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INTEGER GRADED INSTANTON HOMOLOGY GROUPS FOR HOMOLOGY THREE SPHERES

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§1. INTRODUCTION

LET Σ be an oriented integral homology 3-sphere. In this paper we associate to Σ a discrete set $\Lambda_\Sigma \subset \mathbb{R}$ and for each $\mu \in \mathbb{R}_\Sigma = \mathbb{R} \setminus \Lambda_\Sigma$ an abelian group $\mathscr{I}_*^{(\mu)}(\Sigma)$ with a natural $\mathbb{Z}$ grading. Furthermore, $\frac{1}{2}\chi(\mathscr{I}_*^{(\mu)}(\Sigma)) = \lambda(\Sigma)$, where $\lambda(\Sigma)$ is Casson's invariant (see [1]). In particular, $\frac{1}{2}\chi(\mathscr{I}_*^{(\mu)}(\Sigma)) \equiv \mu(\Sigma) \pmod 2$, where $\mu(\Sigma)$ is the Kervaire–Milnor–Rochlin invariant of Σ. These groups will depend on μ only through the interval in $\mathbb{R}_\Sigma$ in which μ lies; i.e. if the interval $[\mu_0, \mu_1] \subset \mathbb{R}_\Sigma$, then as graded groups $\mathscr{I}_*^{(\mu_0)}(\Sigma) = \mathscr{I}_*^{(\mu_1)}(\Sigma)$. Furthermore, the set Λ_Σ is invariant under the translation $r \mapsto r+1$ and $\mathscr{I}_*^{(\mu+1)}(\Sigma) = \mathscr{I}_{*+8}^{(\mu)}(\Sigma)$.

The groups $\mathscr{I}_*^{(\mu)}(\Sigma)$ can be viewed as integer lifts of the Floer homology groups $\mathscr{I}_*(\Sigma)$, $* \in \mathbb{Z}_8$, introduced in [4], in the following sense. Let $\mathscr{R}(\Sigma)$ denote the space of conjugacy classes of representations $a: \pi_1(\Sigma) \to SU(2)$, and let θ denote the class of the trivial representation. For simplicity assume that all nontrivial representations are regular, i.e. that $H^1(\Sigma; ad(a)) = 0$ for every representation $a \in \mathscr{R}^*(\Sigma) = \mathscr{R}(\Sigma) \setminus \{\theta\}$. By carefully analyzing the behavior of the Chern–Simons invariant $c: \tilde{\mathscr{B}}_\Sigma \to \mathbb{R}$ on the infinite cyclic cover $\tilde{\mathscr{B}}_\Sigma$ of the space of gauge equivalence classes of connections $\mathscr{B}_\Sigma$ in the trivial bundle over Σ, we associate in §2 to each $a \in \mathscr{R}(\Sigma)$ and $\mu \in \mathscr{R}_\Sigma$ a well-defined integer $\iota^{(\mu)}(a)$ and define $\mathscr{R}_n^{(\mu)}(\Sigma) = \mathbb{Z}\{a \in \mathscr{R}(\Sigma) \setminus \{\theta\} \mid \iota^{(\mu)}(a) = n\}$. If we let $R_*(\Sigma)$, $* \in \mathbb{Z}_8$ denote Floer's chain groups [4], then $\Sigma_{j \in \mathbb{Z}} \mathscr{R}_{n+8j}^{(\mu)}(\Sigma) \cong R_n(\Sigma)$, for $n \in \mathbb{Z}_8$. After defining a boundary operator $\partial^{(\mu)}$: $\mathscr{R}_*^{(\mu)}(\Sigma) \to \mathscr{R}_{*-1}^{(\mu)}(\Sigma)$ similar to that of Floer's [4] and showing in §2 that $\partial^{(\mu)}\partial^{(\mu)} = 0$, we have the resulting homology groups $\mathscr{I}_*^{(\mu)}(\Sigma)$. In general, for $* \in \mathbb{Z}_8$, $\Sigma_{j \in \mathbb{Z}} \mathscr{I}_{*+8j}^{(\mu)}(\Sigma) \neq I_*(\Sigma)$. However, in §5 we construct a spectral sequence with $\mathscr{I}_*^{(\mu)}(\Sigma)$ as its E^1 term and converging to $I_*(\Sigma)$.

It is not always the case that all $a \in \mathscr{R}(\Sigma)$ are regular, so in §3 we show how to define these instanton homology groups in general by perturbing the Chern–Simons function. Then in §4 we show that these groups are independent of perturbation and are thus topological invariants.

The Floer homology groups satisfy suggestive functorial properties. In particular, let X be an oriented smooth cobordism from Σ_1 to Σ_2. By counting instantons over X, Floer [4] shows that X induces homomorphisms $X_*: I_*(\Sigma_1) \to I_{*+3(b_1(X) - b_2^-(X))}(\Sigma_2)$. In §5 we show that if $\mu \in \mathbb{R} \setminus (\Lambda_{\Sigma_1} \cup \Lambda_{\Sigma_2})$, then X also induces homomorphisms X_*: $\mathscr{I}_*^{(\mu)}(\Sigma_1) \to I_{*+3(b_1(X) - b_2^-(X))}^{(\mu)}(\Sigma_2)$ such that

(1) $(\Sigma \times \mathbb{R})_* = \text{identity}$
(2) $(XY)_* = X_{*+3(b_1(Y) - b_2^-(Y))} Y_*$

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An important feature of Floer's homology groups is presented in forthcoming work of Donaldson which gives relative versions of his polynomial invariants in terms of the Floer homology of X. In particular, if X is an oriented simply connected smooth 4-mainfold with oriented boundary Σ and $S_1, \ldots, S_d \in H_2(X;\mathbb{Z})$, then there are relative Donaldson invariants $q(S_1, \ldots, S_d) \in I_{-3(1+b_2^-(X))-2d}(\Sigma)$. Using the ideas presented in §2 and §5 we construct in §6 for each such X and $S_1, \ldots, S_d \in H_2(X;\mathbb{Z})$ a sequence of invariants which filter the Donaldson invariant.

In §7 we point out that the computations given in [3] of Floer's instanton homology groups $I_*(\Sigma)$, $* \in \mathbb{Z}_8$, for Σ a Brieskorn sphere actually give an algorithm for computing $\mathscr{I}_*^{(\mu)}(\Sigma)$, $* \in \mathbb{Z}$, and we list some explicit computations. These computations indicate some interesting properties of these instanton homology groups. In particular, they are not mod 4 periodic as has been conjectured for the Floer groups.

§2. INSTANTON HOMOLOGY FOR REGULAR REPRESENTATIONS

Let Σ be a homology 3-sphere. Our goal in this section is to describe the integer-graded instanton homology groups under the special assumption that all representations are regular, i.e. that $H^1(\Sigma; ad(a)) = 0$ for every representation $a \in \mathscr{R}(\Sigma)$. This means that the space $\mathscr{R}(\Sigma)$ of conjugacy classes of representations of $\pi_1(\Sigma)$ into $SU(2)$ is finite and that the Chern–Simons function has non-degenerate Hessian at each nontrivial $a \in \mathscr{R} = \mathscr{R}(\Sigma)$. Recall that the Chern–Simons function is defined by fixing a trivialization and thus a trivial connection θ on the trivial $SU(2)$ bundle over Σ. The space of $L_1^4\, SU(2)$-connections, $\mathscr{A}_\Sigma$, over Σ can be identified with $L_1^4(\Omega_\Sigma^1 \otimes su(2))$. For each $a \in \mathscr{A}_\Sigma$ choose a path A from a to θ in $\mathscr{A}_\Sigma$. Identify A as a connection over $\Sigma \times I$ and let $c(a)$ be the Chern–Weil integral of the connection A, i.e.

$$c(a) = \frac{-1}{8\pi^2}\int_{\Sigma \times I} \operatorname{Tr}(F_A \wedge F_A).$$

The map $c\colon \mathscr{A}_\Sigma \to \mathbb{R}$ is the Chern–Simons function (and depends on the choice of the trivial connection θ); its critical points are the flat connections. A gauge transformation over Σ is a map $g\colon \Sigma \to SU(2)$. Let $\mathscr{G}_\Sigma$ denote the Banach Lie group of L_2^4 gauge transformations. For $a \in \mathscr{A}_\Sigma$ and $g \in \mathscr{G}_\Sigma$ we have

$$(*) \qquad c(g(a)) = c(a) + \deg(g).$$

Thus c descends to the Banach manifold $\mathscr{B}_\Sigma = \mathscr{A}_\Sigma/\mathscr{G}_\Sigma$ as a map to $\mathbb{R}/\mathbb{Z}$. Since conjugacy classes of representations can be identified with gauge equivalence classes of flat connections in $\mathscr{B}_\Sigma$, we may view $\mathscr{R}(\Sigma) \subset \mathscr{B}_\Sigma$ as the critical set of $c\colon \mathscr{B}_\Sigma \to \mathbb{R}/\mathbb{Z}$.

Very briefly, Floer's instanton homology groups $I_*(\Sigma)$, $* \in \mathbb{Z}_8$, are defined as follows. (See [4] for details.) The chains are generated by the elements of $\mathscr{R}^* = \mathscr{R}\backslash\{\theta\}$, and for $n \in \mathbb{Z}_8$ the chain group

$$R_n(\Sigma) = \mathbb{Z}\{a \in \mathscr{R}^*(\Sigma) \mid SF(a) \equiv n \pmod 8\}$$

where $SF(a)$ is the spectral flow of the Hessian $\star d_a$ of the Chern–Simons function along any path in $\mathscr{B}_\Sigma$ from a to θ. (The Hodge star operator $\star$ is taken with respect to some fixed metric on Σ.) For $a \in \mathscr{B}_\Sigma$, the spectral flow is defined only mod 8 since gauge transformations g change the value by 8 deg(g).

Let $\mathscr{M}(a, b)$ denote the moduli space of self-dual connections on $\Sigma \times \mathbb{R}$ which tend asymptotically to a and b as $\tau \to \pm\infty$. The dimensions of the components of $\mathscr{M}(a, b)$, $a, b \in \mathscr{R}^*$ are all congruent mod 8. Since the connections in $\mathscr{M}(a, b)$ are translationally (τ)

invariant, we may divide out by this invariance to obtain $\hat{\mathcal{M}}(a, b) = \mathcal{M}(a, b)/\mathbb{R}$. Let $\mathcal{M}^i(a, b)$ denote the union of the components of dimension i, and let $\#(\hat{\mathcal{M}}^1(a, b))$ count the points with orientations $(\pm)$ in $\hat{\mathcal{M}}^1(a, b)$. The boundary operator of Floer's chain complex is defined by

$$\partial a = \sum_{b \in \mathcal{R}^*(\Sigma)} \#(\hat{\mathcal{M}}^1(a, b))b.$$

Floer shows that $\partial\partial = 0$, and the homology groups of this complex are Floer's instanton homology groups $I_*(\Sigma), * \in \mathbb{Z}_8$.

In order to extend this theory to one with an integer grading we make use of the infinite cyclic (universal) cover $\tilde{\mathcal{B}}_\Sigma$ of $\mathcal{B}_\Sigma$ defined as $\tilde{\mathcal{B}}_\Sigma = \mathcal{A}/\mathcal{G}_{\Sigma,0}$ where $\mathcal{G}_{\Sigma,0} \subset \mathcal{G}_\Sigma$ is the group of degree zero gauge transformations. The Chern–Simons and spectral flow functions on $\mathcal{A}$ descend to functions c and $\mathrm{SF}: \tilde{\mathcal{B}}_\Sigma \to \mathbb{R}$. (Recall that c and SF still depend on the choice of the trivial connection θ.) Let Λ_Σ be the image $c(\tilde{\mathcal{R}}^*(\Sigma))$; mod $\mathbb{Z}$, Λ_Σ is a finite set. The equation $(*)$ shows that Λ_Σ is independent of the choice of θ. The set $\mathbb{R}_\Sigma = \mathbb{R} \setminus \Lambda_\Sigma$ consists of the regular values of the Chern–Simons function. For any $\mu \in \mathbb{R}_\Sigma$, we shall form integer graded instanton homology groups $\mathcal{I}_n^{(\mu)}(\Sigma), n \in \mathbb{Z}$. These groups $\mathcal{I}_n^{(\mu)}(\Sigma)$ will depend on μ only through the interval in $\mathbb{R}_\Sigma$ in which μ lies; i.e. if the interval $[\mu_0, \mu_1] \subset \mathbb{R}_\Sigma$, then $\mathcal{I}_*^{(\mu_0)}(\Sigma) = \mathcal{I}_*^{(\mu_1)}(\Sigma)$. From their definition it will be clear that $\mathcal{I}_*^{(\mu+1)}(\Sigma) = \mathcal{I}_{*+8}^{(\mu)}(\Sigma)$.

To begin, let $\mathcal{B}(a, b)$ be defined as follows (c.f. [4, §2]). For $a, b \in \mathcal{B}_\Sigma$ choose any smooth representatives $a, b \in \mathcal{A}_\Sigma$ and a connection A on $\Sigma \times \mathbb{R}$ whose τ-component vanishes and which equals a for large negative τ and equals b for large positive τ. Define $\mathcal{A}_\delta(a, b) = A + L^4_{1,\delta}(\Omega^1_{\Sigma\times\mathbb{R}}(su(2))$. (Here $\|\xi\|_{L^4_{1,\delta}} = \|e_\delta \cdot \xi\|_{L^4_{1,\delta}}$ where $e_\delta: \Sigma \times \mathbb{R} \to (0, \infty)$ is such that $e_\delta(x, \tau) = e^{\delta|\tau|}$ for $|x| \geq 1, \delta > 0$.) It is acted on by the gauge group

$$\mathcal{G}_\delta = \{g \in L^4_{2,\mathrm{loc}}(\Sigma \times \mathbb{R}, SU(2)) | \exists R > 0,\ \xi \in L^4_{2,\delta}(\Omega^0_{\Sigma\times\mathbb{R}}(su(2)) \ni g = \exp(\xi) \text{ for } |\tau| \geq R\}.$$

Then the quotient space $\mathcal{B}_\delta(a, b) = A_\delta(a, b)/\mathcal{G}_\delta$ is a smooth Banach manifold. (We will drop the "δ's" from our notation.)

The next proposition proclaims the existence of the so-called temporal gauge (cf. [4]).

Proposition 2.1. *Each $A \in \mathcal{A}(a, b)$ is gauge equivalent to a connection whose component in the $\mathbb{R}$-direction vanishes.* ∎

This proposition says that we can view $\mathcal{B}(a, b)$ as the space of paths in $\mathcal{B}_\Sigma$ from a to b. For lifts $\hat{a}, \hat{b} \in \tilde{\mathcal{B}}_\Sigma$ of a and b, write $\tilde{\mathcal{B}}(\hat{a}, \hat{b})$ for lifts of connections in $\mathcal{B}(a, b)$.

To define $\mathcal{I}_*^{(\mu)}(\Sigma)$ we need to redefine the chain groups. Given $a \in \mathcal{R}^*(\Sigma) \subset \mathcal{B}_\Sigma$, let $a^{(\mu)} \in \tilde{\mathcal{R}}(\Sigma) \in \tilde{\mathcal{B}}_\Sigma$ be the unique lift of a such that $c(a^{(\mu)}) \in (\mu, \mu + 1)$. Let $\iota^{(\mu)}(a) = SF(a^{(\mu)})$ and define, for $n \in \mathbb{Z}$, the chain group

$$\mathcal{R}_n^{(\mu)}(\Sigma) = \mathbb{Z}\{a \in \mathcal{R}^*(\Sigma) | \iota^{(\mu)}(a) = n\}.$$

Note that by equation $(*)$ the definition of $\iota^{(\mu)}(a)$ is independent of the choice of trivial connection θ used to define c. For if $\bar{\theta}$ is another choice of trivial connection and $g(\theta) = \bar{\theta}$ for some gauge transformation g, then the corresponding choice of lift $\bar{a}^{(\mu)}$ of a is just $g(a^{(\mu)})$.

If $a \in \mathcal{R}_n^{(\mu)}(\Sigma)$, define $\partial^{(\mu)}: \mathcal{R}_n^{(\mu)}(\Sigma) \to \mathcal{R}_{n-1}^{(\mu)}(\Sigma)$ by

$$\partial^{(\mu)} a = \sum_{b \in \mathcal{R}_{n-1}^{(\mu)}(\Sigma)} \#(\hat{\mathcal{M}}^1(a, b))b.$$

We will now show that $\partial^{(\mu)}\partial^{(\mu)} = 0$. The corresponding homology groups are $\mathcal{I}_*^{(\mu)}(\Sigma)$, $* \in \mathbb{Z}$.

We first need an index calculation for the self-duality operator D_A for $A \in \mathscr{B}(a, b)$. This is [4, Proposition 2b.2].

PROPOSITION 2.2 [4]. *If $A \in \mathscr{B}(a,b)$ for $a, b \in \mathscr{R}^*$ and $\hat{A}$ is any lift to $\tilde{\mathscr{B}}(\hat{a}, \hat{b})$, then $\operatorname{ind} D_{\hat{A}} = SF(\hat{a}) - SF(\hat{b})$.* ∎

A key fact in showing $\partial^{(\mu)}\partial^{(\mu)} = 0$ is Floer's observation that the gradient trajectories of the Chern–Simons function are the self-dual connections on $\Sigma \times \mathbb{R}$ and that the Chern–Simons function is nonincreasing along the gradient trajectories in $\tilde{\mathscr{B}}_\Sigma$. Thus if $\tilde{A} \in \tilde{\mathscr{M}}(\hat{a}, \hat{b})$, then $c(\hat{b}) \leq c(\hat{a})$. Now to check that $\partial^{(\mu)}\partial^{(\mu)} = 0$ we follow Floer's argument. If $a \in \mathscr{R}^{(\mu)}_{n+1}(\Sigma)$ then the coefficient of $c \in \mathscr{R}^{(\mu)}_{n-1}(\Sigma)$ in $\partial^{(\mu)}\partial^{(\mu)}(a)$ is

$$\sum_{b \in \mathscr{R}^{(\mu)}_n(\Sigma)} \# \hat{\mathscr{M}}^1(a, b) \cdot \# \hat{\mathscr{M}}^1(b, c). \tag{2.3}$$

According to [4, Prop. 1c.1] the boundary of the 1-dimensional manifold $\hat{\mathscr{M}}^2(a, c) = \mathscr{M}^2(a, c)/\mathbb{R}$ corresponds to splittings $\mathscr{M}^1(a, b) \times \mathscr{M}^1(b, c)$. Each term $\# \hat{\mathscr{M}}^1(a, b) \cdot \# \hat{\mathscr{M}}^1(b, c)$ is the number of oriented boundary points of $\hat{\mathscr{M}}^2(a, b)$ corresponding to splittings where $b \in \mathscr{R}^{(\mu)}_n(\Sigma)$. For any such $b \in \mathscr{R}^{(\mu)}_n(\Sigma)$, there are gradient trajectories $A \in \mathscr{M}^1(a, b)$ and $B \in \mathscr{M}^1(b, c)$. The other end of the corresponding component of the 1-manifold $\hat{\mathscr{M}}^2(a, c)$ corresponds to some splitting $\mathscr{M}^1(a, d) \times \mathscr{M}^1(d, c)$ with gradient trajectories $A' \in \mathscr{M}^1(a, d)$ and $B' \in \mathscr{M}^1(d, c)$. Then $\hat{\mathscr{M}}^2(a, c)$ provides a 1-parameter family of paths in $\mathscr{B}_\Sigma$ from a to c with ends $A \#_\rho B$ and $A' \#_{\rho'} B'$ for appropriate graftings (see [4, 1c.1]). If we lift A to $\tilde{A} \in \tilde{\mathscr{M}}^1(a^{(\mu)}, \hat{b})$, then

$$1 = \operatorname{ind} D_{\tilde{A}} = SF(a^{(\mu)}) - SF(\hat{b}) = n + 1 - SF(\hat{b}).$$

Thus $SF(\hat{b}) = n$; so $b = b^{(\mu)}$, the preferred lift, and $\tilde{A} \in \tilde{\mathscr{M}}^1(a^{(\mu)}, b^{(\mu)})$. Similarly $\tilde{B} \in \hat{\mathscr{M}}^1(b^{(\mu)}, c^{(\mu)})$.

Since $A' \#_{\rho'} B'$ is homotopic rel ends to $A \#_\rho B$ in $\mathscr{B}_\Sigma$, the lift $\tilde{A}' \#_\rho \tilde{B}'$, starting at $a^{(\mu)}$, ends at $c^{(\mu)}$. Say $\tilde{A}' \in \tilde{\mathscr{M}}(a^{(\mu)}, \hat{d})$. Since the Chern–Simons function is nonincreasing along the gradient trajectory $\tilde{A}'$, we get $\mu + 1 > c(a^{(\mu)}) \geq c(\hat{d}) \geq c(c^{(\mu)}) > \mu$. Thus $\hat{d} = d^{(\mu)}$. And again $1 = \operatorname{ind} D_{\tilde{A}} = \iota^{(\mu)}(a^{(\mu)}) - \iota^{(\mu)}(d^{(\mu)})$; so $d^{(\mu)} \in \mathscr{R}^{(\mu)}_n$. As in Floer's proof, $A' \#_{\rho'} B'$ contributes to the sum (2.3) with the opposite sign as does $A \#_\rho B$. So $\partial^{(\mu)}\partial^{(\mu)} = 0$.

§3. PERTURBATIONS

In general, the representation space $\mathscr{R}^*(\Sigma) \subset \mathscr{B}_\Sigma$ may contain degenerate critical points of the Chern–Simons function and the self-dual moduli spaces $\mathscr{M}(a, b)$ may not be manifolds. In this case we must perturb the equations involved. We first describe Floer's construction [4], and then explain how it must be modified to suit our purposes. Floer considers maps

$$\gamma\colon \bigvee_{i=1}^{m} S^1_i \times D^2 \to \Sigma$$

which restrict to smooth embeddings $\gamma_\vartheta\colon \bigvee_{i=1}^m S^1_i \times \{\vartheta\} \to \Sigma$ for each $\vartheta \in D^2$ and $\gamma_i\colon S^1_i \times D^2 \to \Sigma$ for each i. Let Γ_m be the set of these maps. We then get a family of holonomy maps $\bar{\gamma}_\vartheta\colon \mathscr{B}_\Sigma \to L_m = SU(2)^m/ad(SU(2))$. Let $d^2\vartheta$ be a compactly supported volume form on $\operatorname{int} D^2$, and let $C^\infty(L_m, \mathbb{R})$ be the set of smooth $\operatorname{ad}(SU(2))$-invariant $\mathbb{R}$-valued functions on $SU(2)^m$. For each $h \in C^\infty(L_m, \mathbb{R})$, Floer defines $h_\gamma\colon \mathscr{B}_\Sigma \to \mathbb{R}$ by $h_\gamma(a) = \int_{D^2} h(\bar{\gamma}_\vartheta(a))\, d^2\vartheta$. Floer's set of perturbations is $\Pi = \bigcup_{m \in \mathbb{N}} \Gamma_m \times C^\infty(L_m, \mathbb{R})$. He proves [4, §1b.1]:

(3.1) *For each* $\pi = (\gamma, h) \in \Pi$, h_γ *is a smooth function on* $\mathscr{B}_\Sigma$, *and for each smooth metric* σ *on* Σ *there is a smooth section* $\text{grad}_\sigma h_\gamma$ *of* $T\mathscr{B}_\Sigma$ *such that for each* $\xi \in T_a\mathscr{B}_\Sigma$, $\langle \text{grad}_\sigma h_\gamma(a), \xi \rangle = Dh_\gamma(a)\xi$.

For a smooth metric σ on Σ and for $\pi \in \Pi$, let

$$\mathscr{M}_{\sigma,\pi} = \left\{ A \in L_1^4(p^*\Omega_\Sigma^1 \otimes su(2)) \,\middle|\, \frac{\partial A}{\partial \tau} - \star_\sigma F_{A(\tau)} - \text{grad}_\sigma h_\gamma(A(\tau)) = 0,\ l(A) < \infty \right\} \Big/ \mathscr{G}$$

where

$$l(A) = \left\| \frac{\partial A}{\partial \tau} \right\|_{L^2}^2.$$

Note that the condition that a connection A over $\Sigma \times \mathbb{R}$ be self-dual is that

$$\frac{\partial A}{\partial \tau} - \star_\sigma F_a = 0;$$

so $\mathscr{M}_{\sigma,\pi}$ can be viewed as the solution space to the perturbed self-duality equations:

$$\mathscr{M}_{\sigma,\pi} = \{ A \in \mathscr{A}(\Sigma \times \mathbb{R}) \mid F_{\sigma\pi}(A) = 0,\ l(A) < \infty \}/\mathscr{G}$$

where

$$F_{\sigma\pi}(A) = \frac{\partial A}{\partial \tau} - \star_\sigma F_{A(\tau)} - \text{grad}_\sigma h_\gamma(A(\tau)).$$

(The product metric is used on $\Sigma \times \mathbb{R}$). Furthermore, for $a \in \mathscr{B}_\Sigma$ let $c_\pi(a) = c(a) + h_\gamma(a)$. Then Floer checks that c_π is nonincreasing along trajectories in $\mathscr{M}_{\sigma,\pi}$ and proves [4, 2c.2]:

(3.2). *Let* $\mathscr{S}$ *be the space of smooth metrics on* Σ *and let* Π *have the* C^∞*-topology. For a dense set of parameters* $(\sigma, \pi) \in \mathscr{S} \times \Pi$, *the critical set* $\mathscr{R}_\pi$ *of* $c_\pi = c + h_\gamma$ *is nondegenerate, and* $\mathscr{M}_{\sigma,\pi}$ *decomposes into smooth oriented manifolds* $\mathscr{M}(a, b)$ *of regular trajectories connecting* $a, b \in \mathscr{R}_\pi$.

We need to further restrict the allowable perturbations. Since $\mathscr{R}$ is compact and θ is isolated in $\mathscr{R}$, the subspace $\mathscr{R}^*$ is also compact and so is each of the finitely many connected components of $\mathscr{R}^*$, and c is constant on each of these components. Thus, as we asserted in §2, the image $\Lambda_\Sigma = c(\mathscr{R}^*)$ in $\mathbb{R}/\mathbb{Z}$ is a finite set. Let

$$M = \min\{\lambda - \mu, \mu + 1 - \lambda \mid \lambda \in (\mu, \mu + 1) \cap \Lambda_\Sigma\}.$$

For each $\rho \in \mathscr{R}^*$ we can find a neighborhood U_ρ in $\mathscr{B}_\Sigma$ such that

(1) U_ρ is evenly covered in $\tilde{\mathscr{B}}_\Sigma$.
(2) For each $a \in U_\rho$, $|c(a) - c(\rho)| < M/8$.
(3) U_ρ contains no reducible connections.

Since $\mathscr{R}^*$ is compact there is a finite subcover $\{U_{\rho_1}, \ldots, U_{\rho_k}\}$, and by Uhlenbeck's compactness theorem [6] we can find $\varepsilon_1 > 0$ such that if $\|\star F_a\|_{L^4} < \varepsilon_1$, then $a \in \bigcup_{i=1}^k U_{\rho_i}$.

Let $\varepsilon = \min(\varepsilon_1, M/8)$. Our set of allowable perturbations consists of $(\sigma, \pi) \in \mathscr{S} \times \Pi$ such that $\pi = (\gamma, h)$ satisfies the conclusions of (3.1), (3.2), and also

(4) $|h_\gamma(a)| < \varepsilon$ for all $a \in \mathscr{B}_\Sigma$.
(5) $\|\text{grad}_\sigma h_\gamma(a)\|_{L^4} < \varepsilon/2$ and $\|\text{grad}_\sigma h_\gamma(a)\|_{L^2} < \varepsilon/2$ for all $a \in \mathscr{B}_\Sigma$.

That these conditions can be achieved follows from the density statement in (3.2) and the fact that the 0-function lies in $C^\infty(L_m, \mathbb{R})$. Let $\mathscr{P}$ denote the set of $(\sigma, \pi) \in \mathscr{S} \times \Pi$ which satisfy these conditions.

Lemma 3.3. *Suppose $(\sigma, \pi) \in \mathscr{P}$ and let $a \in \mathscr{R}_\pi$, $\alpha \in \mathscr{R}$ with $a \in U_\alpha$. Let $\hat{a}$ and $\hat{\alpha}$ be lifts to $\tilde{\mathscr{B}}_\Sigma$ such that $\hat{a}$ and $\hat{\alpha}$ lie in the same lift of U_α. Then $c_\pi(\hat{a}) \in [\mu + \frac{3M}{4}, \mu + 1 - \frac{3M}{4}]$ if and only if $c(\hat{\alpha}) \in [\mu + M, \mu + 1 - M]$.*

Proof. Since $a \in U_\alpha$, $|c(\hat{a}) - c(\hat{\alpha})| < M/8 \pmod{\mathbb{Z}}$ and since they lie in the same lift of U_α to $\tilde{\mathscr{B}}_\Sigma$, $|c(\hat{a}) - c(\hat{\alpha})| < M/8$. If $c(\hat{\alpha}) \in [\mu + M, \mu + 1 - M]$ then $c(\hat{a}) \in [\mu + \frac{7M}{8}, \mu + 1 - \frac{7M}{8}]$. Since also $|h_\gamma(a)| < \varepsilon < M/8$, we get $c_\pi(\hat{a}) = c(\hat{a}) + h_\gamma(a) \in [\mu + \frac{3M}{4}, \mu + 1 - \frac{3M}{4}]$. The converse is proved similarly. ∎

Thus if $\alpha^{(\mu)}$ is the preferred lift of α and $a \in U_\alpha$, and if $\alpha^{(\mu)}$ and $\hat{a}$ lie in the same lift of U_α, then $a^{(\mu)} = \hat{a}$.

The construction of the integer graded instanton homology groups now proceeds as in §2. For $(\sigma, \pi) \in \mathscr{P}$ we have the perturbed Chern–Simons function defined above: $c_\pi(a) = c(a) + h_\gamma(a)$ for $a \in \mathscr{B}_\Sigma$, and $c_\pi(\hat{a}) = c(\hat{a}) + h_\gamma(a)$ for $\hat{a} \in \tilde{\mathscr{B}}_\Sigma$. Its critical set is $\mathscr{R}_\pi$, the zero set of $f_{\sigma\pi}(a) = \star_\sigma F_a + \mathrm{grad}_\sigma h_\gamma(a)$. If $\mu \in \mathscr{R} \setminus \Lambda_\Sigma$ and $a \in \mathscr{R}_\pi^* = \mathscr{R}_\pi \setminus \{\theta\}$, define $\imath^{(\mu)}(a) = SF(a^{(\mu)})$ where $a^{(\mu)}$ is the lift of a such that $c_\pi(a^{(\mu)}) \in (\mu, \mu + 1)$ and $SF(a^{(\mu)})$ is the spectral flow of $Df_{\sigma\pi}$ along any path from $a^{(\mu)}$ to θ in $\tilde{\mathscr{B}}_\Sigma$. Note that since $(\sigma, \pi) \in \mathscr{P}$, for each $a \in \mathscr{R}_\pi^*$ there is an $\alpha \in \mathscr{R}^*$ with $a \in U_\alpha$; hence $c_\pi(a) \in [\mu + \frac{3M}{4}, \mu + 1 - \frac{3M}{4}] \subset \mathbb{R}/\mathbb{Z}$ by Lemma 3.3.

As in §2 we define $\mathscr{R}_n^{(\mu)}(\Sigma, \sigma, \pi)$ to be the free abelian group generaed by

$$\{a \in \mathscr{R}_\pi^*(\Sigma) \mid \imath^{(\mu)}(a^{(\mu)}) = n\},$$

and if $a \in \mathscr{R}_n^{(\mu)}(\Sigma, \sigma, \pi)$.

$$\partial^{(\mu)} a = \sum_{b \in \mathscr{R}_{n-1}^{(\mu)}(\pi)} \#(\hat{\mathscr{M}}^1_{\sigma\pi}(a, b))b.$$

The proof given in §2 to show $\partial^{(\mu)}\partial^{(\mu)} = 0$ works here as well. It is only necessary to comment that the analogue of Lemma 2.2 holds for the perturbed self-duality operator. That is, let D_A be the operator

$$D_A \colon L_{1,\delta}(\Omega^1_{\Sigma\times\mathbb{R}}(su(2))) \to L^4_{0,\delta}((\Omega^0_{\Sigma\times\mathbb{R}} \oplus \Omega^2_{\Sigma\times\mathbb{R},-}))(su(2))$$

given by $D_A\alpha = D_A^*\alpha, DF_{\sigma\pi}(A)\alpha)$. Then we have the result of Floer:

Lemma 3.4 [4, 2b1-2]. *For sufficiently small positive δ, D_A is a Fredholm operator. If $A \in \mathscr{B}(a, b)$ with $a, b \in \mathscr{R}_\pi^*$ nondegenerate and if $\hat{A}$ is any lift to $\mathscr{B}(\hat{a}, \hat{b})$ then $\mathrm{ind}\, D_A = SF(\hat{a}) - SF(\hat{b})$.* ∎

We denote the homology groups obtained from the above complex by $\mathscr{I}_*^{(\mu)}(\Sigma, \sigma, \pi)$ or simply $\mathscr{I}_*^{(\mu)}(\sigma, \pi)$. The goal of the next section is to show that these groups are independent of $(\sigma, \pi) \in \mathscr{P}$.

§4. INDEPENDENCE OF PERTURBATIONS

In the previous section we obtained for $(\sigma, \pi) \in \mathscr{P}$ and the product metric induced from σ on $\Sigma \times \mathbb{R}$ the perturbed self-duality equation

$$\frac{\partial A}{\partial \tau} - \star_\sigma F_{A(\tau)} - \mathrm{grad}_\sigma h_\gamma(A(\tau)) = 0.$$

This is equivalent to $F_A^- + \pi(A) = 0$ where

$$\pi(A) = \tfrac{1}{2}(\star_\sigma \mathrm{grad}_\sigma h_\gamma(A(\tau); \tau) - \mathrm{grad}_\sigma h_\gamma(A(\tau); \tau) \wedge d\tau).$$

Now consider (σ_0, π_0) and (σ_1, π_1) in $\mathscr{P}$. We may suppose that the set γ of embedded circles for π_0 and π_1 is the same by choosing $h_\gamma^{(i)} = 0$ appropriately. On $\Sigma \times \mathbb{R}$ consider the 1-parameter family (σ_τ, π_τ) that interpolates linearly from (σ_0, π_0) to (σ_1, π_1) and is constant for $\tau \leq 0$ and $\tau \geq 1$. If we define the perturbation π of the self-duality equations on $\Sigma \times \mathbb{R}$ by $\pi(A)(\tau) = \frac{1}{2}(\star_{\sigma_\tau} \mathrm{grad}_{\sigma_\tau} h_{\gamma_\tau}(A(\tau); \tau) - \mathrm{grad}_{\sigma_\tau} h_{\gamma_\tau}(A(\tau); \tau) \wedge d\tau)$, then the perturbed self-duality equations

$$F_A^- + \pi(A) = 0$$

directly generalize the defining equations of $\mathscr{M}_{\sigma\pi}$ in the case of the product $(\sigma_0, \pi_0) \times \mathbb{R}$ and restrict to them over $\Sigma \times (-\infty, 0)$ and $\Sigma \times (1, \infty)$.

Generalizing this, let $\mathscr{P}_0 = \mathscr{P}_0((\sigma_0, \pi_0), (\sigma_1, \pi_1))$ be the set of all (σ, π) where σ is a conformal structure on $\Sigma \times \mathbb{R}$ which restricts to the ends $\Sigma \times \mathbb{R}_-$ and $\Sigma \times \mathbb{R}_+$ in the conformal class of the product metrics σ_0 and σ_1 respectively, and π is a gauge-equivariant perturbation of the self-duality equations on $\Sigma \times \mathbb{R}$ which restricts to those given by π_0 and π_1 on $\Sigma \times \mathbb{R}_-$ and $\Sigma \times \mathbb{R}_+$. Floer shows that there is a dense subset $\mathscr{P}'_0$ of $\mathscr{P}_0$ for which the zero set

$$\mathscr{M}_{\sigma\pi}(a, a') = \{A \in L^4_{1,\delta}(\mathscr{B}_{\Sigma\times\mathbb{R}}(a, a')) \mid F_A^- + \pi(A) = 0,\ l(A) < \infty\}$$

is a manifold of dimension $\iota^{(\mu)}(a) - \iota^{(\mu)}(a') \pmod 8$ for all $a \in \mathscr{R}^*_{\pi_0}$ and $a' \in \mathscr{R}^*_{\pi_1}$.

We need to further restrict the class of perturbations. First consider the class $\mathscr{P}_1 \subset \mathscr{P}_0$ (resp. $\mathscr{P}'_1 \subset \mathscr{P}'_0$) of (σ, π) such that $\int_{\Sigma\times\mathbb{R}_0} \|\pi(A)\|^2_{L^2} < \varepsilon$ for all $A \in \mathscr{B}_{\Sigma\times\mathbb{R}}$, where $\mathbb{R}_0 = \mathbb{R}\backslash\{\mathbb{R}_- \cup \mathbb{R}_+\}$. Note that the perturbation π between (σ_0, π_0) and (σ_1, π_1) given above has for each τ, $\|\pi(A)(\tau)\|_{L^2(\Sigma)} < 2\cdot\varepsilon/2 = \varepsilon$ by condition (5) in the definition of the space $\mathscr{P}$ of allowable perturbations of the Chern–Simons function. Hence if $\varepsilon < 1$, it lies in $\mathscr{P}'_1$. (Here $\mathbb{R}_0 = [0, 1]$.)

Suppose we have (σ, π) and $(\sigma', \pi') \in \mathscr{P}'_1$ and $A \in \mathscr{M}_{\sigma\pi}(a, a')$ and $A' \in \mathscr{M}_{\sigma'\pi'}(a', a'')$ and suppose that (σ, π) on $\Sigma \times \mathbb{R}_+$ agrees with (σ', π') on $\Sigma \times \mathbb{R}_-$. Then we may graft A and A' as in [4] to obtain $A \#_\rho A' \in \mathscr{M}_{\sigma''\pi''}(a, a'')$ where $(\sigma'', \pi'') = (\sigma \#_\rho \sigma', \pi \#_\rho \pi')$. The perturbation (σ'', π'') is, however, no longer in $\mathscr{P}'_1$. So we define new classes of perturbations $\mathscr{P}_2$ (resp. $\mathscr{P}'_2$) to consist of those $(\sigma, \pi) \in \mathscr{P}_0((\sigma_0, \pi_0), (\sigma_2, \pi_2))$ (resp. $\mathscr{P}'_0((\sigma_0, \pi_0), (\sigma_2, \pi_2))$) such that $\mathbb{R}_0$ is the union $I_- \cup I_0 \cup I_+$ of three intervals with (σ, π) restricting to a (constant) $(\sigma_1, \pi_1) \in \mathscr{P}$ on $\Sigma \times I_0$ and with

$$\int_{\Sigma\times(I_-\cup I_+)} \|\pi(A)\|^2_{L^2} < 2\varepsilon.$$

Then if $(\sigma', \pi') \in \mathscr{P}_1((\sigma_0, \pi_0), (\sigma_1, \pi_1))$ (resp. $\mathscr{P}'_1$) and $(\sigma'', \pi'') \in \mathscr{P}_1((\sigma_1, \pi_1), (\sigma_2, \pi_2))$ (resp. $\mathscr{P}'_1$) we have $(\sigma' \#_\rho \sigma'', \pi' \#_\rho \pi'') \in \mathscr{P}_2((\sigma_0, \pi_0), (\sigma_2, \pi_2))$ (resp. $\mathscr{P}'_2$) for large enough ρ.

LEMMA 4.1. *If $A \in \mathscr{M}_{\sigma\pi}(a, a')$ where $(\sigma, \pi) \in \mathscr{P}'_1$ ($\mathscr{P}'_2$) and $\hat{A} \in \tilde{\mathscr{B}}_\Sigma(\hat{a}, \hat{a}')$ is any lift of A, then $c_{\pi_1}(\hat{a}') < c_{\pi_0}(\hat{a}) + 3\varepsilon$ ($< c_{\pi_0}(\hat{a}) + 6\varepsilon$).*

Proof. Suppose $(\sigma, \pi) \in \mathscr{P}'_1$ and for definiteness say $\mathbb{R}_- = (-\infty, 0)$ and $\mathbb{R}_+ = (1, \infty)$. The path $\{A(\tau) \mid \tau \in \mathbb{R}_-\}$ is a gradient trajectory for c_{π_0}, thus $c_{\pi_0}(\hat{a}) \geq c_{\pi_0}(\hat{A}(0))$. Since $A \in \mathscr{M}_{\sigma\pi}(a, a')$ we have $F_A^- + \pi(A) = 0$. Thus $(\sigma, \pi) \in \mathscr{P}'_1$ implies that

$$\int_{\Sigma\times[0,1]} \|F_A^-\|^2_{L^2} < \varepsilon.$$

But

$$\frac{-1}{8\pi^2}\int_{\Sigma\times I} \mathrm{Tr}(F_A \wedge F_A) = \frac{1}{8\pi^2}\int_{\Sigma\times[0,1]} \|F_A^+\|^2 - \|F_A^-\|^2$$

so

$$\frac{-1}{8\pi^2}\int_{\Sigma\times I}\mathrm{Tr}(F_A\wedge F_A)>\frac{1}{8\pi^2}\int_{\Sigma\times[0,1]}\|F_A^+\|^2-\frac{\varepsilon}{8\pi^2}>-\varepsilon.$$

Thus

$$c(\hat{A}(0))=c(\hat{a}')+\frac{-1}{8\pi^2}\int_{\Sigma\times I}\mathrm{Tr}(F_A\wedge F_A)>c(\hat{a}')-\varepsilon.$$

So

$$c_{\pi_0}(\hat{a})\geq c_{\pi_0}(\hat{A}(0))=c(\hat{A}(0))+h_{\gamma_0}(\hat{A}(0))>c(\hat{A}(0))-\varepsilon>c(\hat{a}')-2\varepsilon,$$

and $c_{\pi_1}(\hat{a}')=c(\hat{a}')+h_{\gamma_1}(\hat{a}')<c(\hat{a}')+\varepsilon<c_{\pi_0}(\hat{a})+3\varepsilon$ as required. In case $(\sigma,\pi)\in\mathscr{P}'_2$, the proof follows similarly. ∎

Now let $(\sigma,\pi)\in\mathscr{P}'_1((\sigma_0,\pi_0),(\sigma_1,\pi_1))$ or $\mathscr{P}'_2((\sigma_0,\pi_0),(\sigma_1,\pi_1))$. Following Floer, we define for each n a homomorphism $\gamma_{\sigma\pi}\colon\mathscr{R}_n^{(\mu)}(\pi_0)\to\mathscr{R}_n^{(\mu)}(\pi_1)$ by

$$\gamma_{\sigma\pi}(a)=\sum_{\bar{a}\in\mathscr{R}_n^{(\mu)}(\pi_1)}\#\mathscr{M}^0_{\sigma\pi}(a,\bar{a}))\bar{a}.$$

We first check that $\gamma_{\sigma\pi}$ is a chain map. This is done by using Lemma 4.1 to modify Floer's proof. For $a\in\mathscr{R}_n^{(\mu)}(\pi_0)$ and $\bar{b}\in\mathscr{R}_n^{(\mu)}(\pi_1)$ the coefficient of $\bar{b}$ in $(\partial^{(\mu)}\gamma_{\sigma\pi}-\gamma_{\sigma\pi}\partial^{(\mu)})a$ is the cardinality of the set

$$\bigcup_{b\in\mathscr{R}_{n-1}^{(\mu)}(\pi_0)}-\hat{\mathscr{M}}^1_{\sigma_0\pi_0}(a,b)\times\mathscr{M}^0_{\sigma\pi}(b,\bar{b})\cup\bigcup_{\bar{a}\in\mathscr{R}_n^{(\mu)}(\pi_1)}\mathscr{M}^0_{\sigma\pi}(a,\bar{a})\times\hat{\mathscr{M}}^0_{\sigma_1\pi_1}(\bar{a},\bar{b}).\tag{4.2}$$

However, consider $\mathscr{M}^1_{\sigma\pi}(a,\bar{b})$. Floer shows that the ends of this 1-manifold are in oriented 1–1 correspondence with

$$\bigcup_{d\in\mathscr{R}^*(\pi_0)}-\hat{\mathscr{M}}^1_{\sigma_0\pi_0}(a,d)\times\mathscr{M}^0_{\sigma\pi}(d,\bar{b})\cup\bigcup_{\bar{c}\in\mathscr{R}^*(\pi_1)}\mathscr{M}^0_{\sigma\pi}(a,\bar{c})\times\hat{\mathscr{M}}^1_{\sigma_1\pi_1}(\bar{c},\bar{b}).\tag{4.3}$$

Given any end $A\#_\rho B$ of $\mathscr{M}^1_{\sigma\pi}(a,\bar{b})$ corresponding to a term in (4.2), the other end of the component, $A'\#_\rho B'$, corresponds to a term in (4.3). Say $A'\in\mathscr{M}^0_{\sigma\pi}(a,c)$ and $B\in\hat{\mathscr{M}}^1_{\sigma_1\pi_1}(c,\bar{b})$. We must show that $c\in\mathscr{R}_n^{(\mu)}(\pi_1)$; so that $A'\#_\rho B'$ actually corresponds to a term in (4.2). As in our proof that $\partial^{(\mu)}\partial^{(\mu)}=0$, $\mathscr{M}^1_{\sigma\pi}(a,\bar{b})$ gives a 1-parameter family of paths in $\mathscr{B}_\Sigma$ with fixed end points a and $\bar{b}$, providing a homotopy of paths from $A\#_\rho B$ to $A'\#_\rho B'$ rel endpoints. The lift of $A\#_\rho B$ starting at $a^{(\mu)}$ ends at $\bar{b}^{(\mu)}$; so the same is true of the lift of $A'\#_\rho B'$ starting at $a^{(\mu)}$. Suppose A' lifts to an element of $\mathscr{M}^0_{\sigma\pi}(a^{(\mu)},\hat{c})$. By Lemma 4.1, $c_{\pi_1}(\hat{c})<c_{\pi_0}(a^{(\mu)})+6\varepsilon<\mu+1-\frac{3M}{4}+6\varepsilon<\mu+1$. Also $c_{\pi_1}(\hat{c})>c_{\pi_1}(\bar{b}^{(\mu)})>\mu+\frac{3M}{4}$. So $c_{\pi_1}(\hat{c})\in(\mu,\mu+1)$ and thus $\hat{c}=c^{(\mu)}$, the preferred lift. By Lemma 3.4, $1=\mathrm{ind}D_B=SF(c^{(\mu)})-SF(\bar{b}^{(\mu)})=SF(c^{(\mu)})-(n-1)$. Thus $c\in\mathscr{R}_n^{(\mu)}(\pi_1)$ as required. This shows that $\gamma_{\sigma\pi}$ is a chain map.

Next consider two perturbations $(\sigma',\pi')\in\mathscr{P}'_1((\sigma_0,\pi_0),(\sigma_1,\pi_1))$ and $(\sigma'',\pi'')\in\mathscr{P}'_1((\sigma_1,\pi_1),(\sigma_2,\pi_2))$. Then for large ρ we have $(\sigma,\pi)=(\sigma'\#_\rho\sigma'',\pi'\#_\rho\pi'')\in\mathscr{P}'_2((\sigma_0,\pi_0),(\sigma_2,\pi_2))$. Floer shows that for each compact set K in $\mathscr{M}_{\sigma'\pi'}(a,b)\times\mathscr{M}_{\sigma''\pi''}(b,c)$ there is a $\rho_K>0$ and for all $\rho>\rho_K$ a local diffeomorphism

$$\mathscr{M}_{\sigma'\pi'}(a,b)\times\mathscr{M}_{\sigma''\pi''}(b,c)\supset K\to\mathscr{M}_{\sigma\pi}(a,c).$$

We claim that $\gamma_{\sigma\pi}=\gamma_{\sigma''\pi''}\circ\gamma_{\sigma'\pi'}$ for such ρ.

For each $a\in\mathscr{R}_n^{(\mu)}(\pi_0)$:

$$\gamma_{\sigma\pi}(a)=\sum_{c\in\mathscr{R}_n^{(\mu)}(\pi_2)}\#(\mathscr{M}^0_{\sigma\pi}(a,c))c$$

and

$$\gamma_{\sigma''\pi''} \circ \gamma_{\sigma'\pi'}(a) = \sum_{c \in \mathscr{R}_n^{(\mu)}(\pi_2)} \sum_{b \in \mathscr{R}_n^{(\mu)}(\pi_1)} \#(\mathscr{M}^0_{\sigma'\pi'}(a, b) \times \mathscr{M}^0_{\sigma''\pi''}(b, c))c.$$

The local diffeomorphism given above implies that

$$\# \mathscr{M}^0_{\sigma\pi}(a, c) = \sum_{b \in \mathscr{R}^*_{\pi_1}} \# \mathscr{M}^0_{\sigma'\pi'}(a, b) \times \mathscr{M}^0_{\sigma''\pi''}(b, c)).$$

We must show that when $a \in \mathscr{R}_n^{(\mu)}(\pi_0)$ and $c \in \mathscr{R}_n^{(\mu)}(\pi_2)$ we have $b \in \mathscr{R}_n^{(\mu)}(\pi_1)$ in the above sum. This is essentially the same argument as given above to show the $\gamma_{\sigma\pi}$ is a chain map; it will be left to the reader.

Still following Floer, we let $(\bar{\sigma}, \bar{\pi}) = \{(\sigma^{(\lambda)}, \pi^{(\lambda)}) | 0 \leq \lambda \leq 1\}$ be a smooth family of perturbations on $\Sigma \times \mathbb{R}$, each of which lies in $\mathscr{P}_2((\sigma_0, \pi_0), (\sigma_2, \pi_2))$ and such that $(\sigma^{(\lambda)}, \pi^{(\lambda)}) \in P'_2((\sigma_0, \pi_0), (\sigma_2, \pi_2))$ for $\lambda = 0, 1$. Applying an arbitrarily small perturbation if necessary (fixing $(\sigma^{(0)}, \pi^0)$ and $(\sigma^{(1)}, \pi^{(1)})$), the sets

$$\bar{\mathscr{M}}(a, b) = \{(u, \lambda) | u \in \mathscr{M}_{\sigma^{(\lambda)}\pi^{(\lambda)}}(a, b)\} \subset \mathscr{B}_{\Sigma \times \mathbb{R}}(a, b) \times [0, 1]$$

are regular zero sets of $F_{\bar{\sigma}\bar{\pi}}(A, \lambda) = F_{\sigma^{(\lambda)}\pi^{(\lambda)}}(A)$ and are smooth manifolds. The dimension of the component $\bar{\mathscr{M}}(a, a')$ containing a connection A is $SF(\hat{a}) - SF(\hat{b}) + 1$ where A lifts to $\hat{A} \in \tilde{\mathscr{B}}_{\Sigma \times \mathbb{R}}(\hat{a}, \hat{b})$. This defines a $\mathbb{Z}$-module homomorphism

$$\bar{\gamma} = \bar{\gamma}(\bar{\sigma}, \bar{\pi}) \colon \mathscr{R}_*^{(\mu)}(\pi_0) \to \mathscr{R}_*^{(\mu)}(\pi_2)$$

of degree $+1$ by the formula

$$\bar{\gamma}(a) = \sum_{b \in \mathscr{R}_{n+1}^{(\mu)}(\pi_2)} (\# \bar{\mathscr{M}}^0(a, b))b$$

for $a \in \mathscr{R}_n^{(\mu)}(\pi_0)$.

Consider $\bar{\mathscr{M}}^1(a, c)$ where $a \in \mathscr{R}_n^{(\mu)}(\pi_0)$. The boundary of this 1-dimensional submanifold of $\mathscr{B}_{\Sigma \times \mathbb{R}}(a, c) \times [0, 1]$ consists of $\mathscr{M}^0_{\sigma^{(0)}\pi^{(0)}}(a, c) \times \{0\} \cup \mathscr{M}^0_{\sigma^{(1)}\pi^{(1)}}(a, c) \times \{1\}$, together with terms of the form $\bar{\mathscr{M}}^0(a, d) \times \hat{\mathscr{M}}^1_{\sigma_2\pi_2}(d, c)$ and $\hat{\mathscr{M}}^1_{\sigma_0\pi_0}(a, b) \times \bar{\mathscr{M}}^0(b, c)$. Since $\bar{\mathscr{M}}^0(a, d)$ and $\bar{\mathscr{M}}^0(b, c)$ consist of solutions (u, t) of perturbed self-duality equations lying in formal dimension -1, they can only occur for $0 < t < 1$.

We then have:

$$\begin{aligned} \partial \bar{\mathscr{M}}^1(a, c) = {} & \mathscr{M}^0_{\sigma^{(1)}\pi^{(1)}}(a, c) \times \{1\} - \mathscr{M}^0_{\sigma^{(0)}\pi^{(0)}}(a, c) \times \{0\} \\ & + \bigcup_{d \in \mathscr{R}_{\pi_2}} \bar{\mathscr{M}}^0(a, d) \times \hat{\mathscr{M}}^1_{\sigma_2\pi_2}(d, c) + \bigcup_{b \in \mathscr{R}_{\pi_0}} \mathscr{M}^1_{\sigma_0\pi_0}(a, b) \times \bar{\mathscr{M}}^0(b, c). \end{aligned}$$

Using the 1-manifold $\bar{\mathscr{M}}^1(a, c)$ as a counting device we get

$$\begin{aligned} \# \mathscr{M}^0_{\sigma^{(0)}\pi^{(0)}}(a, c) \times \{0\} = {} & \# \mathscr{M}^0_{\sigma^{(1)}\pi^{(1)}}(a, c) \times \{1\} \\ & + \sum_{d \in \mathscr{R}_{\pi_2}} \bar{\mathscr{M}}^0(a, d) \times \hat{\mathscr{M}}^1_{\sigma_2\pi_2}(d, c) + \sum_{b \in \mathscr{R}_{\pi_0}} \mathscr{M}^1_{\sigma_0\pi_0}(a, b) \times \bar{\mathscr{M}}^0(b, c). \end{aligned}$$

Since each $(\sigma^{(t)}, \pi^{(t)}) \in \mathscr{P}_2$, if $\hat{A} \in \bar{\mathscr{M}}^0(a^{(\mu)}, \hat{d})$ and $\hat{B} \in \hat{\mathscr{M}}^1_{\sigma_2\pi_2}(\hat{d}, c^{(\mu)})$ we can show as usual that $\hat{d} = d^{(\mu)}$. Similarly if $\hat{C} \in \hat{\mathscr{M}}^1_{\sigma_0\pi_0}(a^{(\mu)}, \hat{b})$ and $\hat{D} \in \bar{\mathscr{M}}^0(\hat{b}, c^{(\mu)})$, we can show that $\hat{b} = b^{(\mu)}$. Thus

$$\begin{aligned} \# \mathscr{M}^0_{\sigma^{(0)}\pi^{(0)}}(a, c) \times \{0\} = {} & \# \mathscr{M}^0_{\sigma^{(1)}\pi^{(1)}}(a, c) \times \{1\} \\ & + \sum_{d \in \mathscr{R}_{n+1}^{(\mu)}(\pi_2)} \bar{\mathscr{M}}^0(a, d) \times \hat{\mathscr{M}}^1_{\sigma_2\pi_2}(d, c) + \sum_{b \in \mathscr{R}_{n-1}^{(\mu)}(\pi_0)} \mathscr{M}^1_{\sigma_0\pi_0}(a, b) \times \bar{\mathscr{M}}^0(b, c). \end{aligned}$$

Summing over all $c \in \mathscr{R}_n^{(\mu)}(\pi_2)$ we get

$$\gamma_{\sigma^{(0)}\pi^{(0)}}(a) = \gamma_{\sigma^{(1)}\pi^{(1)}}(a) + \bar{\gamma}\partial^{(\mu)}_{\sigma_0\pi_0}(a) + \partial^{(\mu)}_{\sigma_2\pi_2}\bar{\gamma}(a).$$

Hence $(\gamma_{\sigma^{(1)}\pi^{(1)}})_* = (\gamma_{\sigma^{(0)}\pi^{(0)}})_*$ as homomorphisms $\mathscr{I}_*^{(\mu)}(\Sigma, \sigma_0, \pi_0) \to \mathscr{I}_*^{(\mu)}(\Sigma, \sigma_2, \pi_2)$.

We are now ready to prove the main theorem of this section.

THEOREM 4.4. *For (σ_0, π_0) and $(\sigma_1, \pi_1) \in \mathscr{P}$, $\mathscr{I}_*^{(\mu)}(\Sigma, \sigma_0, \pi_0) \cong \mathscr{I}_*^{(\mu)}(\Sigma, \sigma_2, \pi_2)$.*

Proof. Let (σ, π) be the perturbation of the self-duality equations defined at the beginning of this section. It is obtained from a 1-parameter family (σ_t, π_t) interpolating from (σ_0, π_0) to (σ_1, π_1). Recall $(\sigma, \pi) \in \mathscr{P}'_1$. Reverse the τ-direction on $\Sigma \times \mathbb{R}$ to get the perturbation (σ', π') from (σ_1, π_1) to (σ_0, π_0). Then $(\sigma \#_\rho \sigma', \pi \#_\rho \pi') \in \mathscr{P}'_2$ and it induces $\gamma_{\sigma'\pi'} \circ \gamma_{\sigma\pi} = \gamma_{\sigma\#_\rho\sigma' \pi\#_\rho\pi'}: \mathscr{I}_*^{(\mu)}(\Sigma, \sigma_0, \pi_0) \to \mathscr{I}_*^{(\mu)}(\Sigma, \sigma_0, \pi_0)$. There is, as above, a smooth family $(\bar{\sigma}, \bar{\pi}) = (\sigma^{(\lambda)}, \pi^{(\lambda)})_{0 \le \lambda \le 1}$ (with each level lying in $\mathscr{P}_2$) such that $(\sigma^{(0)}, \pi^{(0)}) = (\sigma \#_\rho \sigma', \pi \#_\rho \pi')$ and $(\sigma^{(1)}, \pi^{(1)})$ is the perturbation on $\Sigma \times \mathbb{R}$ induced from (σ_0, π_0). From this last perturbation we see that $\gamma_{\sigma^{(1)}\pi^{(1)}} = \mathrm{id}$, since if $a, b \in \mathscr{R}_n(\pi_0)$, then $\mathscr{M}^0_{\sigma_0\pi_0} = 0$ unless $a = b$. (If $a \ne b$, then translational invariance, $\dim \mathscr{M}_{\sigma_0\pi_0}(a^{(\mu)}, b^{(\mu)}) \ge 1$.) Thus we have

$$(\gamma_{\sigma'\pi'})_* \circ (\gamma_{\sigma\pi})_* = \mathrm{id}: \mathscr{I}_*^{(\mu)}(\Sigma, \sigma_0, \pi_0) \to \mathscr{I}_*^{(\mu)}(\Sigma, \sigma_0, \pi_0)$$

Similarly $(\gamma_{\sigma\pi})_* \circ (\gamma_{\sigma'\pi'})_* = \mathrm{id}$.

Thus $(\gamma_{\sigma\pi})_*: \mathscr{I}_*^{(\mu)}(\Sigma, \sigma_0, \pi_0) \to \mathscr{I}_*^{(\mu)}(\Sigma, \sigma_1, \pi_1)$ is an isomorphism. ∎

§5. RELATIONSHIP WITH FLOER HOMOLOGY

Let Σ be a homology 3-sphere. In this section we will show how, for each $\mu \in \mathbb{R}_\Sigma$, the instanton homology groups $\mathscr{I}_*^{(\mu)}(\Sigma)$, $* \in \mathbb{Z}$, determine the Floer homology groups $\mathscr{I}_*(\Sigma)$, $* \in \mathbb{Z}_8$. This will be accomplished by filtering the Floer chain complex. Recall that $\mathscr{P}$ denotes the allowable perturbations of the flatness equations given in §3. For $(\sigma, \pi) \in \mathscr{P}$, let $(R_*(\Sigma, \sigma, \pi), \partial)$ denote Floer's chain complex. Then for $\mu \in \mathbb{R}_\Sigma$, $n \in \mathbb{Z}_8$ and $s \in \mathbb{Z}$ with $s \equiv n$ (mod 8), define the free abelian groups

$$F_s^{(\mu)} R_n(\Sigma, \sigma, \pi) = \sum_{j \ge 0} \mathscr{R}_{s+8j}^{(\mu)}(\Sigma, \sigma, \pi).$$

Then

$$\cdots \subset F_{s+8}^{(\mu)} R_n(\Sigma, \sigma, \pi) \subset F_s^{(\mu)} R_n(\Sigma, \sigma, \pi) \subset F_{s-8}^{(\mu)} R_n(\Sigma, \sigma, \pi) \subset \cdots \subset R_n(\Sigma, \sigma, \pi)$$

is a finite length decreasing filtration of $R_n(\Sigma, \sigma, \pi)$. Furthermore, since the (perturbed) Chern–Simons functional is non-increasing along gradient trajectories (see §2), it follows that Floer's boundary operator $\partial: F_s^{(\mu)} R_n(\Sigma, \sigma, \pi) \to F_{s-1}^{(\mu)} R_{n-1}(\Sigma, \sigma, \pi)$ preserves the filtration. Thus Floer's $\mathbb{Z}_8$-graded complex $(R_*(\Sigma, \sigma, \pi), \partial)$ has a decreasing bounded filtration $(F_s^{(\mu)} R_*(\Sigma, \sigma, \pi), \partial)$. For $n \in \mathbb{Z}_8$ and $s \equiv n$ (mod 8) let $\mathscr{I}_{s,n}^{(\mu)}(\Sigma, \sigma, \pi)$ denote the homology of the complex

$$\cdots \xrightarrow{\partial} F_{s+1}^{(\mu)} R_{n+1}(\Sigma, \sigma, \pi) \xrightarrow{\partial} F_s^{(\mu)} R_n(\Sigma, \sigma, \pi) \xrightarrow{\partial} F_{s-1}^{(\mu)} R_{n-1}(\Sigma, \sigma, \pi) \xrightarrow{\partial} \cdots.$$

We then have a bounded filtration on $I_n(\Sigma) = I_n(\Sigma, \sigma, \pi)$ defined by

$$F_s^{(\mu)} I_n(\Sigma, \sigma, \pi) = \mathrm{im}[\mathscr{I}_{s,n}^{(\mu)}(\Sigma, \sigma, \pi) \to I_n(\Sigma)]$$

with

$$\cdots \subset F_{s+8}^{(\mu)} I_n(\Sigma, \sigma, \pi) \subset F_s^{(\mu)} I_n(\Sigma, \sigma, \pi) \subset F_{s-8}^{(\mu)} I_n(\Sigma, \sigma, \pi) \subset \cdots \subset I_n(\Sigma, \sigma, \pi).$$

Observe that

$$F_s^{(\mu)} R_n(\Sigma, \sigma, \pi)/F_{s+8}^{(\mu)} R_n(\Sigma, \sigma, \pi) = \mathscr{R}_s^{(\mu)}(\Sigma, \sigma, \pi)$$

and that the induced chain map is precisely $\partial^{(\mu)}$.

THEOREM 5.1. *There is a spectral sequence* $(E^r_{s,n}(\Sigma), d^r)$ *with*

$$E^1_{s,n}(\Sigma) \cong \mathscr{I}_s^{(\mu)}(\Sigma)$$

($n \in \mathbb{Z}_8$ *and* $s \in \mathbb{Z}$ *with* $s \equiv n \pmod 8$)) *and*

$$E^\infty_{s,n}(\Sigma) \cong F_s^{(\mu)} I_n(\Sigma)/F_{s+8}^{(\mu)} I_n(\Sigma)$$

Furthermore, the groups $E^r_{s,n}(\Sigma)$ *are topological invariants.*

Proof. It is a standard consequence of the existence of the filtration $F^{(\mu)}$ given above that there is a spectral sequence $(E^r_{s,n}(\Sigma, \sigma, \pi), d^r)$ with $E^1_{s,n}(\Sigma, \sigma, \pi) \equiv \mathscr{I}_s^{(\mu)}(\Sigma, \sigma, \pi) = \mathscr{I}_s^{(\mu)}(\Sigma)$ and $E^\infty(\Sigma, \sigma, \pi)$ isomorphic to the bigraded module associated to the filtration $F^{(\mu)}$ of $I_n(\Sigma, \sigma, \pi) = I_n(\Sigma)$ $(n \in \mathbb{Z}_8)$ given above (see [5]). Since the grading is unusual we will list the explicit definitions of the desired groups and homomorphisms.

$$Z^r_{s,n}(\Sigma, \sigma, \pi) = \{c \in F_s^{(\mu)} R_n(\Sigma, \sigma, \pi) \mid \partial c \in F^{(\mu)}_{s-1+8r} R_{n-1}(\Sigma, \sigma, \pi)\}$$

$$Z^\infty_{s,n}(\Sigma, \sigma, \pi) = \{c \in F_s^{(\mu)} R_n(\Sigma, \sigma, \pi) \mid \partial c = 0\}$$

$$E^r_{s,n}(\Sigma, \sigma, \pi) = Z^r_{s,n}(\Sigma, \sigma, \pi)/(Z^{r-1}_{s+8,n}(\Sigma, \sigma, \pi) + \partial Z^{r-1}_{s+1-8(r-1),n+1}(\Sigma, \sigma, \pi))$$

$$E^\infty_{s,n}(\Sigma, \sigma, \pi) = Z^\infty_{s,n}(\Sigma, \sigma, \pi)/(Z^\infty_{s+8,n}(\Sigma, \sigma, \pi) + \partial R_{n+1}(\Sigma, \sigma, \pi) \cap F_s^{(\mu)} R_n(\Sigma, \sigma, \pi)).$$

Then ∂ induces

$$d^r\colon E^r_{s,n}(\Sigma, \sigma, \pi) \to E^r_{s+8r-1,n-1}(\Sigma, \sigma, \pi).$$

What remains to be shown is that $E^r_{s,n}(\Sigma, \sigma, \pi)$ is independent of the perturbation $(\sigma, \pi) \in \mathscr{P}$. But this follows from the proof of Theorem 4.4. For if (σ_0, π_0) and $(\sigma_1, \pi_1) \in \mathscr{P}$, there is the perturbation $(\sigma, \pi) \in \mathscr{P}'_1$ given in the proof of Theorem 4.4 which induces a chain isomorphism $\gamma_{\sigma\pi}\colon R_*(\Sigma, \sigma_0, \pi_0) \to R_*(\Sigma, \sigma_1, \pi_1)$ which respects the filtrations and induces an isomorphism on the E^1 terms. It is now a standard consequence that $\gamma_{\sigma\pi}$ induces an isomorphism on all the E^r. ∎

Remark. All the $E^r(\Sigma)$ provide potentially interesting invariants for the homology 3-sphere Σ. In particular, the minimal r for which $E^r(\Sigma) = E^\infty(\Sigma)$ should be meaningful.

Let X be an oriented smooth cobordism from Σ_1 to Σ_2, i.e. $\partial X = \Sigma_2 - \Sigma_1$. By counting instantons over X, Floer [4] shows that X induces homomorphisms $X_*\colon I_*(\Sigma_1) \to I_{*+3(b_1(X)-b_2^-(X))}(\Sigma_2)$ such that

(1) $(\Sigma \times \mathbb{R})_* = \text{identity}$
(2) $(XY)_* = X_{*+3(b_1(Y)-b_2^-(Y))} Y_*$.

In the remainder of this section we will show that X actually defines a homomorphism

$$X_s^{(\mu)}\colon E^r_{s,n}(\Sigma_1) \to E^r_{s+3(b_1-b_2^-(X)),n+3(b_1-b_2^-(X))}(\Sigma_2)$$

for $\mu \in \mathbb{R}_{\Sigma_1} \cap \mathbb{R}_{\Sigma_2}$ and we will investigate its properties. For simplicity suppose that $\mathscr{R}^*(\Sigma_1)$ and $\mathscr{R}^*(\Sigma_2)$ are regular. The general case is handled as in [4] using restricted perturbations analogous to those given in §3 and §4. Let $\mu \in \mathbb{R}_{\Sigma_1} \cap \mathbb{R}_{\Sigma_2}$ and fix a trivial connection $\Theta_{\Sigma_1} \in \tilde{\mathscr{B}}_{\Sigma_1}$. Homotopy classes of extensions of θ_{Σ_1} to a trivial connection Θ over all of X

are in 1-1 correspondence with elements of $H^3(X, \Sigma_1)$. The composition $H^3(X, \Sigma_1) \to H^3(X) \to H^3(\Sigma_2)$ is Poincaré dual to $H_1(X, \Sigma_2) \to H_1(X, \Sigma_1 \cup \Sigma_2) \to H_0(\Sigma_2)$. This last composition is the boundary of the long exact sequence of the pair (X, Σ_2); thus it is trivial. It follows that no matter which extension Θ of θ_{Σ_1} is chosen, the corresponding restrictions $\theta_{\Sigma_2} = \Theta|_{\Sigma_2}$ are all homotopic. Thus the choice of θ_{Σ_1} determines integer-valued indices on $\mathscr{R}^*(\Sigma_2)$ as well as on $\mathscr{R}^*(\Sigma_1)$. For a generic conformal structure on $(-\infty, 0] \times \Sigma_1 \cup X \cup \Sigma_2 \times [0, \infty)$ which is a product on the ends, moduli spaces of self-dual connections will be manifolds. Fix a metric in such a conformal structure, and for $a \in \mathscr{R}(\Sigma_1)$ and $b \in \mathscr{R}(\Sigma_2)$ let $\# \mathscr{M}^0_X(a, b)$ denote the oriented count of instantons on the zero-dimensional moduli space of self-dual connections over $(-\infty, 0] \times \Sigma_1 \cup X \cup \Sigma_2 \times [0, \infty)$ with asymptotic conditions a and b. For $a \in F^{(\mu)}_s R_n(\Sigma_1)$, the cobordism-induced map $X^{(\mu)}_s$ is defined by

$$X^{(\mu)}_s(a) = \sum_{b \in R^*(\Sigma_2)} \# \mathscr{M}^0_X(a, b) b.$$

This is simply the restriction to $F^{(\mu)}_s R_n(\Sigma_1)$ of Floer's map, and therefore it is a chain map.

THEOREM 5.2. *The map $X^{(\mu)}_s$ preserves the filtrations on $R_*(\Sigma_1)$ and $R_*(\Sigma_2)$; i.e.*

$$X^{(\mu)}_s \colon F^{(\mu)}_s R_n(\Sigma_1) \to F^{(\mu)}_{s+3(b_1(X)-b_2^-(X))} R_{n+3(b_1(X)-b_2^-(X))}(\Sigma_2).$$

Proof. Suppose that we have $a \in F^{(\mu)}_s R_n(\Sigma_1)$; so $a \in \mathscr{R}^{(\mu)}_{s+8j}(\Sigma_1)$ for some $j \geq 0$. Also suppose $\# \mathscr{M}^0_X(a, b) \neq 0$ with $A \in \mathscr{M}^0_X(a, b)$. Since any gauge transformation over Σ_1 extends over X we may suppose that A has asymptotic limits $a^{(\mu)}$ and $\hat{b}$. Choose any $\hat{A}_1 \in \tilde{\mathscr{B}}_{\Sigma_1}(\theta_{\Sigma_1}, a^{(\mu)})$ and $\hat{A}_2 \in \tilde{\mathscr{B}}_{\Sigma_2}(\hat{b}, \theta_{\Sigma_2})$. Then the Chern–Weil integrand associated to $A_1 \# A \# A_2$ computes the relative Pontrjagin number of the trivial SO(3)-bundle over X corresponding to the trivialization $\theta_{\Sigma_1} \amalg \theta_{\Sigma_2}$ of ∂X. Since this trivialization extends over X, the relative Pontrjagin number is 0. Thus

$$0 = -c(a^{(\mu)}) + \frac{-1}{8\pi^2} \int_X \operatorname{Tr}(F_A \wedge F_A) + c(\hat{b}),$$

and since A is self-dual, its Chern–Weil integral is nonnegative; so $c(\hat{b}) \leq c(a^{(\mu)}) < \mu + 1$. Thus $\mu - k < c(\hat{b}) < \mu + 1 - k$ for some nonnegative integer k. This means that $b^{(\mu)} = g(\hat{b})$ where g is a gauge transformation of degree k.

Since the Pontrjagin charge of the connection $A_1 \# A \# A_2$ is 0, the index of the self-duality operator $D_{A_1 \# A \# A_2}$ equals the index D_Θ. Hence

$$-3 - (s + 8j) + 0 + SF(\hat{b}) = -3(1 - b_1(X) + b_2^-(X)).$$

So

$$\iota^{(\mu)}(b) = SF(b^{(\mu)}) = SF(g(\hat{b})) = SF(\hat{b}) + 8k \geq s + 8j + 3(b_1(X) - b_2^-(X)).$$

Hence $b \in F^{(\mu)}_{s+3(b_1(X)-b_2^-(X))} R_{n+3(b_1(X)-b_2^-(X))}(\Sigma_2)$. ∎

Note that $X^{(\mu)}_s$ is just the restriction of Floer's chain map $X_n\colon R_n(\Sigma_1) \to R_{n-3b_2^-(X)}(\Sigma_2)$ to $F^{(\mu)}_s R_n(\Sigma_1)$, and Theorem 5.2 says that this chain map preserves the filtration. We now have

THEOREM 5.3. *Let X be an oriented smooth cobordism from Σ_1 to Σ_2. For $\mu \in \mathbb{R}_{\Sigma_1} \cap \mathbb{R}_{\Sigma_2}$ there are homomorphisms*

$$X^{(\mu)}_s \colon E^r_{s,n}(\Sigma_1) \to E^r_{s+3(b_1(X)-b_2^-(X)),\, n+3(b_1(X)-b_2^-(X))}(\Sigma_2)$$

such that

(1) $(\Sigma \times \mathbb{R})_s =$ *identity*
(2) $(XY)_s = X_{s+3(b_1(Y)-b_2^-(Y))} Y_s$.

Proof. We need to prove that the induced homomorphisms

$$X_s^{(\mu)}\colon E^r_{s,n}(\Sigma_1) \to E^r_{s+3(b_1(X)-b_2^-(X)),n+3(b_1(X)-b_2^-(X))}(\Sigma_2)$$

are independent of the metric chosen on X. But this follows from the arguments preceding the proof of Theorem 4.4 which show that if X is the product cobordism, then the induced homomorphism is independent of the metric on X. Again the key observation is that whenever a moduli space is nonempty, then the corresponding Chern–Weil integrand is non-negative.

To get an idea of why this mechanism works, let us suppose that we have a very simple situation where X is simply connected and a class in $E^1_{s,n}(\Sigma_1) = \mathcal{I}_s^{(\mu)}(\Sigma_1)$ is represented by a single flat connection $a \in \mathcal{R}_s^{(\mu)}(\Sigma_1)$. Then $X_s^{(\mu)}(a) \in E^1_{s-3b_2^-(X),n-3b_2^-(X)}(\Sigma_2) = \mathcal{I}^{(\mu)}_{s-3b_2^-(X)}(\Sigma_2)$. So

$$X_s^{(\mu)}(a) = \sum_{b \in \mathcal{R}^{(\mu)}_{s-3b_2^-(X)}(\Sigma_2)} \# \mathcal{M}^0_X(a, b) b.$$

Let g_0 be the given metric on X, and let $\{g_t\}$ be a one-parameter family of metrics beginning at g_0. The value of the cobordism inducéd homomorphism $X^{(\mu)}_{s,g_t}(a)$ will remain unchanged at the chain level as long as there are no moduli spaces $\mathcal{M}^{-1}_{X,g_t}(a, d)$ or $\mathcal{M}^{-1}_{X,g_t}(d, b)$ of formal dimension -1 for $0 < t < 1$. For simplicity assume that only one such moduli space occurs $-\mathcal{M}^{-1}_{X,g_u}(a, d)$, where $0 < u < 1$. Then it is easy to see that $X^{(\mu)}_{s,g_1}(a) = X^{(\mu)}_{s,g_0}(a) \pm \sum_{b \in \mathcal{R}^{(\mu)}_{s-3b_2^-(X)}(\Sigma_2)} \# \mathcal{M}^{-1}_{X,g_u}(a, d) \cdot \# \hat{\mathcal{M}}^1_{\Sigma_2}(d, b) b$. If we knew that $d \in R^{(\mu)}_{s-1-3b_2^-(X)}(\Sigma_2)$ then we would have $X^{(\mu)}_{s,g_1}(a) = X^{(\mu)}_{s,g_0}(a) \pm \partial d$ and see that the value remains unchanged in E^1.

If $B \in \mathcal{M}^{-1}_{X,g_u}(a, d)$ and $C \in \mathcal{M}^1_{\Sigma_2}(d, b)$, since any gauge transformation of Σ_1 extends over X, we may assume that B has asymptotic limits $a^{(\mu)}$ and $\hat{d}$ and that C has limits $\hat{d}$ and $\hat{b}$. We also have a grafted self-dual connection $B \# C$ on X with index 0. As in the proof of Theorem 5.2 we see that the index of the self-duality operator $D_{A_1 \# (B \# C) \# A_2}$ equals the index of D_Θ. If g is a gauge transformation such that $\hat{b} = g(b^{(\mu)})$ we have:

$$-3 - s + 0 + s - 3b_2^-(X) - 8\deg(g) = -3(1 + b_2^-(X)).$$

Hence $\deg(g) = 0$ and $\hat{b} = b^{(\mu)}$. Also, since the Chern–Weil integrand of $A_1 \# (B \# C) \# A_2$ vanishes we get

$$0 = -c(a^{(\mu)}) + \frac{-1}{8\pi^2} \int_X \operatorname{Tr}(F_B \wedge F_B) + c(\hat{d}).$$

Since B is self-dual this implies that $\mu + 1 > c(a^{(\mu)}) > c(\hat{d})$, and in turn, $c(\hat{d}) > c(b^{(\mu)}) > \mu$ since the Chern–Simons function decreases along the gradient trajectory C. Hence $\hat{d} = d^{(\mu)}$. Now since a 1-dimensional moduli space connects $d^{(\mu)}$ and $b^{(\mu)}$ we see that $d \in R^{(\mu)}_{s-1-3b_2^-(X)}(\Sigma_2)$ as required.

It is quite simple to repeat this argument when we allow perturbations of the Chern–Simons function and the self-duality equation. Also, a similar argument works to show that the homomorphism induced on E^r is independent of perturbations and metrics. ∎

§6. RELATIVE DONALDSON POLYNOMIALS

Let X be an oriented simply connected 4-manifold whose boundary Σ is a homology sphere. An important feature of Floer's homology groups $I_n(\Sigma)$ is presented in forthcoming work of Donaldson which gives relative versions of his polynomial invariants with values in $I_n(\Sigma)$. Let us begin by quickly previewing the construction of these relative polynomial invariants. We will assume that $\mathscr{R}^*(\Sigma)$ is regular. (One of the main points of the forthcoming work of Donaldson is to treat the situation when $\mathscr{R}^*(\Sigma)$ is not regular. Presumably the arguments presented in this work can be modified using the perturbations presented in our §3 and §4.) The moduli space $\mathscr{M}_X(a)$ of self-dual connections on the trivial $SU(2)$ bundle over X with asymptotic condition $a \in \mathscr{R}^*(\Sigma)$ has connected components with formal dimension $D(a) \equiv -3(1 + b_2^-(X)) - SF(a) \pmod 8$. For a generic choice of Riemannian metric on X the components of the moduli space $\mathscr{M}_X(a)$, if nonempty, will be manifolds of these dimensions. Given homology classes $z_1, \ldots, z_d \in H_2(X;\mathbb{Z})$, represent them by surfaces $S_1, \ldots, S_d$ in X. Donaldson [2] associates to each S_i a codimension 2 submanifold V_{S_i} of $\mathscr{B}_X$ such that the intersection $V_{S_1} \cap \cdots \cap V_{S_d} \cap \mathscr{M}_X(a)$ is transverse and in the case that $D(a) \equiv 2d \pmod 8$ the intersection with the components of $\mathscr{M}_X(a)$ of dimension $2d$ (for sufficiently large d) will be a finite collection of points. The chain

$$q_X(z_1, \ldots, z_d) = \sum \#(V_{S_1} \cap \cdots \cap V_{S_d} \cap \mathscr{M}_X^{2d}(a))a$$

can be shown to be a Floer cycle, and it induces the Donaldson invariant $q_X(z_1, \ldots, z_d) \in I_{-3(1+b_2^-(X))-2d}(\Sigma)$.

Since X is simply connected there is (up to homotopy) a unique trivial connection Θ on the $SU(2)$-bundle over X, and it restricts to a trivial connection θ over Σ. Fix $\mu \in \mathbb{R}_\Sigma$. Define $\mathscr{N}_X^{2d,(\mu),r}(a)$ to be the subset of equivalence classes of connections $A \in \mathscr{M}_X^{2d}(a)$ which satisfy the inequality

$$\frac{-1}{8\pi^2}\int_{X \cup \Sigma \times [0,\infty)} \mathrm{Tr}(F_A \wedge F_A) < r - (\mu + 1 - [\mu + 1])$$

where $[\mu + 1]$ denotes the integer part of $\mu + 1$.

THEOREM 6.1. *For classes $z_1, \ldots, z_d \in H_2(X;\mathbb{Z})$, the formula*

$$q_X^{(\mu),r}(z_1, \ldots, z_d) = \sum \#(V_{S_1} \cap \cdots \cap V_{S_d} \cap \mathscr{N}_X^{2d,(\mu),r}(a))a$$

define a class in $E^r_{s,n}$ where $s = -3(1 + b_2^-(X)) - 2d + 8[\mu + 1]$ and $n = -3(1 + b_2^-(X)) - 2d$. This class is invariant under change of generic metric on X.

Proof. Suppose that the gauge equivalence class of A lies in $V_{S_1} \cap \cdots \cap V_{S_d} \cap \mathscr{N}_X^{2d,(\mu),r}(a)$ and that A has limiting value $\hat{a} \in \tilde{\mathscr{R}}^*(\Sigma)$. Then $2d + SF(\hat{a}) = -3(1 + b_2^-(X))$, and $c(\hat{a}) + \dfrac{-1}{8\pi^2}\displaystyle\int \mathrm{Tr}(F_A \wedge F_A) = 0$. Thus $c(\hat{a}) < 0$ since A is self-dual. Let $\bar{a} \in \tilde{\mathscr{R}}(\Sigma)$ be the unique flat connection gauge equivalent to $\hat{a}$ and such that $\mu - [\mu + 1] < c(\bar{a}) < \mu + 1 - [\mu + 1]$. If g is the gauge transformation such that $\bar{a} = g(\hat{a})$ then $c(\hat{a}) < 0$ implies that $\deg(g) \geq 0$. Hence $SF(\bar{a}) = SF(\hat{a}) + 8\deg(g) \geq SF(\hat{a}) = n$. Since $SF(a^{(\mu)}) = SF(\bar{a}) + 8[\mu + 1]$, this shows that $q_X^{(\mu),r}(z_1, \ldots, z_d)$ lies in $F_s R_n(\Sigma)$.

Next we show that $q_X^{(\mu),r}(z_1, \ldots, z_d) \in Z^r_{s,n}$. For if a occurs in the sum as above, then if $B \in \mathscr{M}_{\frac{1}{2}}(a, b)$ and $\hat{B}$ is a lift with limiting values $\hat{a}$ and $\hat{b}$, then $A \# \hat{B}$ represents an end of the 1-dimensional cut-down moduli space $V_{S_1} \cap \cdots \cap V_{S_d} \cap \mathscr{M}_X^{2d+1}(b)$. There must be another end represented, say, by $A' \# \hat{B}'$, where A' has limiting value $\hat{a}' \in \tilde{\mathscr{R}}^*(\Sigma)$ and $\hat{B}'$ has limiting values $\hat{a}'$ and $\hat{b}$ and is a lift of $B' \in \mathcal{M}^1_{\Sigma}(a', b)$. Since $a \in F_s R_n(\Sigma)$ we have $b \in F_{s-1} R_{n-1}(\Sigma)$. Suppose that $b \notin F_{s-1+8r} R_{n-1}(\Sigma)$; hence $b \in \mathcal{R}_{s-1+8j}(\Sigma)$ where $0 \le j \le r-1$. We need to show that $A' \in \mathcal{N}^{2d,(\mu),r}_X(a')$. If h is the gauge transformation such that $h(\hat{b}) = b^{(\mu)}$ then $SF(\hat{b}) = n-1$ implies that

$$\deg(h) = \tfrac{1}{8}((s-1+8j)-(n-1)) = j + [\mu+1].$$

Since B' is self-dual, $c(\hat{a}) \ge c(\hat{b}) = c(\hat{b}) = c(b^{(\mu)}) - \deg(h) > \mu - (r-1) - [\mu+1]$. Then

$$\frac{-1}{8\pi^2}\int \mathrm{Tr}(F'_A \wedge F'_A) = -c(\hat{a}')$$

implies that $A' \in \mathcal{N}^{2d,(\mu),r}_X(a')$ as desired.

Similar arguments show that the image $q^{(\mu),r}_X(z_1, \ldots, z_d) \in E^r_{s,n}$ is actually independent of the choice of generic metric on X. (C.f. (5.3)). ∎

These invariants are in theory easier to compute than the Donaldson invariants since fewer boundary maps will occur. At any rate they could be useful in distinguishing manifolds with the same relative Donaldson invariant.

§7. BRIESKORN HOMOLOGY SPHERES

In [3] we presented an algorithm for computing the Floer homology groups $I_n(\Sigma(a_1,a_2,a_3))$, $n \in \mathbb{Z}_8$, for the Brieskorn homology 3-spheres $\Sigma(a_1,a_2,a_3) = \{z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0 \mid (z_1,z_2,z_3) \in \mathbb{C}^3\} \cap S^5$, or, equivalently, the Seifert-fibered homology 3-sphere with exceptional orbits of order a_1, a_2 and a_3. In this section we will show how to enhance this algorithm to compute Λ_Σ and for any $\mu \in \mathbb{R}_\Sigma$ the groups $\mathcal{I}^{(\mu)}_n(\Sigma(a_1,a_2,a_3))$, $n \in \mathbb{Z}$. We will end this section with some explicit computations.

We assume familiarity with [3]. In that paper we worked with $SO(3)$ rather than $SU(2)$ connections, and for that reason it will be best for us to use $SO(3)$ connections in this section as well. Thus we now have a Chern–Simons function taking values in $\mathbb{R}/4\mathbb{Z}$ rather than in $\mathbb{R}/\mathbb{Z}$. In the end there is no change to the instanton homology.

For the Seifert fibrations $\Sigma(a_1,a_2,a_3)$ we showed in [3] that $\mathcal{R}(\Sigma(a_1,a_2,a_3))$ contained only regular representations, so was a finite set. Furthermore the Chern–Simons function $c\colon \mathcal{R}(\Sigma(a_1,a_2,a_3)) \to \mathbb{R}/4\mathbb{Z}$ is given by $c(a) \equiv 2e_a^2/a_1a_2a_3 \pmod 4$ where $e_a \in \mathbb{Z}$ is the Euler class of the representation a. The algorithm to determine e_a is one of the main points of [3]. It is shown in [3] that $a \in R_n$ where

$$n \equiv \frac{2e_a^2}{a_1a_2a_3} + \sum_{l=1}^{3} \frac{2}{a_l} \sum_{k=1}^{a_l-1} \cot\left(\frac{\pi a k}{a_l^2}\right)\cot\left(\frac{\pi k}{a_l}\right)\sin^2\left(\frac{\pi e_a k}{a_l}\right) \qquad \pmod 8.$$

Let $e_a^{(\mu)} \in (\mu, 4+\mu)$ be defined by the congruence $e_a^{(\mu)} \equiv 2e_a^2/a_1a_2a_3 \pmod 4$. Then

$$\iota^{(\mu)}(a) = \frac{2(e_a^{(\mu)})^2}{a_1a_2a_3} + \sum_{l=1}^{3} \frac{2}{a_l} \sum_{k=1}^{a_l-1} \cot\left(\frac{\pi a k}{a_l^2}\right)\cot\left(\frac{\pi k}{a_l}\right)\sin^2\left(\frac{\pi e_a^{(\mu)} k}{a_l}\right)$$

Since $\iota^{(\mu)}(a)$ is always odd (see the proof of Proposition 3.10 of [3]), we have

$$\mathcal{I}^{(\mu)}_n(\Sigma(a_1,a_2,a_3)) \cong \mathcal{R}^{(\mu)}_n(\Sigma(a_1,a_2,a_3)).$$

We conclude by listing some explicit computations of the Poincaré–Laurent polynomials $p(a_1,a_2,a_3)(t)$ of the homology groups $\mathcal{I}^{(0)}_n(\Sigma(a_1,a_2,a_3))$.

$$p(2,3,5) = t + t^5$$
$$p(2,3,11) = t + t^3 + t^5 + t^7$$
$$p(2,3,17) = t + t^3 + 2t^5 + t^7 + t^9$$
$$p(2,3,23) = t + 2t^3 + 2t^5 + 2t^7 + t^9$$
$$p(2,3,29) = t + t^3 + 3t^5 + 2t^7 + 2t^9 + t^{11}$$
$$p(2,3,35) = t + 2t^3 + 3t^5 + 3t^7 + 2t^9 + t^{11}$$
$$p(2,3,41) = t + t^3 + 4t^5 + 3t^7 + 3t^9 + 2t^{11}$$
$$p(2,3,47) = t + 2t^3 + 3t^5 + 4t^7 + 3t^9 + 2t^{11} + t^{13}$$
$$p(2,3,53) = t + t^3 + 4t^5 + 4t^7 + 4t^9 + 3t^{11} + t^{13}$$
$$p(2,3,59) = t + 2t^3 + 3t^5 + 5t^7 + 4t^9 + 3t^{11} + 2t^{13}$$
$$p(2,3,65) = t + t^3 + 4t^5 + 5t^7 + 5t^9 + 4t^{11} + 2t^{13}$$
$$p(2,3,71) = t + 2t^3 + 3t^5 + 5t^7 + 5t^9 + 4t^{11} + 3t^{13} + t^{15}$$
$$p(2,3,7) = t^{-1} + t^3$$
$$p(2,3,13) = t^{-1} + t + t^3 + t^5$$
$$p(2,3,19) = 2t^{-1} + t + 2t^3 + t^5$$
$$p(2,3,25) = 2t^{-1} + 2t + 2t^3 + 2t^5$$
$$p(2,3,31) = 2t^{-1} + 2t + 3t^3 + 2t^5 + t^7$$
$$p(2,3,37) = t^{-1} + 3t + 3t^3 + 2t^5 + 2t^7$$
$$p(2,3,37) = 2t^{-1} + 3t + 4t^3 + 3t^5 + 2t^7$$
$$p(2,3,43) = t^{-1} + 4t + 4t^3 + 4t^5 + 3t^7$$
$$p(2,3,49) = 2t^{-1} + 3t + 5t^3 + 4t^5 + 3t^7 + t^9$$
$$p(3,4,13) = t^{-5} + 2t^{-3} + 3t^{-1} + 2t + 2t^3$$
$$p(4,5,21) = 4t^{-9} + 5t^{-7} + 8t^{-5} + 6t^{-3} + 5t^{-1} + t + t^3$$
$$p(5,6,31) = 2t^{-15} + 10t^{-13} + 11t^{-11} + 15t^{-9} + 3t^{-7} + 8t^{-5} + 5t^{-3} + 4t^{-1} + t + t^3$$

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