of the radii in the linkage corresponding to σ have decreasing length. This linkage consists of three radii spanning from $x_1=e^{il_1(\pi/a_1)}$ to 1 in S^3. It has four vertices with two pinned down at 1 and X_1 and the other two, X_1X_2 and $X_1X_2X_3$, free to move.

To see the linkage, identify S^3 with $\mathbb{R}^3_+=\mathbb{R}^3\cup\{\infty\}$ using stereographic projection σ: $S^3\to\mathbb{R}^3_+$. Then 2-spheres in S^3 correspond to 2-spheres in $\mathbb{R}^3_+$; but the concept of a 'linkage' is somewhat changed since spheres of the same radius in S^3 can project to spheres with different radii in $\mathbb{R}^3_+$. The linkage at hand has three radii, r_2, r_3, r_4, and stereographic projection takes the spheres X_1S_2 (of radius r_2) and S_4 (of radius r_4) to spheres $\sigma(X_1S_2)$ of radius r_2' and $\sigma(S_4)$ of radius r_4'. The arm $\langle\sigma(X_1X_2),\sigma(X_1X_2X_3)\rangle$ has varying length depending on the centre $\sigma(X_1X_2)$ of the 2-sphere $\sigma(X_1X_2S_3)$.

Consider the radius r_2 of X_1S_2 as it moves in S^3 from one of the two poles $(X_1S_2\cap S^1)$ to the other. We view this radius as giving us a height function for our configuration space. We may suppose that $\sigma(S^1)=\{z\text{-axis}\}$. Then we watch the radius r_2' move from pole to pole on $\sigma(X_1S_2)\cap\{yz\text{-plane}\}$. If there is a linkage in $\mathbb{R}^3_+$ with $\sigma(X_1X_2)$ at a pole, then the only free motion in that linkage is that of the vertex $\sigma(X_1X_2X_3)$ by rotations perpendicular to the z-axis. (See Fig. 4 and notice that r_3' is a function of the 'height' of $\sigma(X_1X_2)$; so its length is fixed once we fix $\sigma(X_1X_2)$.) These rotations correspond to rotations in S^3 which fix S^1. A configuration such as those shown in Fig. 4 is 'rigid' in the sense that keeping the radius r_2' at the fixed height, the only allowable motion of the linkage is perpendicular to the z-axis (and so will be divided out by conjugation).

If one of the poles $\sigma(X_1S_2)$ does not lie on a configuration of the linkage, then at the closest height to the pole at which there is an allowable configuration, it once again is rigid (Fig. 5). The contribution to $\mathcal{R}_\alpha$ is a single point.

At any height between the extremes, the configurations (divided out by rotations perpendicular to the z-axis) form a circle in $\mathcal{R}_\alpha$ (Fig. 6). Note that the

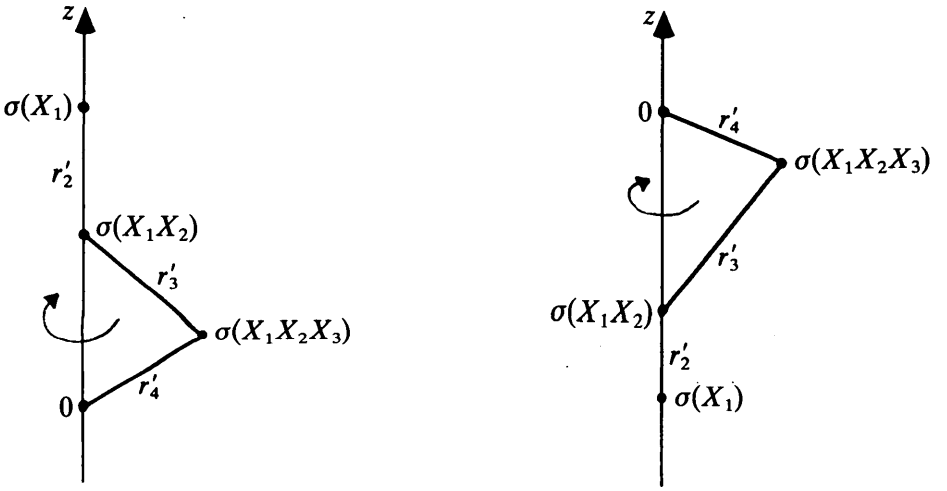

FIG. 4

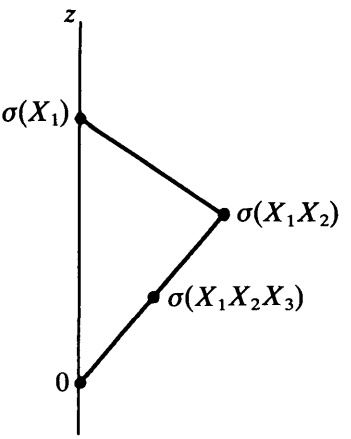

FIG. 5

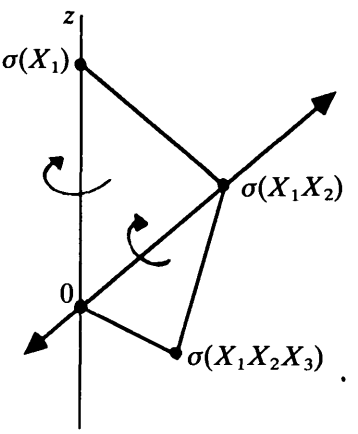

FIG. 6

only alternative would be for this configuration to be rigid. But that would give a point in $\mathcal{R}_\alpha$, implying that $\mathcal{R}_\alpha$ is not a manifold and contradicting Proposition 2.7.

This Morse function on $\mathcal{R}_\alpha$ gives a 2-sphere component. It is conceivable that there will be more than one connected component of $\mathcal{R}(\Sigma)$ corresponding to our given data, but each will be a 2-sphere. It thus follows from Theorem 4.2, using Morse functions with critical points only of index 0 and 2, that the only non-zero instanton chain groups are those of even grading. Thus the boundary operator is trivial. We conjecture that this is true for any Seifert fibration.

CONJECTURE. Let Σ be a Seifert fibred homology three sphere and α: $\pi_1(\Sigma) \to SU(2)$ a representation. Then the connected component containing α in $\mathcal{R}(\Sigma)$ has a Morse function with critical points only of even indices.†

Consider now $\Sigma(2,3,5,7)$. We first list (Table 1) the conjugacy classes of rotations which send some x_i to ± 1. These representations have $m=3$ in Proposition 2.5 and so are isolated.

TABLE 1

	(l_1, l_2, l_3, l_4)	e	$R(e)$(mod 8)	Grading
$h \mapsto -1$				
	(1, 0, 2, 2)	249	7	6
	(1, 0, 2, 4)	309	5	0
	(1, 0, 2, 6)	369	1	4
	(1, 0, 4, 4)	393	7	6
	(1, 2, 0, 2)	305	7	6
	(1, 2, 0, 4)	365	5	0
	(1, 2, 2, 0)	329	7	6
	(1, 2, 4, 0)	413	1	4
$h \mapsto +1$				
	(0, 2, 2, 2)	284	1	4
	(0, 2, 2, 4)	344	7	6
	(0, 2, 2, 6)	404	3	2
	(0, 2, 4, 2)	368	3	2
	(2, 2, 2, 2)	74	5	0
	(2, 2, 2, 4)	134	3	2
	(2, 2, 4, 4)	218	5	0
	(2, 2, 4, 6)	278	1	4

Next we list (Table 2) the representations which have 2-dimensional components. As above, these can be seen to be 2-spheres. By Theorem 4.2, each will contribute two generators to the instanton chain complex, one in dimension $-3-R(e)$ and the other in dimension $-1-R(e)$. Both numbers are listed in the last column.

We count 28 generators for the chain complex. Since the Casson invariant $\lambda(\Sigma)=14$, there are no 'hidden' components which we have missed. Thus $I_*(\Sigma(2,3,5,7))$ is free of rank 7 in even dimensions and is 0 in odd dimensions.

† This conjecture has been proved recently by P. Kirk and E. Klassen, 'Representation spaces of Seifert fibered homology spheres', preprint. See also S. Bauer and C. Okonek, 'The algebraic geometry of representation spaces associated to Seifert fibered homology 3-spheres', preprint.

TABLE 2

	(l_1, l_2, l_3, l_4)	e	$R(e)$ (mod 8)	Grading
$h \mapsto -1$				
	$(1,2,2,2)$	389	3	2,4
	$(1,2,2,4)$	29	1	4,6
	$(1,2,2,6)$	89	5	0,2
	$(1,2,4,2)$	53	5	0,2
	$(1,2,4,4)$	113	3	2,4
	$(1,2,4,6)$	173	7	6,0

5. *An invariant for homology 3-spheres*

The instanton homology of a homology 3-sphere Σ does not appear to carry any homology cobordism information. For example, $I_*(\Sigma(2, 3, 13)) \cong I_*(\Sigma(2, 3, 11))$, but $\Sigma(2, 3, 13)$ can be shown to bound a contractible manifold and $\Sigma(2, 3, 11)$ does not bound any acyclic manifold (see **[10]**). However, it is the case that the integral that appears in the computation of the instanton grading for a flat connection (see (3.5)) carries some homology cobordism information, as we shall now demonstrate.

Let Σ be an arbitrary homology 3-sphere and let $\mathcal{R}(\Sigma)$ denote the space of conjugacy classes of irreducible representations of Σ in SO(3). For $\alpha \in \mathcal{R}(\Sigma)$ let

$$CS(\alpha) = \frac{1}{8\pi^2} \int_{\Sigma \times \mathbb{R}} \operatorname{Tr}(F_A \wedge F_A),$$

where A is an SO(3)-connection over $\Sigma \times \mathbb{R}$ which is trivial near $+\infty$ and which is the flat connection a_α near $-\infty$. Up to sign, this is the Chern–Simons invariant of the connection a_α and is well-defined as an element of $\mathbb{R}/4\mathbb{Z}$.

Noting that $\mathcal{R}(\Sigma)$ is compact, define

$$\tau(\Sigma) = \min\{CS(\alpha) \mid \alpha \in \mathcal{R}(\Sigma)\} \in [0, 4).$$

Explicit in the computations of the instanton homology of Brieskorn spheres $\Sigma(p, q, r)$ is the computation of $\tau(\Sigma(p, q, r))$. For, to each representation $\alpha \in \mathcal{R}(\Sigma(p, q, r))$ there is associated an Euler number e and $CS(\alpha) \equiv e^2/pqr \bmod 4\mathbb{Z}$. Note that $pqr\tau(\Sigma(p, q, r)) \in \mathbb{Z}$.

For example, for p and q relatively prime,

$$\tau(\Sigma(p, q, pqk - 1)) = 1/(pq(pqk - 1)),$$

for $k \geq 1$. One way to see this is to use the algorithm presented in § 2 and § 3 to find a representation with associated Euler number $e = 1$. A rather curious way to see this is to consider the orbifold W which is the mapping cylinder of the orbit map $\Sigma(p, q, pqk - 1) \to S^2$ and consider the moduli space $\mathcal{M}$ of asymptotically trivial self-dual connections in the V-bundle $E = L_e \oplus \mathbb{R}$ over W, where e is the generator of $H^2(W; \mathbb{Z}) \cong \mathbb{Z}$. Then $\dim \mathcal{M} = R(e) = 1$ (cf. **[10]**), so that (perhaps after a compact perturbation) there is a component of $\mathcal{M}$ which is an arc with one endpoint corresponding to the reducible self-dual connection. Because the energy of a self-dual connection on E is $1/(pq(pqk - 1))$ which is less than 4, $4/p$, $4/q$,

or $4/(pq(k-1))$, it follows that no sequence of self-dual connections on E can converge to an instanton, either interior to W or at a cone point. Thus the non-compact end of $\mathcal{M}$ consists of connections of the form $B\#_\rho C$ as in [**13**, 1c.1], where C is a self-dual connection over $Y=\Sigma\times\mathbb{R}$ which is asymptotically trivial near $+\infty$ and tends asymptotically to a flat connection a_α at $-\infty$, and where B is a self-dual connection over W which tends asymptotically to a_α. For any asymptotically trivial connection A in E over W we have

$$\frac{1}{8\pi^2}\int_W \mathrm{Tr}(F_A\wedge F_A)=\frac{e^2}{(pq(pqk-1))}=\frac{1}{(pq(pqk-1))}.$$

Then

$$CS(\alpha,\theta)=\frac{1}{8\pi^2}\int_Y \mathrm{Tr}(F_C\wedge F_C)\leqslant\frac{1}{(pq(pqk-1))},$$

so that $CS(\alpha,\theta)=1/(pq(pqk-1))$.

Let Θ_3^H denote the group of oriented homology 3-spheres modulo the equivalence relation of oriented homology cobordism. In [**10**] it is shown that if $R(p,q,r)=R(p,q,r;1)\geqslant 1$, then $\Sigma(p,q,r)$ has infinite order in Θ_3^H. (For the background for this argument, and for those below, we refer the reader to [**10**].)

THEOREM 5.1. *Let p and q be pairwise relatively prime integers. The collection of homology 3-spheres $\{\Sigma(p,q,pqk-1)|\ k\geqslant 1\}$ is linearly independent over $\mathbb{Z}$ in Θ_3^H.*

Proof. Fix $k\geqslant 2$ and suppose that $\Sigma(p,q,pqk-1)=\sum_{j=1}^k n_j\Sigma(p,q,pqj-1)$ in Θ_3^H, where $n_j\in\mathbb{Z}$ and $n_k\leqslant 0$. Then there is a cobordism Y between $\Sigma(p,q,pqk-1)$ and the disjoint union $\coprod_{j=1}^k n_j\Sigma(p,q,pqj-1)$ with Y having the cohomology of a $(1+\sum|n_j|)$-punctured 4-sphere. Now cap off the $-n_k$ copies of $-\Sigma(p,q,pqk-1)$ by adjoining to Y the positive definite canonical resolutions Z bounded by $-\Sigma(p,q,pqk-1)$. Let X be the resulting positive definite 4-manifold. Let W denote the mapping cylinder of the orbit map $\Sigma(p,q,pqk-1)\to S^2$. Finally, let $\tilde{X}=W\cup_{\Sigma(p,q,pqk-1)}X$ and consider the SO(3)-V-bundle $E=L_e\oplus\mathbb{R}$ over the positive definite orbifold $\tilde{X}$, where

$$e\in H^2(\tilde{X};\mathbb{Z})=H^2(W;\mathbb{Z})\oplus\left(\bigoplus_{i=1}^{-n_k}H^2(Z;\mathbb{Z})\right)$$

is a generator of $H^2(W;\mathbb{Z})=\mathbb{Z}$. For any asymptotically trivial connection A in E over $\tilde{X}$ we have

$$\frac{1}{8\pi^2}\int_{\tilde{X}}\mathrm{Tr}(F_A\wedge F_A)=e^2/(pq(pqk-1))=1/(pq(pqk-1)).$$

The moduli space $\mathcal{M}$ of asymptotically trivial self-dual connections in E has dimension $R(p,q,pqk-1)=1$, so that (perhaps after a compact perturbation) there is a component of $\mathcal{M}$ which is an arc with one endpoint corresponding to the reducible self-dual connection. Then as in the argument above, the non-compact end of $\mathcal{M}$ consists of connections of the form $B\#_\rho C$ where C is self-dual connection over $Y=\pm\Sigma(p,q,pqj-1)\times\mathbb{R}$, for some j, which is asymptotically trivial near $+\infty$ and is asymptotically a flat connection a_α near $-\infty$, and B is a

self-dual connection over $\bar{X}$ which is asymptotically a flat connection at the ends of $\bar{X}$ so that

$$\frac{1}{8\pi^2}\int_{\bar{X}} \mathrm{Tr}(F_B \wedge F_B) \geq 0.$$

However,

$$\begin{aligned}\frac{1}{(pq(pqk-1))} &= \frac{1}{8\pi^2}\int_{\bar{X}} \mathrm{Tr}(F_A \wedge F_A)\\ &= \frac{1}{8\pi^2}\int_{Y} \mathrm{Tr}(F_C \wedge F_C) + \frac{1}{8\pi^2}\int_{\bar{X}} \mathrm{Tr}(F_B \wedge F_B)\\ &> \frac{1}{pq(pqk-1)} + \frac{1}{8\pi^2}\int_{\bar{X}} \mathrm{Tr}(F_B \wedge F_B),\end{aligned}$$

so that

$$\frac{1}{8\pi^2}\int_{\bar{X}} \mathrm{Tr}(F_B \wedge F_B) < 0,$$

a contradiction.

This theorem was originally proved by Furuta [**14**], using a similar technique.

Other non-cobordism relationships can be detected by the explicit computations of $\tau(\Sigma(p, q, r))$. For example, $\tau(\Sigma(2, 3, 7)) = 25/42$ and $\tau(-\Sigma(2, 3, 7)) = 4 - 121/42 = 47/42$. Thus, the proof of Theorem 5.1 shows that $\Sigma(2, 3, 5)$ is not a multiple of $\Sigma(2, 3, 7)$.

In Theorem 5.1 we have only used information that involves the integral term of the computation of the instanton grading of a given flat connection α, i.e. only involving $CS(\alpha, \theta)$. More information can be obtained from the full computation.

For an arbitrary homology 3-sphere Σ and representation $\alpha \in \mathcal{R}(\Sigma)$ let $I(\alpha)$ denote the formal dimension of the moduli space of self-dual connections C over $\Sigma \times \mathbb{R}$ that are asymptotically trivial near $+\infty$ and are asymptotically the flat connection a_α near $-\infty$ and such that

$$\frac{1}{8\pi^2}\int_{\Sigma\times\mathbb{R}} \mathrm{Tr}(F_C \wedge F_C) = CS(\alpha) \in [0, 4).$$

THEOREM 5.2. *Suppose $R(a_1, \ldots, a_n) \geq 1$. If $\Sigma(a_1, \ldots, a_n)$ is homology cobordant to a homology 3-sphere Σ, then*

(1) $\tau(E) \leq \tau(\Sigma(a_1, \ldots, a_n)) = 1/a_1 \ldots a_n$, *and*

(2) $1 \leq I(\alpha) \leq R(a_1, \ldots, a_n)$ *for some representation* $\alpha \in \mathcal{R}(\Sigma)$ *with* $CS(\alpha) \leq 1/a_1 \ldots a_n$.

Furthermore, $\Sigma(a_1, \ldots, a_n)$ is not homology cobordant via a simply-connected homology cobordism to any other Σ.

Proof. Following the proof of Theorem 5.1 we obtain a reducible (asymptotically trivial) self-dual connection on the bundle $E = L_e \oplus \mathbb{R}$ over the union $\bar{X}$ of the mapping cylinder W of $\Sigma(a_1, \ldots, a_n)$ and the homology cobordism. Again let $\mathcal{M}$ be the moduli space of asymptotically trivial self-dual connections on E. Any asymptotically trivial connection A on E satisfies

$$\frac{1}{8\pi^2}\int_{\bar{X}} \mathrm{Tr}(F_A \wedge F_A) = \frac{1}{a_1 \ldots a_n}.$$

This means, as above, that any non-compact ends of $\mathcal{M}$ are composed of connections of the form $B\#_\rho C$ where C is a self-dual connection over $\Sigma\times\mathbb{R}$ which is asymptotically trivial near $+\infty$ and is asymptotically a flat connection a_α near $-\infty$, and B is a self-dual connection over $\tilde{X}$ which tends asymptotically to a_α. It follows further that B is irreducible. Since $\dim\mathcal{M}=R(a_1,\ldots,a_n)$ is odd, we may use the technique of [**12**] for cutting down the moduli space to find a 1-dimensional submanifold $\mathcal{N}$ of $\mathcal{M}$ which is non-compact and has one endpoint corresponding to the reducible self-dual connection.

Now apply the proof of Theorem 5.1 to $\mathcal{N}$ to verify claim (1) and to show the existence of a self-dual connection C on $\Sigma\times\mathbb{R}$ which is asymptotically trivial near $+\infty$ and is asymptotically some flat connection a_α near $-\infty$, for $\alpha\in\mathcal{R}(\Sigma)$. This implies that $I(\alpha)\geqslant 1$ because of translational invariance in the $\mathbb{R}$-factor (cf. [**13**]). The other inequality also follows from Theorem 5.1.

The last statement follows as in Proposition 1.7 of [**16**]. Let U be a simply-connected homology cobordism from $\Sigma(a_1,\ldots,a_n)$ to Σ, and let V be the simply-connected homology cobordism from $\Sigma(a_1,\ldots,a_n)$ to itself obtained by doubling U along Σ. We obtain a reducible (asymptotically trivial) self-dual connection on the bundle $E=L_e\oplus\mathbb{R}$ over the union $\tilde{X}$ of the mapping cylinder W of $\Sigma(a_1,\ldots,a_n)$ with infinitely many copies of V adjoined. Again let $\mathcal{M}$ be the moduli space of asymptotically trivial self-dual connections on E. Now, since $\tilde{X}$ has a simply-connected end, $\mathcal{M}$ is compact [**16**]. As above, since $\dim\mathcal{M}=R(a_1,\ldots,a_n)>0$, we can cut down to obtain a compact moduli space with one end point, a contradiction.

For example, $\tau(2,7,15)=\tau(2,3,35)$ and $R(2,3,35)=1$. However, for the unique α in $\mathcal{R}(\Sigma(2,7,15))$ with $CS(\alpha)=\tau(\Sigma(2,7,15))$, $I(\alpha)=-5$, so that $\Sigma(2,7,15)$ is not homology cobordant to $\Sigma(2,3,35)$. These and other homology cobordism phenomena that are derived from the computations of instanton homology groups will be discussed in a later manuscript.

6. *Examples*

In this section we list a few computations of instanton homology groups I_i of Brieskorn homology three spheres $\Sigma(p,q,r)$. As shown in § 3, I_i is free over $\mathbb{Z}$ and vanishes for odd i, so we denote the instanton homology $I_*(\Sigma(p,q,r))$ of $\Sigma(p,q,r)$ as an ordered 4-tuple (f_0,f_1,f_2,f_3) where f_i is the rank of $I_{2i}(\Sigma(p,q,r))$.

$$I_*(\Sigma(2,3,6k\pm1))=\begin{cases}(\tfrac12(k\mp1),\tfrac12(k\pm1),\tfrac12(k\mp1),\tfrac12(k\pm1) & \text{for } k \text{ odd},\\ (\tfrac12k,\tfrac12k,\tfrac12k,\tfrac12k) & \text{for } k \text{ even};\end{cases}$$

$$I_*(\Sigma(2,5,10k\pm1))=\begin{cases}(\tfrac12(3k\mp1),\tfrac12(3k\pm1),\tfrac12(3k\mp1),\tfrac12(3k\pm1)) & \text{for } k \text{ odd},\\ (\tfrac12(3k),\tfrac12(3k),\tfrac12(3k),\tfrac12(3k)) & \text{for } k \text{ even};\end{cases}$$

$$I_*(\Sigma(2,5,10k\pm3))=\begin{cases}(\tfrac12(3k\pm1),\tfrac12(3k\pm1),\tfrac12(3k\pm1),\tfrac12(3k\pm1)) & \text{for } k \text{ odd},\\ (\tfrac12(3k\pm2),\tfrac12(3k),\tfrac12(3k\pm2),\tfrac12(3k)) & \text{for } k \text{ even};\end{cases}$$

$$I_*(\Sigma(2,7,14k\pm1))=\begin{cases}(3k\mp1,3k\pm1,3k\mp1,3k\pm1) & \text{for } k \text{ odd},\\ (3k,3k,3k,3k) & \text{for } k \text{ even};\end{cases}$$

$$I_*(\Sigma(2,7,14k\pm3))=(3k,3k\pm1,3k,3k\pm1);$$

$$I_*(\Sigma(2,7,14k\pm5))=(3k\pm1,3k\pm1,3k\pm1,3k\pm1);$$

$$I_*(\Sigma(3,4,12k\pm 1)) = \begin{cases} (\tfrac{1}{2}(5k\mp 1), \tfrac{1}{2}(5k\pm 1), \tfrac{1}{2}(5k\mp 1), \tfrac{1}{2}(5k\pm 1)) & \text{for } k \text{ odd,} \\ (\tfrac{1}{2}(5k), \tfrac{1}{2}(5k), \tfrac{1}{2}(5k), \tfrac{1}{2}(5k)) & \text{for } k \text{ even;} \end{cases}$$

$$I_*(\Sigma(3,4,12k-5)) = \begin{cases} (\tfrac{1}{2}(5k-1), \tfrac{1}{2}(5k-3), \tfrac{1}{2}(5k-1), \tfrac{1}{2}(5k-3)) & \text{for } k \text{ odd,} \\ (\tfrac{1}{2}(5k-2), \tfrac{1}{2}(5k-2), \tfrac{1}{2}(5k-2), \tfrac{1}{2}(5k-2)) & \text{for } k \text{ even;} \end{cases}$$

$$I_*(\Sigma(3,5,15k\pm 2)) = \begin{cases} (4k, 4k\pm 1, 4k, 4k\pm 1) & \text{for } k \text{ odd,} \\ (4k\pm 1, 4k, 4k\pm 1, 4k) & \text{for } k \text{ even.} \end{cases}$$

Added in proof, January 1990. The definition given just before the statement of Theorem 5.2 is only correct for a non-degenerate representation α. When α is degenerate the correct definition is $I(\alpha) = \dim \mathcal{M} + h_\alpha$, where $\dim \mathcal{M}$ is the formal dimension mentioned. Theorem 5.2 still holds. For further information see the authors' forthcoming paper 'Invariants for homology 3-spheres'.

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