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2-TORSION INSTANTON INVARIANTS

RONALD FINTUSHEL AND RONALD J. STERN

Dedicated to the memory of Andreas Floer

1. INTRODUCTION

In [D2] S. Donaldson introduced new and useful invariants for smooth, closed, simply connected 4-manifolds by evaluating certain elements of the rational cohomology of the space of connections in an $SU(2)$-bundle over a 4-manifold M on a homology class represented by the moduli space of self-dual connections. In this paper we utilize torsion in the cohomology of the space of connections to define a collection of mod 2 polynomial invariants for smooth, simply connected, closed spin 4-manifolds and show that this collection of polynomials is stable under connect sums with $S^2 \times S^2$ (Theorems 1.1 and 1.3). This contrasts with the vanishing theorems for Donaldson's integral polynomials for connected sums [D2]. It will then follow that if one could find two homotopy equivalent simply connected, smooth, closed, and spin 4-manifolds with Donaldson polynomials having different parity, then these manifolds are not diffeomorphic and remain nondiffeomorphic after connect summing with one or two copies of $S^2 \times S^2$. At present, no such example is known. However, we will show that the Donaldson polynomial invariants have limited utility in this vein. In fact, using the relation between the usual Donaldson invariants and the mod 2 polynomial invariants (Theorem 1.1) and through a detailed understanding of how moduli spaces decompose for manifolds which are connected sums, we are able obtain severe restrictions on when the Donaldson polynomials reduced mod 2 can be nonzero. (See Theorem 1.6.)

In order to describe these mod 2 polynomial invariants, recall that Donaldson's (integral) polynomial invariant $q_{\ell,M}$ is defined for a closed oriented simply connected 4-manifold M with b_M^+ odd > 1 (and for ℓ a large enough positive integer) and has degree $d = 4\ell - \frac{3}{2}(1+b_M^+)$. Consider the Banach manifold $\mathscr{B}_{M,\ell}^*$ of equivalence classes of irreducible connections of charge ℓ. Donaldson's invariant is defined on homology classes $z_1, \dots, z_d \in H_2(M;\mathbf{Z})$ by evaluating the cup product of the d cohomology classes $\mu(z_i) \in H^2(\mathscr{B}_{M,\ell}^*;\mathbf{Z})$ (μ is defined in [D1]) against the fundamental class of the $2d$-dimensional

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moduli space $\mathscr{M}_{M,\ell}$ of anti-self-dual $SU(2)$ connections of charge ℓ. See §2 for a more complete description. In case $b_M^+ > 1$ is even, we have $\dim \mathscr{M}_{M,k} = 8k - 3(1 + b_M^+) = 2d + 1$; so a similar polynomial invariant can be defined if there is a nontrivial 1-dimensional cohomology class in $\mathscr{B}_{M,k}^*$. If k is even and M is spin there is a nontrivial class $u_1 \in H^1(\mathscr{B}_{M,k}^*; \mathbf{Z}_2)$ (cf. [D1] and §2). Thus in this case there is (for large enough even k) a polynomial invariant $q_{k,u_1,M}$ of degree d in $H_2(M; \mathbf{Z}_2)$ and defined with values in $\mathbf{Z}_2$. (See [D3] for a general discussion of such invariants.) As in the case of $q_{\ell,M}$, the mod 2 invariant $q_{k,u_1,M}$ is an invariant of the smooth structure of M.

Now suppose that M is a closed oriented simply connected spin 4-manifold with b_M^+ odd > 1 and has a Donaldson polynomial invariant $q_{\ell,M}$ of degree d and with ℓ odd. Then $M\#S^2 \times S^2$ has b^+ even and the moduli space $\mathscr{M}_{M\#S^2\times S^2,\ell+1}$ has dimension $2d+5$. Since $\ell + 1$ is even, we have the mod 2 invariant $q_{\ell+1,u_1,M\#S^2\times S^2}$ of degree $d+2$ in $H_2(M\#S^2 \times S^2; \mathbf{Z}_2)$. Our main theorem is

Theorem 1.1. *Suppose that M is a closed simply connected spin 4-manifold with a Donaldson polynomial $q_{\ell,M}$ of degree d, where ℓ is odd. Then $q_{\ell+1,u_1,M\#S^2\times S^2}$ is defined and for any classes $z_1, \dots, z_d \in H_2(M; \mathbf{Z})$ and for $x = [S^2 \times 0]$ and $y = [0 \times S^2]$ in $H_2(S^2 \times S^2; \mathbf{Z})$ we have*

$$q_{\ell,M}(z_1, \dots, z_d) \equiv q_{\ell+1,u_1,M\#S^2\times S^2}(z_1, \dots, z_d, x, y) \mod 2.$$

In order to explain the relevance of this theorem to the problem at hand, let us make the following definition. Suppose that M_1 and M_2 are homotopy equivalent simply connected 4-manifolds. We shall say that their degree d Donaldson invariants q_{ℓ,M_1} and q_{ℓ,M_2} *have the same parity* if for each isomorphism of intersection forms $f : H_2(M_1; \mathbf{Z}) \to H_2(M_2; \mathbf{Z})$ and for all $z_1, \dots, z_d \in H_2(M_1; \mathbf{Z})$ we have

$$q_{\ell,M_1}(z_1, \dots, z_d) \equiv q_{\ell,M_2}(f(z_1), \dots f(z_d)) \mod 2.$$

Theorem 1.2. *Let M_1 and M_2 be homotopy equivalent closed simply connected spin 4-manifolds. If $M_1\#S^2 \times S^2$ is diffeomorphic to $M_2\#S^2 \times S^2$ then q_{ℓ,M_1} and q_{ℓ,M_2} have the same parity for all odd ℓ.*

This theorem is actually a corollary of Theorem 1.1 and Wall's work on the diffeomorphisms of 4-manifolds [W1]. For if there is an odd ℓ and an isomorphism f of intersection forms as above, then consider the isomorphism $f \oplus 1 : H_2(M_1\#S^2 \times S^2; \mathbf{Z}) \to H_2(M_2\#S^2 \times S^2; \mathbf{Z})$. Let $\Phi : M_1\#S^2 \times S^2 \to M_2\#S^2 \times S^2$ be a diffeomorphism; so $\Phi_*^{-1} \circ (f \oplus 1) = g$ is an automorphism of $H_2(M_1\#S^2 \times S^2; \mathbf{Z})$. Then by [W1, Theorem 2], $g = G_*$ where G is a self-diffeomorphism of $M_1\#S^2 \times S^2$. Then $f \oplus 1$ is induced from the diffeomorphism $\Phi \circ G$. Theorem 1.2 now follows from Theorem 1.1 and the naturality of Donaldson's invariant.

If b_N^+ is odd > 1, so $\dim \mathscr{M}_{N,k}$ is even, we may write $\dim \mathscr{M}_{N,k} = 2d+2$. For k odd and N spin, $\pi_1(\mathscr{B}_{N,k}^*) = 0$ and there is a nontrivial class $u_2 \in H^2(\mathscr{B}_{N,k}; \mathbf{Z}_2)$ arising from the 2-torsion in $H_2(\mathscr{B}_{N,k}; \mathbf{Z})$. (Again, see [D1].) Thus we may form the degree d polynomial invariant $q_{k,u_2,N}$ with values in $\mathbf{Z}_2$. We get

Theorem 1.3. *Let N be a closed simply connected spin 4-manifold with the degree d Donaldson invariant $q_{k,u_1,N}$ with values in $\mathbf{Z}_2$. (Thus b_N^+ is even and k is even.) Then the degree $d+2$ invariant $q_{k+1,u_2,N\#S^2\times S^2}$ is defined and for any $z_1, \ldots, z_d \in H_2(N; \mathbf{Z}_2)$*

$$q_{k,u_1,N}(z_1, \ldots, z_d) \equiv q_{k+1,u_2,N\#S^2\times S^2}(z_1, \ldots, z_d, x, y) \mod 2,$$

where $x = [S^2 \times 0]$ and $y = [0 \times S^2]$.

A basic theorem in the theory of smooth 4-manifolds, due to C. T. C. Wall [W1,2], states that given two homotopy equivalent simply connected smooth closed 4-manifolds M_1 and M_2, there is an integer k such that $M_1\#k(S^2\times S^2)$ is diffeomorphic to $M_2\#k(S^2 \times S^2)$, i.e. M_1 and M_2 are stably diffeomorphic. A natural problem is to determine the minimal such integer k, denoted $sd(M_1, M_2)$. It now follows from Theorems 1.1, 1.2 and 1.3 that if M_1 and M_2 are homotopy equivalent closed simply connected spin 4-manifolds with Donaldson invariants q_{ℓ,M_1} and q_{ℓ,M_2} (ℓ odd) which do not have the same parity then actually $sd(M_1, M_2) \geq 3$. At present no such examples are known. A possible reason is given by Theorem 1.6 where it is shown that many Donaldson invariants are even.

For many closed simply connected 4-manifolds it is known that "homotopy equivalent implies diffeomorphic after a single connected sum with $S^2 \times S^2$." For example, work of Mandelbaum [M] and Gompf [G] shows that this is true for simply connected elliptic surfaces. Hence

Theorem 1.4. *Let M_1 and M_2 be homotopy equivalent closed simply connected spin elliptic surfaces. Then any Donaldson polynomial invariants q_{ℓ,M_1} and q_{ℓ,M_2}, for ℓ odd, have the same parity.*

Theorem 1.4 also follows from the explicit computations for spin elliptic surfaces given by Friedman and Morgan [FM].

Let $\mathrm{Sym}_R^d(H_2(M; \mathbf{Z}))$ be the set of d-linear symmetric functions on $H_2(M; \mathbf{Z})$ with values in a ring R. The symmetric product $q_1q_2 \in \mathrm{Sym}_R^{d_1+d_2}(H_2(M; \mathbf{Z}))$ of the symmetric functions $q_1 \in \mathrm{Sym}_R^{d_1}(H_2(M; \mathbf{Z}))$ and $q_2 \in \mathrm{Sym}_R^{d_2}(H_2(M; \mathbf{Z}))$ is defined by the rule

$$q_1q_2(x_1, \ldots, x_{p+q}) = \frac{1}{d_1!d_2!} \sum_{\sigma\in S_{d_1+d_2}} q_1(x_{\sigma(1)}, \ldots, x_{\sigma(d_1)})q_2(x_{\sigma(d_1+1)}, \ldots, x_{\sigma(d_1+d_2)}).$$

The degree d Donaldson invariant $q_{\ell,M}$ is an element of $\mathrm{Sym}^d_{\mathbf{Z}}(H_2(M;\mathbf{Z}))$. Similarly the intersection form Q_M of M is an element of $\mathrm{Sym}^2_{\mathbf{Z}}(H_2(M;\mathbf{Z}))$. Define

$$Q_M^{(p)} = \frac{1}{p!}Q_M^p.$$

Reducing mod 2 we consider the algebra

$$\mathrm{Sym}^*_{\mathbf{Z}_2}(H_2(M;\mathbf{Z})) = \bigoplus_d \mathrm{Sym}^d_{\mathbf{Z}_2}(H_2(M;\mathbf{Z})).$$

In §8 we prove a vanishing theorem for $q_{\ell+1,u_1,M\#S^2\times S^2}$ (Theorem 8.1) reminiscent of Donaldson's connected sum theorem [D2] which together with our calculations of $q_{\ell+1,u_1,X} \in \mathrm{Sym}^{d+2}_{\mathbf{Z}_2}(H_2(X;\mathbf{Z}))$ and its invariance under the orthogonal transformations of $H_2(X;\mathbf{Z})$ induced from diffeomorphisms of $X = M\#S^2 \times S^2$ will prove

Theorem 1.5. *Let M be a closed simply connected spin 4-manifold with a Donaldson invariant $q_{\ell,M}$ with ℓ odd. Then $q_{\ell,M} \equiv \epsilon_{\ell,M}Q_M^{(p)} \bmod 2$ for some integer p and $\epsilon_{\ell,M} \in \mathbf{Z}_2$.*

Combining this result with the thesis of Y. Ruan [R], we obtain strong restrictions on the possibility of Donaldson's invariants $q_{\ell,M}$ taking odd values.

Theorem 1.6. *Let M be a closed simply connected spin 4-manifold with a Donaldson invariant $q_{\ell,M}$ of degree d with ℓ odd. If $b^+_M \not\equiv 3 \bmod 8$, or if $d > \mathrm{rank}(H_2(M;\mathbf{Z}))$, then $q_{\ell,M} \equiv 0 \bmod 2$.*

The proof of the main Theorem 1.1 is accomplished via a degeneration of metrics argument of the sort utilized by Donaldson in [D2]. Two routes are available for carrying out this argument. The first is to split $M\#S^2 \times S^2$ along an obvious S^3. Then the argument is a modification of the arguments of [D2], and we shall outline such an approach in a moment. We have chosen instead to view $M\#S^2 \times S^2$ as the result of surgery along a circle in M. Then the interface between $M\backslash\{$ tubular neighborhood of the circle $\}$ and the handle $S^2 \times D^2$ is the 3-manifold $S^2 \times S^1$. We study the result of stretching a tubular neighborhood of this $S^2 \times S^1$ to infinite length. The tool for comparing the resulting moduli spaces of anti-self-dual connections with the original moduli space is the thesis of Tom Mrówka [Mr]. Techniques from Mrówka's thesis are becoming increasingly important in gauge theory (cf. [GM] and [MMR]), and we felt that it would be interesting to carry out our argument from Mrówka's point of view.

For the experts, we outline the alternative approach using Donaldson's work from [D1] and [D2]. Let S^3 be a 3-sphere in $M\#S^2 \times S^2$ whose complement is the disjoint union of $M \setminus B^4$ and $S^2 \times S^2 \setminus B^4$. Suppose that one has a 1-parameter family of metrics $\{g_t\}$ on $M\#S^2 \times S^2$ such that in $(M\#S^2 \times S^2, g_t)$ our S^3 has diameter less than $d(t)$ and $d(t) \to 0$ as $t \to \infty$. Thus in a reasonable sense, the sequence of Riemannian manifolds $(M\#S^2 \times S^2, g_t)$

converges to the one-point union $(M, g_M) \vee (S^2 \times S^2, g_{S^2\times S^2})$, and we choose the g_t so that both g_M and $g_{S^2\times S^2}$ are "generic". For each $i = 1, \dots, d$, let V_i be a codimension 2 subvariety of the appropriate space of connections, such that V_i is a cocycle representative of $\mu(z_i)$. (See §2.) Similarly choose V_x and V_y. Then the invariant $q_{\ell+1, u_1, M\#S^2\times S^2}(z_1, \dots, z_d, x, y)$ is found by evaluating $u_1[V_1 \cap \dots \cap V_d \cap V_x \cap V_y \cap \mathscr{M}_{M\#S^2\times S^2, \ell+1}]$, and $q_{\ell, M}(z_1, \dots, z_d)$ is the algebraic intersection number $\# V_1 \cap \dots \cap V_d \cap \mathscr{M}_{M, \ell}$.

Suppose that $\{A_n\}$ is a sequence of connections such that for each n we have $A_n \in V_1 \cap \dots \cap V_d \cap V_x \cap V_y \cap \mathscr{M}_{M\#S^2\times S^2, \ell+1}(g_{t_n})$ where $t_n \to \infty$. Then $\{A_n\}$ converges to a pair of connections $A \in \mathscr{M}_{M,k}(g_M)$ and $B \in \mathscr{M}_{S^2\times S^2, j}(g_{S^2\times S^2})$ together with possible instanton bubbles, and $k + j + \#(\text{bubbles}) \leq \ell + 1$. Counting arguments as in [D2] and §5 below show that the only possibility is that $A \in V_1 \cap \dots \cap V_d \cap \mathscr{M}_{M,\ell}(g_M)$, $B \in \mathscr{M}_{S^2\times S^2, 0}(g_{S^2\times S^2})$ (so B is the trivial connection $\Theta_{S^2\times S^2}$). Furthermore, a single bubble must occur at an intersection point of generic surfaces S_x and S_y representing x and y. One then needs to argue that the calculation of $V_1 \cap \dots \cap V_d \cap V_x \cap V_y \cap \mathscr{M}_{M\#S^2\times S^2, \ell+1}(g_{t_n})$ for large n will follow from the solution of the gluing problem: To an $A \in V_1 \cap \dots \cap V_d \cap \mathscr{M}_{M,\ell}(g_M)$ glue the connection $\Theta_{S^2\times S^2}\#I$ on $S^2 \times S^2$ obtained from starting with $\Theta_{S^2\times S^2}$ and grafting in a charge 1 instanton I at a point of $S_x \cap S_y$.

The gluing problem at hand is quite similar to the one considered by Donaldson in his proof of Theorem B of [D1]. In that case one needs to graft a pair of charge 1 instantons to the trivial connection on a $b^+ = 1$ manifold. Here there is only one instanton on the $b^+ = 1$ manifold $S^2 \times S^2$, but there is a second gluing parameter coming from gluing the connection $\Theta_{S^2\times S^2}\#I$ on $S^2 \times S^2$ to the connection A on M. Finally, one needs to modify the argument of [D1,§V] to see that for each $A \in V_1 \cap \dots \cap V_d \cap \mathscr{M}_{M,\ell}(g_M)$ and each intersection point of S_x and S_y, one obtains a circle of connections of the form $A\#(\Theta_{S^2\times S^2}\#I)$ on which u_1 evaluates nontrivially. Since $x \cdot y = 1$ is odd, Theorem 1.1 will follow. This is discussed further in the proof of Theorem 8.1 below.

Here is an outline of the paper. In §2 we review Donaldson's invariant and describe $q_{k, u_1, N}$ in more detail. In §3 we present the necessary results of Mrówka [Mr] and of Taubes [T3] concerning gauge theory on manifolds with cylindrical ends. Mrówka's thesis [Mr] is discussed in §4. In §5 we begin serious consideration of Theorem 1.1, whose proof is there reduced to a single calculation. This calculation is then carried out in §6. In §7 we study the invariant $q_{k, u_2, M}$ and prove Theorem 1.3. Finally, in §8 we combine our gauge-theoretic calculations with some algebra to prove Theorems 1.5 and 1.6.

2. Some torsion instanton invariants

Let M be a closed oriented simply connected 4-manifold and P a principal $SU(2)$-bundle over M. The bundle P is classified topologically by its second Chern class, $c_2(P) = k$. Let $\mathscr{A} = \mathscr{A}(P)$ be the L^2_3-Sobolev space of connections on P. It is acted on by the Hilbert Lie group $\mathscr{G} = \mathscr{G}(P)$ of

L_4^2-gauge transformations. The quotient is $\mathscr{B}_{M,k} = \mathscr{B}(P)$, the space of equivalence classes of connections. Let $\mathscr{B}^*_{M,k}$ denote the irreducible connections in $\mathscr{B}_{M,k}$. (We do not distinguish in notation between a connection $A \in \mathscr{A}$ and its equivalence class $A \in \mathscr{B}_{M,k}$.) The moduli space of equivalence classes of anti-self-dual connections on P is denoted by $\mathscr{M}_{M,k}$ or by $\mathscr{M}_{M,k}(g)$ when making explicit the Riemannian metric g on M. For $k > 0$ and a generic choice of g, this moduli space is a manifold, which, if nonempty, has dimension $8k - 3(1 + b_M^+)$. (See [FU] for details.)

If $b_M^+ > 1$ is odd and if $k > \frac{3}{4}(1 + b_M^+)$ then the Donaldson polynomial invariant $q_{k,M}$ is defined as follows. The dimension of $\mathscr{M}_{M,k}$ is $8k - 3(1 + b_M^+) = 2d$, and $q_{k,M} \in \mathrm{Sym}^d_{\mathbf{Z}}(H_2(M; \mathbf{Z}))$. For a generic surface Σ in M, the restriction of an irreducible anti-self-dual connection over Σ is again irreducible; let $r_\Sigma : \mathscr{M}_{M,k} \to \mathscr{B}^*_\Sigma$ denote the restriction map. Donaldson defines a complex line bundle $\mathscr{L}_\Sigma$ over $\mathscr{B}^*_\Sigma \cup \{\theta_\Sigma\}$ (where θ_Σ denotes the trivial connection on Σ) together with a section so that when pulled back by r_Σ it gives a section of $r^*_\Sigma(\mathscr{L}_\Sigma)$ whose zero set V_Σ is a codimension 2 submanifold of $\mathscr{B}^*_{M,k} \cup \{\theta\}$ which meets all of the moduli spaces $\mathscr{M}_{M,l}$, $l \le k$, transversely [D1]. We shall call V_Σ "the divisor associated to Σ".

Given homology classes $z_1, \dots, z_d \in H_2(M; \mathbf{Z})$, represent them by generic surfaces $\Sigma_1, \dots, \Sigma_d$ in general position. The intersection $V_{\Sigma_1} \cap \dots \cap V_{\Sigma_d} \cap \mathscr{M}_{M,k}$ will then be discrete, and the condition $k > \frac{3}{4}(1 + b_M^+)$ will imply that is is compact. (The V_{Σ_i} are also chosen to have transverse multiple intersections.) Donaldson's polynomial invariant is defined to be

$$q_{k,M}(z_1, \dots, z_\alpha) = \#(V_{\Sigma_1} \cap \dots \cap V_{\Sigma_d} \cap \mathscr{M}_{M,k})$$

where "#" denotes a count with signs. Donaldson [D2] proves that $q_{k,M}$ depends only on the smooth structure of M. More formally, $q_{k,M}$ can be viewed as follows. There is a homomorphism $\mu : H_2(M; \mathbf{Z}) \to H^2(\mathscr{B}^*_{M,k}; \mathbf{Z})$ defined in [D1], and under the hypotheses on k, the above intersection is compact. Then $q_{k,M}(z_1, \dots, z_d) = \mu(z_1) \cup \dots \cup \mu(z_d)[\mathscr{M}_{M,k}]$. In fact, the divisor V_{Σ_i} is a cocycle representative for $\mu(z_i)$. The next proposition is well known.

Proposition 2.1. *Let M be a closed simply connected 4-manifold. Then $\pi_1(\mathscr{B}^*_{M,k}) = 0$ unless M is spin and k is even, in which case $\pi_1(\mathscr{B}^*_{M,k}) \cong \mathbf{Z}_2$.*

Proof. This proof can be ferretted out of [FU] as follows. Let $\widehat{\mathscr{G}} = \mathscr{G}/\pm 1$. Then $\mathscr{A}^*$ is a principal $\widehat{\mathscr{G}}$-bundle over $\mathscr{B}^*_{M,k}$; so $\pi_1(\mathscr{B}^*_{M,k}) \cong \pi_0(\widehat{\mathscr{G}})$. If $\mathscr{G}^0$ is the based gauge group (of gauge transformations restricting to the identity over a basepoint) then $\mathscr{G}$ fibers over $SU(2)$ with fiber $\mathscr{G}^0$; so $\pi_0(\mathscr{G}) \cong \pi_0(\mathscr{G}^0)$. It is easy to see as in [FU, §5] that $\pi_0(\mathscr{G}^0) \cong [M, S^3]$, and by the Steenrod Classification Theorem this is $\mathbf{Z}_2$ if M is spin and 0 if M is not spin. We have

$$\mathbf{Z}_2 \xrightarrow{j_*} \pi_0(\mathscr{G}) \to \pi_0(\widehat{\mathscr{G}}) \to 0;$$

so $\pi_0(\widehat{\mathscr{G}}) = 0$ if M is not spin. If M is spin then [FU] shows that j_* is onto in the case that $k = 1$ by identifying its image with a generator $u \in \pi_4(S^3) \cong \pi_0(\mathscr{G}^0) \cong \pi_0(\mathscr{G})$. For general k the argument shows that its image is $g^*(u)$, where $g : S^4 \to S^4$ is a degree k map; hence j_* is onto if and only if k is odd, completing the proof. $\square$

Now consider a simply connected spin 4-manifold M with b_M^+ even. The moduli space of anti-self-dual connections on the $SU(2)$-bundle over M with $c_2 = k$ has formal dimension $8k - 3(1 + b_M^+) = 2d + 1$. Let homology classes $z_1, \ldots, z_d \in H_2(M; \mathbf{Z})$ be represented by generic surfaces $\Sigma_1, \ldots, \Sigma_d$ as above. If $k > \frac{3}{4}(1 + b_M^+) + \frac{1}{4}$ then the intersection $V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap \mathscr{M}_{M,k}$ will be a compact 1-manifold in $\mathscr{B}_{M,k}^*$ (for a generic metric on M). Suppose k is even. This intersection represents a class in $H_1(\mathscr{B}_{M,k}^*; \mathbf{Z}) \cong \mathbf{Z}_2$. Just as for $q_{k,M}$ this class can be shown to depend only on $z_1, \ldots, z_d$ and the smooth structure of M. This defines an invariant which Donaldson calls $q_{k,u_1,M} \in \mathrm{Sym}^d_{\mathbf{Z}_2}(H_2(M; \mathbf{Z}))$. Donaldson's definition of this invariant in [D3] is basically the same. It goes as follows. Coupling the Dirac operator $\not{D}$ on M to connections on P gives a family of operators and hence a virtual bundle $\mathrm{Ind}\,\not{D}_A$ over $\mathscr{A}$. The operator $\not{D}_A$ can be regarded as a real operator, and so there is a real line bundle $\det \mathrm{Ind}_{\mathbf{R}} \not{D}_A$ over $\mathscr{A}$. When k is even this bundle descends to a real line bundle η over $\mathscr{B}_{M,k}^*$ (see [D1]). The class $u_1 \in H^1(\mathscr{B}_{M,k}^*; \mathbf{Z}_2)$ is defined to be $u_1 = w_1(\eta)$. Then for k even and in the range above, $q_{k,u_1,M}(z_1, \ldots, z_d) = \mu(z_1) \cup \cdots \cup \mu(z_d) \cup u_1[\mathscr{M}_{M,k}]$.

Similarly for M spin, b_M^+ odd, and k odd one obtains a class $u_2 \in H^2(\mathscr{B}_{M,k}^*; \mathbf{Z}_2)$ as in [D1]. Then for $k > \frac{3}{4}(1 + b_M^+) + \frac{1}{2}$ one obtains an invariant $q_{k,u_2,M} \in \mathrm{Sym}^d_{\mathbf{Z}_2}(H_2(M; \mathbf{Z}))$ where $\dim \mathscr{M}_{M,k} = 8k - 3(1 + b_M^+) = 2d + 2$ defined by

$$q_{k,u_2,M}(z_1, \ldots, z_d) = \mu(z_1) \cup \cdots \cup \mu(z_d) \cup u_2[\mathscr{M}_{M,k}].$$

3. Gauge theory on manifolds with cylindrical ends

Given a simply connected oriented 4-manifold X with nonempty boundary ∂X, we let X_+ denote the extended manifold $X \cup (\partial X \times [-1, \infty))$. We shall only consider Riemannian metrics on X_+ which differ on $\partial X \times [1, \infty)$ from a product metric by an exponentially decaying term. We call such metrics "asymptotically cylindrical". Gauge theory on these manifolds has been studied by several authors, notably Mrówka [Mr], Taubes [T2,3], Morgan, Mrówka, and Ruberman [MMR], and Floer [F]. The appropriate theory in our context is based on connections which decay faster than $e^{-\delta t}$ for a fixed small constant δ.

Let $\mathscr{A}(P)$ denote the space of smooth connections on P with finite energy, i.e. $\int_{X_+} \mathrm{Tr}(F_A \wedge F_A) < \infty$ with the coarsest topology compatible with smooth convergence on compact sets and the continuity of $A \mapsto \int_{X_+} \mathrm{Tr}(F_A \wedge F_A)$. The gauge group $\mathcal{G}$ consists of gauge transformations on P with the topology of smooth convergence on compact sets. In $\mathcal{A}(P)$ consider the finite energy anti-self-dual connections

$$\mathbf{m}_{X,k} = \left\{A \in \mathcal{A} \mid \frac{-1}{8\pi^2}\int_{X_+} \mathrm{Tr}(F_A \wedge F_A) = k < \infty \text{ and } *F_A = -F_A\right\}.$$

Its quotient by $\mathcal{G}$ is the moduli space $\mathcal{M}_{X,k} \subset \mathcal{B}$. The "charge" k need not be an integer.

In auspicious circumstances the moduli space $\mathcal{M}_{X,k}$ has nice local properties and one can proceed without worries. This is explained in [T3, Theorem 1.8] which we shall now paraphrase. We need to work with based moduli spaces; so we fix a basepoint $x_0 \in \partial X \times \{0\}$ and let $\mathcal{G}^0$ be the subgroup of gauge transformations in $\mathcal{G}$ which restrict to the identity on the fiber over x_0. Set $\mathcal{B}^0 = \mathcal{A}^0/\mathcal{G}^0$ and $\mathcal{M}^0_{X,k} = \mathbf{m}_{X,k}/\mathcal{G}^0$. Consider an $\alpha \in \mathcal{R}(\partial X)$, the representation space $\mathrm{Hom}(\pi_1(\partial X), SU(2))$. The action of $SU(2)$ by conjugation induces an effective action of $SO(3)$ on $\mathcal{R}(\partial X)$ whose quotient is the character variety $\boldsymbol{\chi}(\partial X)$. Let Γ_α be the isotropy group of α and let U_α be a slice in $\mathcal{R}(\partial X)$ to this action at α; so $U_\alpha \times_{\Gamma_\alpha} SO(3)$ models a neighborhood of the $SO(3)$ orbit of α in $\mathcal{R}(\partial X)$.

Theorem 3.1 (Taubes [T3; Theorem 1.8], Mrówka [Mr]). *There is a locally constant continuous function R on $\mathcal{M}^0_X$ with values in the set of connected components of $\mathcal{R}(\partial X)$ with the following significance. If K is a connected component of $\mathcal{R}(\partial X)$ such that for each $\alpha \in K$ the dimension of U_α equals the dimension of the twisted cohomology $H^1(\partial X; \mathrm{ad}\,\alpha)$ and $k_0 \in \mathbf{Z}^+$, then for each $k \le k_0$:*

(i) *Each $A \in \mathcal{M}^0_{X,k} \cap R^{-1}(K)$ has a representative in $\mathbf{m}_{X,k}$ such that if $A_t = A|_{\partial X \times \{t\}}$ then $\lim_{t\to\infty} A_t$ exists, is a flat connection, and the assignment $A \to r^0(A) = \lim_{t\to\infty} A_t$ defines an $SO(3)$-equivariant continuous map*

$$r^0 : \mathcal{M}^0_{X,k} \cap R^{-1}(K) \to \mathcal{R}(\partial X)$$

which descends to

$$r: \mathcal{M}_{X,k} \cap R^{-1}(K)/SO(3) \to \boldsymbol{\chi}(\partial X).$$

(ii) *For a dense set of asymptotically cylindrical metrics on X, the moduli space defined by $\mathcal{M}_{X,k}(\alpha) = \mathcal{M}_{X,k} \cap r^{-1}(\alpha)$ is a manifold (except at reducible connections in case X has a negative definite intersection form). If nonempty, $\mathcal{M}_{X,k}(\alpha)$ has dimension $8k - 3(E(X) + \mathrm{sign}(X)) - \frac{1}{2}h_\alpha - \frac{1}{2}\rho_\alpha$, where $E(X)$ and $\mathrm{sign}(X)$ denote the Euler characteristic and signature (with compact supports) of X, ρ_α is the ρ-invariant of [APS2] and h_α is the sum of the 0th and 1st betti numbers of $H^*(\partial X; \mathrm{ad}\,\alpha)$.*

(iii) *If W is an open subset of $\mathcal{R}(\partial X)$ which contains only smooth points, then for a dense set of asymptotically cylindrical metrics on X, $\mathcal{M}^0_{X,k} \cap (r^0)^{-1}(W)$ is a manifold (except at reducible connections).*

(iv) *For a dense set of asymptotically cylindrical metrics,* $r^0 : \mathscr{M}^0_{X,k} \cap \mathbf{r}^{-1}(K) \to \mathscr{R}(\partial X)$ *is an* $SO(3)$*-equivariant generic map. In particular, it can be made transverse to any* $SO(3)$ *invariant complex.*

This follows since in this situation there is a $\delta > 0$ such that, for $A \in \mathscr{M}^0_{X,k} \cap R^{-1}(K)$, $e^{\delta t}\operatorname{Tr}(F_A \wedge F_A)$ is integrable. As examples, note that if ∂X is a Brieskorn homology sphere then $\boldsymbol{\chi}(\partial X)$ is discrete, and if $\alpha \in \boldsymbol{\chi}(\partial X)$ is nontrivial, then $H^1(\partial X; \operatorname{ad}\alpha) = 0$ (cf. [FS1]). For the trivial representation, $H^1(\partial X; \operatorname{ad}\alpha) = H^1(\partial X; \mathbf{R}^3) = 0$ as well. Thus the hypothesis of Theorem 3.1 holds for all of $\mathscr{R}(\partial X)$. Of course, this already appears in the work of Floer [F].

In this paper we are especially interested in the case $\partial X \cong S^2 \times S^1$. We have

$$\mathscr{R}(S^2 \times S^1) = \operatorname{Hom}(\mathbf{Z}, SU(2)) \cong S^3,$$

and

$$\boldsymbol{\chi}(S^2 \times S^1) \cong SU(2)/SO(3) \cong [-1, 1].$$

Let α be a representation corresponding to a point in the open interval $(-1, 1)$. Then $\Gamma_\alpha \cong S^1$, U_α is 1-dimensional, and

$$U_\alpha \times_{\Gamma_\alpha} SO(3) \cong (\text{interval}) \times_{S^1} SO(3) \cong (\text{interval}) \times S^2.$$

Now $H^1(S^2 \times S^1; \operatorname{ad}\alpha)$ is the group cohomology $H^1(\mathbf{Z}; \operatorname{ad}\alpha) = H^1(S^1; \operatorname{ad}\alpha)$. But the twisted Euler characteristic of S^1 is the untwisted Euler characteristic with coefficients in $\mathbf{R}^3$, namely 0; so $\dim H^1(S^2 \times S^1; \operatorname{ad}\alpha) = \dim H^0(S^1; \operatorname{ad}\alpha) = \dim \Gamma_\alpha = 1 = \dim U_\alpha$. For $\tau = \pm 1 \in \boldsymbol{\chi}(S^2 \times S^1)$ we have $\Gamma_\tau \cong SO(3)$ and so U_τ is 3-dimensional. But the same argument as above shows

$$\dim H^1(S^2 \times S^1; \operatorname{ad}\tau) = \dim H^0(S^2 \times S^1; \operatorname{ad}\tau) = \dim \Gamma_\tau = 3 = \dim U_\tau.$$

Thus $K = \mathscr{R}(S^2 \times S^1)$ satisfies the hypothesis of (3.1).

4. Mrówka's thesis

Our technique for proving Theorems 1.1 and 1.3 requires an understanding of what happens when a metric on M degenerates along a codimension 1 submanifold. In particular, suppose we have a fixed metric $g \in M$ and a codimension 1 submanifold Y of M. Suppose that Y splits M into submanifolds X_1 and X_2; so $M = X_1 \cup_Y X_2$. We assume that g is close to a metric on M which is a product in a neighborhood "tube" $Y \times [-1, 1]$. We then wish to study the effect of changing g in a family $\{g_t \mid t \geq 1\}$ where g_t is within ε_t of the metric on M which agrees with g off $Y \times [-1, 1]$ but has stretched the tube to $Y \times [-t, t]$, and $\lim_{t\to\infty} \varepsilon_t = 0$. Suppose further, for simplicity, that all components of $\mathscr{R}(\partial Y)$ satisfy the hypothesis of Theorem 3.1. For t large, we need to understand how $\mathscr{M}_{M,k}(g_t)$ relates to $\mathscr{M}_{X_{1+},k_1}(g_{t,1})$ and $\mathscr{M}_{X_{2+},k_2}(g_{t,2})$.

In one direction, there is Uhlenbeck's Compactness Theorem [U]. In this situation it yields the following result.

Theorem 4.1 (Uhlenbeck (cf. [Mr])). *Consider an increasing unbounded sequence of integers* $\{t_i\}$. *Suppose* A_i *is a sequence with* $A_i \in \mathcal{M}_{M,k}(g_{t_i})$. *Then there are finite sets of points* $\{x_i\}_{i=1,\ldots,m}$ *in* X_1 *and* $\{y_j\}_{j=1,\ldots,n}$ *in* X_2 *such that after passing to subsequences,* $\{A_i\}$ *converges uniformly in* C^∞ *on compact subsets of* $X_1 \setminus \{x_1, \ldots, x_m\}$ *and* $X_2 \setminus \{y_1, \ldots, y_n\}$ *to anti-self-dual connections.*

This phenomenon is called "weak convergence".

As for the other direction, when Y is a homology 3-sphere, the relationship can be studied via the Floer homology of Y; for example see [F, FS 1,2, A]. In this case, the Donaldson invariant of M can be computed via a pairing of relative Donaldson invariants on X_{1+} and X_{2+} which take their values in the Floer homology of Y.

The general case is studied in the thesis of T. Mrówka [Mr] and by Taubes [T3]. (See the forthcoming work of Morgan, Mrówka, and Ruberman [MMR] for further details.) We next proceed to give a synopsis of some of the results of Mrówka's thesis.

Let U^0 be an open subset of the smooth points of $\mathcal{R}(Y)$ and for $j = 1, 2$ let $\mathcal{N}_j^0$ be a precompact open subset of $(r_j^0)^{-1}(U^0)$ where $r_j^0 : \mathcal{M}^0_{X_{j+}, k_j} \to \mathcal{R}(Y)$ is given by Theorem 3.1. Let $\mathcal{N}^0$ be the fibered product (with $SO(3)$-action coming from the diagram):

$$
\begin{tikzcd}
 & \mathcal{N}^0 \arrow[dl] \arrow[dr] & \\
\mathcal{N}_1^0 \arrow[dr, "r_1^0"'] & & \mathcal{N}_2^0 \arrow[dl, "r_2^0"] \\
 & U^0 &
\end{tikzcd}
\tag{4.2}
$$

Then Mrówka shows that for generic (asymptotically cylindrical) metrics on X_{1+} and X_{2+} the restriction maps r_1^0 and r_2^0 are transverse; so $\mathcal{N}^0$ is a manifold.

For $i = 1, 2$, let $X_{i,t} = X_i \cup Y \times (-t, t) \subset M(g_t)$. Then as in [D1,2] we say that an $A \in \mathcal{B}^*_{M, k_1+k_2}$ is "(η, t)-close" to $(A_1, A_2) \in \mathcal{N}_1 \times \mathcal{N}_2$ if for $i = 1, 2$ we have

$$\| A_{X_{i,t}} - A_{i, X_{i,t}} \|_{L^4(X_{i,t})} < \eta ,$$

where the subscripts denote restriction (and $\mathcal{N}_i = \mathcal{N}_i^0 / SO(3)$).

Mrówka's Theorem 4.3 (Part I) [Mr]. *If* k_1 *and* k_2 *are both positive then there are real numbers* η_0 *and* t_0 *such that for all* $t \ge t_0$ *and* $0 < \eta \le \eta_0$ *there is an* $SO(3)$*-equivariant map* $\gamma_t^0 : \mathcal{N}^0 \to \mathcal{M}^0_{M, k_1+k_2}(g_t)$ *satisfying:*

(a) *The image of* γ_t^0 *is open in* $\mathcal{M}^0_{M, k_1+k_2}(g_t)$ *and contains all points* (A, ξ) *such that* A *is* (η, t)*-close to a point of* $\mathcal{N}_1 \times \mathcal{N}_2$.

(b) γ_t^0 *is a homeomorphism onto its image.*

(c) *For fixed* $((A_1, \xi_1), (A_2, \xi_2)) \in \mathcal{N}^0$, *the sequence* $\gamma_t^0((A_1, \xi_1), (A_2, \xi_2))$ *converges weakly to*

$$[(A_1, \xi_1), (A_2, \xi_2)] \in \mathcal{N}_1^0 \times \mathcal{N}_2^0.$$

Here elements of $\mathscr{B}^0$ are given by gauge equivalence classes of pairs (A, ξ) where $A \in \mathscr{A}$ and $\xi \in P_{x_0}$, the fiber over the basepoint.

The theorem above should be compared with [D1, Theorem 4.53] and [D2, Proposition 4.6]. The map γ_t^0 is obtained from an $SO(3)$-equivariant map $\beta_t^0 : \mathcal{N}^0 \to \mathscr{B}^{*,0}_{M,k_1+k_2}$ where $\beta_t^0((A_1, \xi_1), (A_2, \xi_2))$ is equal to A_1 on X_1 and A_2 on X_2 and patches the two connections together on the tubes $\partial X_i \times [0, 2t]$ and has framing $\xi_1 \xi_2^{-1}$. The map γ_t^0 is obtained from a deformation of β_t^0. In particular, the image $\beta_t(\mathcal{N})$ is homologous to $\gamma_t(\mathcal{N})$ in $\mathscr{B}^*_{M,k_1+k_2}$.

There is also a version of Mrówka's theorem which holds when one of the moduli spaces has zero charge. Assume that $k_1 > 0$ and $k_2 = 0$. Then $\mathscr{M}^0_{X_2+,k_2}$ can be identified with the representation space $\mathscr{R}(X_2)$ and we can consider $\mathcal{N}_2^0$ as an open subset of the smooth points of $\mathscr{R}(X_2)$. Thus, as the $\alpha \in \mathcal{N}_2^0$ vary, the $H^2_+(X_{2+}; \operatorname{ad}\alpha)$ fit together to form an $SO(3)$-equivariant bundle $\Xi_2^0 \to \mathcal{N}_2^0$. Pull back Ξ_2^0 to an $SO(3)$-equivariant bundle Ξ^0 over $\mathcal{N}^0$. Mrówka's result in this case is

Mrówka's Theorem 4.4 (Part II) [Mr]. *There are real numbers* η_0 *and* t_0 *such that for all* $t \geq t_0$ *and* $0 < \eta \leq \eta_0$ *there is an* $SO(3)$*-equivariant map* $\gamma_t^0 : \mathcal{N}^0 \to \mathscr{B}^0_{M,k_1}$ *and an* $SO(3)$*-equivariant section* $s^0 : \mathcal{N}^0 \to \Xi^0$ *such that*

(a) $\gamma_t^0(\mathcal{N}^0) \cap \mathscr{M}^0_{M,k_1}(g_t)$ *is open in* $\mathscr{M}^0_{M,k_1}(g_t)$ *and contains all points* (A, ξ) *such that* A *is* (η, t)*-close to a point of* $\mathcal{N}_1 \times \mathcal{N}_2$.

(b) $\gamma_t^0((s^0)^{-1}(0)) = \gamma_t^0(\mathcal{N}^0) \cap \mathscr{M}^0_{M,k_1}(g_t)$, *and* $\gamma_t^0|_{(s^0)^{-1}(0)}$ *is a homeomorphism onto its image.*

(c) *For fixed* $[(A_1, \xi_1), (A_2, \xi_2)] \in \mathcal{N}^0$, *the sequence* $\gamma_t^0((A_1, \xi_1), (A_2, \xi_2))$ *converges weakly to*

$$[(A_1, \xi_1), (A_2, \xi_2)] \in \mathcal{N}_1^0 \times \mathcal{N}_2^0.$$

We next need to look more closely at the divisor V_Σ corresponding to an oriented surface $\Sigma \subset X$. For any such surface which is generic in the sense that restriction induces $r_\Sigma : \mathscr{M}^*_{X,k} \to \mathscr{B}^*_\Sigma$ for all k (i.e. an irreducible anti-self-dual connection over X restricts to an irreducible connection over Σ) the divisor V_Σ is defined to be the zero set of a generic section of $r^*_\Sigma(\mathscr{L}_\Sigma)$ where $\mathscr{L}_\Sigma = \det \operatorname{Ind} D_{\Sigma,A}$ and $D_{\Sigma,A}$ is the Dirac operator on Σ coupled to connections. The operator $D_{\Sigma,A}$ has numerical index 0; thus generic fibers of the index virtual bundle $\operatorname{Ind} D_{\Sigma,A}$ are 0-dimensional, however at certain "jumping points" $A \in \mathscr{B}^*_\Sigma$, $\ker D_{\Sigma,A} \cong \operatorname{coker} D_{\Sigma,A} \neq 0$. These points constitute the divisor of the determinant line bundle $\det \operatorname{Ind} D_{\Sigma,A}$. It is shown in [DK] that if $\mathscr{V}_\Sigma$ is defined by this divisor, then $\mathscr{V}_\Sigma$ is also a cocycle representative for

$\mu[\Sigma]$. Thus for our purposes, it suffices to identify V_Σ with $\mathscr{V}_\Sigma$. Then an anti-self-dual connection A over X lies in V_Σ if and only if its restriction $r_\Sigma(A)$ over Σ is a jumping point for the index virtual bundle of the Dirac operator on Σ coupled to connections.

Proposition 4.5. *In the situation of Theorem* 4.3 *suppose that* $\Sigma \subset X_1$. *Then* $\beta_t(A_1, A_2) \in V_\Sigma$ *if and only if* $A_1 \in V_\Sigma$.

Proof. Let " $'$ " denote restriction of a connection to Σ; so $A_1' = \beta_t(A_1, A_2)'$. Then $\beta_t(A_1, A_2)' \in V_\Sigma$ if and only if $\ker D_{\Sigma, A'} \neq 0$, i.e. if and only if $A_1 \in V_\Sigma$. $\square$

It is more interesting to ask about a surface Σ in X such that $\Sigma = \Sigma_1 \cup \Sigma_2$ with $\Sigma_i \subset X_i$ and such that the homology class of Σ is represented neither in X_1 nor X_2. Assume that $\Sigma_1 \cap \Sigma_2$ is a circle, and let $\Sigma_{i+} = \Sigma_i \cup (\partial\Sigma_i \times [0, \infty)) \subset X_{i+}$. The surface Σ_{i+} has a Dirac operator D_{Σ_i} defined on L^2_δ-sections. The following proposition is discussed at length and proved in [MMR].

Theorem 4.6 (Morgan, Mrówka, and Ruberman). *Let* $((A_1, \xi_1), (A_2, \xi_2)) \in \mathscr{N}^0$. *Suppose that* $-1 \notin U^0$. *Then there is a* t_0 *such that, for all* $t \geq t_0$, $\beta_t^0((A_1, \xi_1), (A_2, \xi_2)) \in V_\Sigma$ *if and only if* $(A_1, \xi_1) \in V_{\Sigma_{1+}}$ *or* $(A_2, \xi_2) \in V_{\Sigma_{2+}}$.

5. Connected sums with $S^2 \times S^2$

Let M be a smooth closed simply connected spin 4-manifold with $b_M^+ \geq 3$ and odd, and let ℓ be an odd integer, $\ell > \frac{3}{4}(1 + b_M^+)$. The moduli space of anti-self-dual connections on the $SU(2)$-bundle P over M with $c_2(P) = \ell$ gives rise to a Donaldson polynomial $q_{\ell, M} \in \mathrm{Sym}^d_{\mathbf{Z}}(H_2(M; \mathbf{Z}))$ where $d = 4\ell - \frac{3}{2}(1 + b_M^+)$. (Notice that the conditions placed on b_M^+ and ℓ imply that $d > \frac{3}{2}(1 + b_M^+) \geq 6$.) Let $X = M \# S^2 \times S^2$ and consider the $SU(2)$ bundle over X with $c_2 = \ell + 1$. The formal dimension of the moduli space $\mathscr{M}_{X, \ell+1}$ is $8(\ell + 1) - 3(1 + (b_M^+ + 1)) = 2d + 5$. Since X is spin and $\ell + 1$ is even and $> \frac{3}{4}(b_X^+ + 1) + \frac{1}{4}$, the $\mathbf{Z}_2$-polynomial invariant $q_{\ell+1, u_1, X}$ of degree $d + 2$ is defined. Let $z_1, \dots, z_d \in H_2(M, \mathbf{Z})$ be represented by generic oriented surfaces $\Sigma_1, \dots, \Sigma_d$ in M as in §2 and let x and y be the classes represented by $S^2 \times 0$ and $0 \times S^2$. Here we are viewing $H_2(M, \mathbf{Z})$ as a subgroup of $H_2(X, \mathbf{Z})$. We mean to evaluate $q_{\ell+1, u_1, X}(z_1, \dots, z_d, x, y)$.

It is useful to view the process of taking connected sums with $S^2 \times S^2$ as the result of surgery along a homotopically trivial circle. Let C denote a circle in M, and let M_0 be the simply connected manifold $M \setminus N(C)$, where $N(C)$ is an open tubular neighborhood of C. Let C' be a circle in S^4, and let $K = S^4 \setminus N(C') \cong S^2 \times D^2$; then $X = M_0 \cup_{S^2 \times S^1} K$. Let g_{M_0} and g_K be asymptotically cylindrical generic metrics on $M_{0,+}$ and K_+ and let $g = g_1$ be a generic metric on X such that the family of metrics $\{g_t \mid t \geq 1\}$ on X, obtained as in §4, converges to $g_{M_0} \amalg g_K$.

Since the character variety $\boldsymbol{\chi}(S^2 \times S^1) = [-1, 1]$ is connected, all Chern-Simons invariants of flat $SU(2)$-connections over $S^2 \times S^1$ are trivial. Thus a connection A with finite action on the restricted bundle over M_0 has integral charge $(-1/8\pi^2) \int_{M_0} \mathrm{Tr}(F_A \wedge F_A)$.

Recall from Theorem 3.1 the restriction map $r_{M_0} : \mathscr{M}_{M_0} \to \boldsymbol{\chi}(S^2 \times S^1)$. Let $\mathscr{M}_{M_0, m}(\alpha)$ denote the moduli space of anti-self-dual connections A over M_0 with charge equal to m and with $r_{M_0}(A) = \alpha$.

Proposition 5.1. *The formal dimension of* $\mathscr{M}_{M_0, m}(\alpha)$ *is*

$$\dim \mathscr{M}_{M_0, m}(\alpha) = \begin{cases} 2d + 8(m - \ell) - 1, & \alpha \neq \pm 1, \\ 2d + 8(m - \ell) - 3, & \alpha = \pm 1. \end{cases}$$

Proof. The Atiyah, Patodi, Singer Theorem gives

$$\dim \mathscr{M}_{M_0, m}(\alpha) = 8m - 3(1 + b_M^+) - \frac{1}{2} h_\alpha - \frac{1}{2} \rho_\alpha$$

where ρ_α is the ρ-invariant of [APS2] and h_α is the sum of the 0th and 1st betti numbers of $H^*(S^2 \times S^1, \mathrm{ad}\,\alpha)$. To compute ρ_α, note that the representation α of $\pi_1(S^2 \times S^1)$ extends as a representation of $\pi_1(D^3 \times S^1)$. Thus $\rho_\alpha = 3\ \mathrm{sign}(D^3 \times S^1) - \mathrm{sign}_{\mathrm{ad}\,\alpha}(D^3 \times S^1) = 0$.

The calculation at the end of §3 shows that $h_\alpha = 2$ if $\alpha \neq \pm 1$ and $h_\alpha = 6$ if $\alpha = \pm 1$. Now $8\ell - 3(1 + b_M^+) = 2d$; so we have $\dim \mathscr{M}_{M_0, m}(\alpha) = 2d + 8(m - \ell) - \frac{1}{2} h_\alpha$, as desired. □

Corollary 5.2. *If* $2d + 8(m - \ell) \geq 1$ *the moduli space* $\mathscr{M}_{M_0, m}$ *(with respect to the generic metric* g_{M_0}*) is a manifold whose formal dimension is* $2d + 8(m - \ell)$.

Proof. Since $\mathscr{R}(S^2 \times S^1) \cong S^3$ is a smooth manifold, Theorem 3.1(iii) implies that $\mathscr{M}^0_{M_0, m}$ is a manifold. Since this moduli space contains no reducible connections, the group $SO(3)$ acts freely on $\mathscr{M}^0_{M_0, m}$. Also $SO(3)$ acts on $\mathscr{R}(S^2 \times S^1)$ with S^2 as principal orbit type. Thus the quotient $\mathscr{M}_{M_0, m}$ is also a manifold, and according to (3.1) $r^0_{M_0}$ is a generic $SO(3)$-equivariant map. Thus for a generic metric, $r_{M_0} : \mathscr{M}_{M_0, m} \setminus r^{-1}_{M_0}(\pm 1) \to (-1, 1)$ is transverse, and so $\dim \mathscr{M}_{M_0, m} = 1 + \dim \mathscr{M}_{M_0, m}(\alpha)$ where $\alpha \neq \pm 1$. □

Similarly we have

Proposition 5.3. *The formal dimension of* $\mathscr{M}_{K, n}(\alpha)$ *is*

$$\dim \mathscr{M}_{K, n}(\alpha) = \begin{cases} 8n - 4, & \alpha \neq \pm 1, \\ 8n - 6, & \alpha = \pm 1. \end{cases}$$

Corollary 5.4. *If* $n \geq 1$ *the moduli space* $\mathscr{M}_{K, n}$ *is a manifold whose formal dimension is* $8n - 3$.

With the metric g_{M_0} on M_0, the moduli spaces $\mathscr{M}_{M_0, m}(\alpha)$ will either be empty or manifolds of dimension given in Proposition 5.1. Now let $V_1, \dots, V_d$

be the divisors associated with the surfaces $\Sigma_i \subset M_0$, $[\Sigma_i] = z_i$. Let $\mathcal{I}_{M_0}$ denote the zero-dimensional intersection $\mathcal{I}_{M_0} = V_1 \cap \cdots \cap V_d \cap \mathcal{M}_{M_0, \ell}$.

Proposition 5.5. *If $A \in \mathcal{I}_{M_0}$ then $r_{M_0}(A) \neq \pm 1$.*

Proof. The subcomplex $r_{M_0}^{-1}(\pm 1)$ of $\mathcal{M}_{M_0, \ell}$ has dimension $2d - 3$ and is met transversely by the codimension $2d$ submanifold $V_1 \cap \cdots \cap V_d$. $\square$

Proposition 5.6. *$\mathcal{I}_{M_0}$ is compact.*

Proof. If not, there is a sequence of connections $\{A_n\}$ in $\mathcal{I}_{M_0}$ converging weakly to an $A_\infty \in \mathcal{M}_{M_0, m}$, $m < \ell$, together with instantons at points $x_1, \dots, x_r \in M_0$, and perhaps instantons on tubes $S^2 \times S^1 \times \mathbf{R}$. (See e.g. [FS2].) Since the Chern-Simons invariant of any flat connection on $S^2 \times S^1$ is $0 \in \mathbf{R}/\mathbf{Z}$, the moduli spaces containing the instantons on tubes account for an integral total charge $T \geq 0$. If $A_\infty \notin \mathcal{M}_{M_0, \ell}$ then $r + T > 0$. The surfaces Σ_i are in general position; so the points $x_1, \dots, x_r$ lie on at most $2r$ of the surfaces. Suppose that $0 < m < \ell$. Then A_∞ lies in at least $d - 2r$ of the codimension 2 varieties V_i. So by Corollary 5.2 and transversality, $2d + 8(m - \ell) \geq 2(d - 2r)$. But also counting charge we get

$$\ell \geq m + r + T \geq \ell + \frac{r}{2} + T.$$

This gives a contradiction unless $r = T = 0$.

In case $A_\infty \in \mathcal{M}_{M_0, 0}$, then $A_\infty = \theta$, the trivial connection. But the formal dimension of $\mathcal{M}_{M_0, 0}$ is negative by Proposition 5.1. Thus transversality implies that $\theta \notin V_i$ for any i. This means that each Σ_i contains some x_j, and so $2r \geq d = 4\ell - \frac{3}{2}(1 + b_M^+)$. Again counting charge, we have $\ell \geq r + T \geq 2\ell - \frac{3}{4}(1 + b_M^+) + T$. This contradicts our basic assumption that $\ell > \frac{3}{4}(1 + b_M^+)$. Thus $\mathcal{I}_{M_0}$ is compact. $\square$

Proposition 5.7. *The intersection $\mathcal{I}_{M_0}$ consists of a finite number of points; modulo 2 this number is $q_{\ell, M}(z_1, \dots, z_d)$.*

Proof. First we apply Mrówka's Theorem 4.4 to $M = M_0 \cup N(C)$. Since $\mathcal{R}(S^2 \times S^1) \cong S^3$, we can take $U^0 = \mathcal{R}(S^2 \times S^1) \setminus \{\pm 1\}$. The tubular neighborhood $N(C) \cong \mathbf{R}^3 \times S^1$; so $\mathcal{R}(N(C))$ can be identified with $\mathcal{R}(S^2 \times S^1)$, and the fibered product $\mathcal{N}^0$ can be identified with $(r_{M_0}^0)^{-1}(U^0)$. The obstruction bundle $\Xi_{M_0}^0$ has fiber $H_+^2(\mathbf{R}^3 \times S^1; \operatorname{ad} \rho)$ for $\rho \in \mathcal{R}(\mathbf{R}^3 \times S^1)$. This cohomology group vanishes; so Theorem 4.4 implies that for large t we get maps $\mathcal{N} = \mathcal{N}^0/SO(3) \to \mathcal{M}_{M, \ell}(g_t)$ which are homeomorphisms onto their (open) image.

The intersection $\mathcal{I}_{M_0} \subset r_{M_0}^{-1}(-1, 1)$ by Propositions 5.5 and 5.6. Hence we may consider $\mathcal{I}_{M_0} \subset \mathcal{N}$. We claim that for large t the homeomorphism γ_t identifies $\mathcal{I}_{M_0}$ with $\mathcal{I}_M(t) = \mathcal{M}_{M, \ell}(g_t) \cap V_1 \cap \cdots \cap V_d$. (Cf. [D2; Proof of Theorem 4.8].) First, if $A \in \mathcal{I}_{M_0}$, then by Theorem 4.4(c) $\{\gamma_t(A)\}$ converges to

$[A, r_{M_0}(A)] \in \mathscr{M}_{M_0,\ell} \times \mathscr{R}(\mathbf{R}^3 \times S^1)$; so as $t \to \infty$, $\gamma_t(A) \in \mathscr{M}_{M,\ell}(g_t)$ converges to a point of $V_1 \cap \cdots \cap V_d$. But $V_1 \cap \cdots \cap V_d$ intersects $\mathscr{M}_{M,\ell}(g_t)$ transversely in a dimension 0 submanifold. So for large enough t by Theorem 4.4(a) there is a unique point of $\mathscr{I}_M(t)$ close to $\gamma_t(A)$. We claim that for large enough t these points comprise all of $\mathscr{I}_M(t)$. If not, then there is a sequence $A_n \in \mathscr{I}_M(t_n)$, $t_n \to \infty$, which fails to converge strongly to some connection $[A, r_{M_0}(A)]$. In other words, $\{A_n\}$ converges to $[A_{M_0}, A_C] \in \mathscr{M}_{M_0,k_1} \times \mathscr{M}_{N(C),k_2}$ together with instantons at points $x_1, \ldots, x_r \in M_0$ and $y_1, \ldots, y_s \in N(C)$, and perhaps also loses some integral charge T on the tube $S^2 \times S^1 \times \mathbf{R}$. (See the proof of Proposition 5.6.) Since we are assuming that $\{A_n\}$ fails to converge strongly to some $[A, r_{M_0}(A)]$, we must have $k_2+r+s+T > 0$. But the standard counting argument as used in Proposition 5.6 implies that $k_2 + r + s + T = 0$. This means that for large t we can identify $\mathscr{I}_{M_0}$ with $\mathscr{I}_M(t)$.

The Donaldson invariant $q_{\ell,M}(z_1, \ldots, z_d)$ is calculated by computing the signed number (finite) of points in $\mathscr{I}_M(t)$; so our proposition follows. □

Notice that we have not yet used reduction mod 2 in a substantial way. Given a homology orientation [D2], $q_{\ell,M}(z_1, \ldots, z_d)$ is the count of signed points in $\mathscr{I}_{M_0}$.

Let Σ_x and Σ_y be generic surfaces representing the homology classes $x = [S^2 \times 0]$ and $y = [0 \times S^2]$ in $X = M\#S^2 \times S^2$, and with divisors V_x and V_y. We are viewing $X = M_0 \cup (S^2 \times D^2)$, and we may clearly assume $\Sigma_x \subset S^2 \times D^2$. We need to focus our attention on Σ_y. The surface Σ_y restricts to M_0 and K to give bounded surfaces Σ_{y,M_0} and $\Sigma_{y,K}$. We also use this same notation to denote the corresponding surfaces with cylindrical ends in M_{0+} and K_+.

For $A \in \mathscr{B}^*_{M_0,k_1}$ and $B \in \mathscr{B}^*_{K,k_2}$ let A' and B' denote the restrictions to Σ_{y,M_0} and $\Sigma_{y,K}$. Since $\mathscr{I}_{M_0}$ consists of a finite number of connections, and since the divisor $V_{\Sigma_{y,M_0}}$ can be chosen transverse to $\mathscr{I}_{M_0}$, we have $\mathscr{I}_{M_0} \cap V_{\Sigma_{y,M_0}} = \varnothing$. Then it follows from Proposition 4.6 that if $((A, \xi), (B, \eta)) \in \mathscr{N}^0$, then for large enough t, the image $\beta_t^0((A, \xi), (B, \eta)) \in V_{\Sigma_y}$ if and only if $(B, \eta) \in V_{\Sigma_{y,K}}$.

For $(A, \xi) \in \mathscr{M}^0_{M_0,\ell}$ and $(B, \eta) \in \mathscr{M}^0_{K,1}$ with $r^0_{M_0}(A, \xi) = \alpha = r^0_K(B, \eta)$, we have in the pullback diagram (4.2) restricted to the fibers of (A, ξ) and (B, η):

$$
\begin{tikzcd}
SO(3)\cdot A \arrow[dr] & & SO(3)\cdot B \arrow[dl] \\
& SU(2)/\Gamma_\alpha \cong S^2 &
\end{tikzcd}
\tag{5.8}
$$

where $SO(3) \cdot A = \{(A, \xi) \mid \xi \in P_{M_0, x_0}\}$ and similarly for $SO(3) \cdot B$.

Now let $\mathscr{I}_K$ denote the intersection $\mathscr{M}_{K,1} \cap V_x \cap V_{y,K}$. By Corollary 5.4 this is a 1-manifold. Let $\mathscr{I}^0_{M_0} * \mathscr{I}^0_K$ denote the $SO(3)$-equivariant fibered product over $\mathscr{R}(S^2 \times S^1)$.

$$\begin{tikzcd}
 & \mathscr{I}^0_{M_0} * \mathscr{I}^0_K \arrow[dl] \arrow[dr] & \\
\mathscr{I}^0_{M_0} \arrow[dr, "r^0_{M_0}"'] & & \mathscr{I}^0_K \arrow[dl, "r^0_K"] \\
 & \mathscr{R}(S^2 \times S^1) &
\end{tikzcd}$$

We see from (5.8) that if $(A, \xi) \in \mathscr{I}^0_{M_0}$ and $\alpha = r^0_{M_0}(A, \xi)$ then the corresponding fiber of $\mathscr{I}^0_{M_0} * \mathscr{I}^0_K$ is an $SO(3)$-equivariant $\Gamma_\alpha \cong S^1$-bundle over the 3-manifold

$$\mathscr{I}^0_K \cap (r^0_K)^{-1}(SU(2)/\Gamma_\alpha) = \mathscr{I}^0_K(\alpha).$$

Taking the quotient by $SO(3)$ we get the 1-manifold $(\mathscr{I}^0_{M_0} * \mathscr{I}^0_K)/SO(3) = \mathscr{I}_{M_0} * \mathscr{I}_K$ which is an S^1-fibration over $\mathscr{I}_K \cap r_K^{-1} r_{M_0}(\mathscr{I}_{M_0})$. Note that $\mathscr{I}_{M_0} * \mathscr{I}_K$ is not a fibered product. It is easy to see, using Uhlenbeck's compactness theorem [U], that each $\mathscr{I}_K(\alpha)$ is a compact 0-manifold; thus $\mathscr{I}_{M_0} * \mathscr{I}_K$ consists of a finite number of the isotropy circles Γ_α, $\alpha \in r_M(\mathscr{I}_{M_0})$.

We are going to count the circles in $\mathscr{I}_{M_0} * \mathscr{I}_K$. Our already cumbersome notation will be kept simpler by assuming that for $A_1, A_2 \in \mathscr{I}_{M_0}$, we have $r_{M_0}(A_1) \neq r_{M_0}(A_2)$. Since $\mathscr{I}_{M_0}$ is a finite set, there is no loss in making this assumption. The general case will follow by keeping track of multiplicities.

Proposition 5.9. *For large enough* t, *the image* $\gamma_t(\mathscr{I}_{M_0} * \mathscr{I}_K)$ *is homologous to* $\mathscr{I}_X(t) = \mathscr{M}_{X, \ell+1}(g_t) \cap V_1 \cap \cdots \cap V_d \cap V_x \cap V_y$ *in* $\mathscr{B}^*_{X, \ell+1}$.

Proof. Let $\alpha \in r_{M_0}(\mathscr{I}_{M_0})$ and let U be a small interval in $\boldsymbol{\chi}(S^2 \times S^1) \setminus \{\pm 1\}$ such that $U \cap r_{M_0}(\mathscr{I}_{M_0}) = \{\alpha\}$. Our running assumption is that $r^{-1}_{M_0}(\alpha) = \{A\}$ is a single connection. Let $\mathscr{N}_{M_0} = \mathscr{M}_{M_0, \ell} \cap r^{-1}_{M_0}(U)$ and $\mathscr{N}_K = \mathscr{M}_{K,1} \cap r_K^{-1}(U)$ and form the $SO(3)$-equivariant fibered product:

$$\begin{tikzcd}
 & \mathscr{N}^0 \arrow[dl] \arrow[dr] & \\
\mathscr{N}^0_{M_0} \arrow[dr] & & \mathscr{N}^0_K \arrow[dl] \\
 & U^0 &
\end{tikzcd}$$

For large enough t, Mrówka's theorem 4.3 gives $SO(3)$-equivariant homeomorphisms $\gamma^0_t : \mathscr{N}^0 \to \mathscr{M}^0_{X, \ell+1}(g_t)$ onto open subsets. Now

$$((\mathscr{I}^0_{M_0} * \mathscr{I}^0_K) \cap \mathscr{N}^0)/SO(3) = (\mathscr{I}_{M_0} * \mathscr{I}_K) \cap \mathscr{N}$$

is an S^1-fibration over $\mathscr{I}_K \cap r_K^{-1}(r_{M_0}(\mathscr{I}_{M_0}) \cap U) = \mathscr{I}_K(\alpha)$. If $B \in \mathscr{I}_K(\alpha)$, denote the fiber over B by $\Gamma_\alpha \cdot B$. Thus $\mathscr{N} \cap (\mathscr{I}_{M_0} * \mathscr{I}_K) = \Gamma_\alpha \cdot \mathscr{I}_K(\alpha)$.

By Proposition 4.5 and Theorem 4.6 we have that for large enough t, $\beta_t^0((A, \xi), (B, \eta)) \in V_1 \cap \cdots \cap V_d \cap V_x \cap V_{y,K}$ if and only if $A \in V_1 \cap \cdots \cap V_d$ and $B \in V_x \cap V_y$, that is, if and only if $((A, \xi), (B, \eta)) \in \mathscr{I}_{M_0}^0 * \mathscr{I}_K^0$. Taking the quotient by the action of $SO(3)$ we see that $\beta_t(\mathscr{N}) \cap V_1 \cap \cdots \cap V_d \cap V_x \cap V_y = \beta_t(\Gamma_\alpha \cdot \mathscr{I}_K(\alpha))$ which is homologous to $\gamma_t(\Gamma_\alpha \cdot \mathscr{I}_K(\alpha))$. But again by the definition of γ_t and the fact that intersections with the divisors V_Σ are transverse, the intersection $\beta_t(\mathscr{N}) \cap V_1 \cap \cdots \cap V_d \cap V_x \cap V_y$ is homologous to $\gamma_t(\mathscr{N}) \cap V_1 \cap \cdots \cap V_d \cap V_x \cap V_y = \gamma_t(\mathscr{N}) \cap \mathscr{I}_X(t)$.

To complete the proof, we must show (as in Proposition 5.7) that for large enough t there are no other points of $\mathscr{I}_X(t)$. If there are other points then there is a sequence $A_n \in \mathscr{I}_X(t_n)$, $t_n \to \infty$, which fails to converge strongly to a point $(A, B, h) \in \mathscr{I}_{M_0} \times \mathscr{I}_K \times \Gamma_\alpha$ where $r_{M_0}(A) = r_K(B) = \alpha$. Thus $\{A_n\}$ converges weakly to some $(A', B') \in \mathscr{M}_{M_0, k_1} \times \mathscr{M}_{K, k_2}$ together with instantons at points $x_1, \ldots, x_r \in M_0$ and $y_1, \ldots, y_s \in K$ and with an integral charge $T \geq 0$ lost on the tube $S^2 \times S^1 \times \mathbf{R}$. By assumption $r + s + T > 0$. We now proceed with a counting argument as in Proposition 5.6, but there are a few more complications. First assume that $k_1 > 0$ and $k_2 > 0$. Then we get $2d+8(k_1-\ell) \geq 2(d-2r)$, i.e. $k_1 \geq \ell - \frac{1}{2}r$, and $8k_2 - 3 \geq (1-2s)$, i.e. $k_2 \geq \frac{1}{2} - \frac{1}{4}s$. By a count of charge we have: $\ell + 1 \geq k_1 + k_2 + r + s + T \geq \ell + \frac{1}{2} + \frac{1}{2}r + \frac{3}{4}s + T$, which implies that $s + T = 0$ and $r = 1$. But if $r = 1$, then $k_1 \geq \ell - \frac{1}{2}$; so $k_1 \geq \ell$. Also $s = 0$ implies that $k_2 \geq \frac{1}{2}$; so $k_2 \geq 1$. The charge count now gives $\ell + 1 \geq k_1 + k_2 + r + s + T \geq \ell + 1 + 1$, a contradiction.

In case $k_1 = 0$ and $k_2 > 0$, we have (as in Proposition 5.6) $r \geq 2\ell - \frac{3}{4}(1 + b_M^+)$ and $k_2 \geq \frac{1}{2} - \frac{1}{4}s$. Now the basic charge count yields $\ell + 1 \geq (\frac{1}{2} - \frac{1}{4}s) + (2\ell - \frac{3}{4}(1 + b_M^+)) + s + T$. Thus $\ell \leq \frac{3}{4}(1 + b_M^+) + \frac{1}{2} - (\frac{3}{4}s + T)$. However our basic assumption on ℓ which guarantees the existence of the Donaldson invariant $q_{\ell, M}$ is that $\ell > \frac{3}{4}(1 + b_M^+)$. Combining this with the above inequality for ℓ we have $0 < \frac{1}{2} - (\frac{3}{4}s + T)$. Thus $s = T = 0$, and so $k_2 \geq \frac{1}{2}$; so in fact $k_2 \geq 1$. Recalculate the basic charge count: $\ell + 1 \geq 1 + r \geq 1 + 2\ell - \frac{3}{4}(1 + b_M^+)$, which implies that $\ell \leq \frac{3}{4}(1 + b_M^+)$, a contradiction.

Next consider the case $k_1 > 0$, $k_2 = 0$. Then $k_1 \geq \ell - \frac{1}{2}r$ and $s \geq 1$. We get $\ell + 1 \geq k_1 + r + s + T \geq \ell + \frac{1}{2}r + 1 + T$; so $r = T = 0$. But then $A' \in \mathscr{I}_{M_0}$, and $B' = \Theta$, and since $T = 0$, $r(A') = r(B') = 1$, which contradicts Proposition 5.5.

Finally suppose that $k_1 = k_2 = 0$. Then $r \geq 2\ell - \frac{3}{4}(1 + b_M^+)$ and $s \geq 1$. The charge count gives $\ell + 1 \geq 2\ell - \frac{3}{4}(1 + b_M^+) + 1 + T$; so $\ell \leq \frac{3}{4}(1 + b_M^+) - T$, which contradicts the basic assumption on ℓ. □

Recall our standing hypothesis that r_{M_0} is 1-1 when restricted to $\mathscr{I}_{M_0}$.

Proposition 5.10. $q_{\ell+1, u_1, X}(z_1, \ldots, z_d, x, y) \equiv \sum\{\#\mathscr{I}_K(\alpha) \mid \alpha \in r_{M_0}(\mathscr{I}_{M_0})\}$ mod 2.

Proof. Consider the isotropy circles $\gamma_t(\Gamma_\alpha \cdot B)$. Referring to (5.8) we first study the family of connections $SO(3) \cdot A$. Since M_0 is spin, the Dirac operator gives us a family of real operators over $SO(3)\cdot A$. The index bundle of this family can be pulled back over $SU(2)$ where it becomes trivial. Now $-1\in SU(2)$ acts nontrivially on spinors, and the index bundle of the Dirac family $\mathrm{Ind}_{\mathbf{R}}(\not{D}, SO(3)\cdot A)$ is the quotient of this trivial bundle. (See [D1].) It follows that as a class in $KO(SO(3))$ the index bundle is $m_1\cdot\eta_1$ where $m_1 = 2\,\mathrm{ind}(\not{D}_{M_0})+\ell$ is the numerical index and η_1 is the class of the Hopf (real) line bundle. Similarly over $SO(3)\cdot B$ we obtain $m_2\cdot\eta_2$, where $m_2 = 2\,\mathrm{ind}(\not{D}_K)+1$.

If $A\in\mathcal{I}^0_{M_0}$ and $B\in\mathcal{I}^0_K$, the connected component of the fibered product $\mathcal{I}^0_{M_0} * \mathcal{I}^0_K$ obtained from (5.8) is $S^1\times SO(3)$. Now fix $\alpha\in r^0_{M_0}(SO(3)\cdot A) = r^0_K(SO(3)\cdot B)$. Then, if $A'\in(r^0_{M_0})^{-1}(\alpha)\cap SO(3)\cdot A$, we have $(r^0_{M_0})^{-1}(\alpha)\cap SO(3)\cdot A = \Gamma_\alpha\cdot A'\cong S^1$ and similarly $(r^0_K)^{-1}(\alpha)\cap SO(3)\cdot B = \Gamma_\alpha\cdot B'\cong S^1$. (Note that $\Gamma_\alpha\cdot B'\subset SO(3)\cdot B$ is not to be confused with $\Gamma_\alpha\cdot B\subset\mathcal{N}$.) The pullback in $S^1\times SO(3)$ of $\Gamma_\alpha\cdot A'$ and $\Gamma_\alpha\cdot B'$ is a copy of $S^1\times S^1$. Under the quotient $\mathcal{N}^0\to\mathcal{N}$ this $S^1\times S^1$ family projects to $\Gamma_\alpha\cdot B\subset\mathcal{I}_{M_0} * \mathcal{I}_K\subset\mathcal{N}$.

We compute $\mathrm{Ind}_{\mathbf{R}}(\not{D}, \gamma_t(S^1\times S^1))$ by using the excision property for indices. Pull the connection B' back over $\Gamma_\alpha\cdot A'$ to get an S^1 family of connections in $\mathcal{N}^0$. Then pull back over $S^1\times S^1$; so we may think of this as an $S^1\times S^1$ family, $\mathcal{F}_{M_0}$ of connections (constant in one direction). The excision property implies that $\mathrm{Ind}_{\mathbf{R}}(\not{D}, S^1\times S^1) - \mathrm{Ind}_{\mathbf{R}}(\not{D}, \mathcal{F}_{M_0})$ is the pullback of

$$\mathrm{Ind}_{\mathbf{R}}(\not{D}, \Gamma_\alpha\cdot B') - \mathrm{Ind}_{\mathbf{R}}(\not{D}, B') = m_2\cdot\eta_2 - m_2\cdot 1,$$

where $(\not{D}, B')$ denotes the Dirac operator on K_+ twisted over a constant S^1-family of connections. Similarly,

$$\begin{aligned}\mathrm{Ind}_{\mathbf{R}}(\not{D}, \mathcal{F}_{M_0}) &= (\mathrm{Ind}_{\mathbf{R}}(\not{D}, \Gamma_\alpha\cdot A') - \mathrm{Ind}_{\mathbf{R}}(\not{D}, A')) + \mathrm{Ind}_{\mathbf{R}}(\not{D}, (A', B'))\\ &= m_1\cdot\eta_1 - m_1\cdot 1 + (m_1+m_2)\cdot 1 = m_1\cdot\eta_1 + m_2\cdot 1.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{Ind}_{\mathbf{R}}(\not{D}, \gamma_t(S^1\times S^1)) &= (\mathrm{Ind}_{\mathbf{R}}(\not{D}, \Gamma_\alpha\cdot B') - \mathrm{Ind}_{\mathbf{R}}(\not{D}, B')) + \mathrm{Ind}_{\mathbf{R}}(\not{D}, \mathcal{F}_{M_0})\\ &= m_1\cdot\eta_1 + m_2\cdot\eta_2.\end{aligned}$$

We get a transversal of $S^1\times S^1\to\Gamma_\alpha\cdot B\subset\mathcal{N}$ by fixing the first factor. Over this transversal we have the index bundle $m_2\cdot\eta_2 + m_1\cdot 1$. Restricted to the family $\Gamma_\alpha\cdot B$ the determinant is $\det(m_2\cdot\eta_2)=\eta_2$ since m_2 is odd. For large t the restriction of u_1 to the family $\gamma_t(\Gamma_\alpha\cdot B)$ is the first Stiefel-Whitney class of the restricted index bundle; i.e. over $\Gamma_\alpha\cdot B$, we have $u_1 = w_1(\eta_2)\neq 0$, since Γ_α is an essential circle in $SO(3)$; so the Hopf bundle is twisted over it. This means that the isotropy circles $\gamma_t(\Gamma_\alpha\cdot B)$ represent the nontrivial element of $\pi_1(\mathcal{B}_{X,\ell+1})$.

Thus $q_{\ell+1,u_1,X}(z_1,\dots,z_d,x,y)$ is the mod 2 count of these circles in $\mathcal{I}_{M_0} * \mathcal{I}_K$, and this is given above. □

We need now to make one final appeal to Mrówka's thesis. We have $r_K^0 : \mathscr{M}_{K,1}^0 \to \mathscr{R}(S^2 \times S^1) \cong S^3$ which according to Theorem 3.1(iv) is an $SO(3)$-equivariant generic map. Taking the quotient by $SO(3)$, this means that $r_K : r_K^{-1}(-1, 1) \cap \mathscr{M}_{K,1} \to (-1, 1)$ is transverse to subcomplexes. In particular, consider $\alpha_0 < \alpha_1 \in (-1, 1)$. For any $\alpha \in (-1, 1)$, the 4-manifold $\mathscr{M}_{K,1}^0(\alpha)$ is compact since such an α cannot be the restriction of a flat connection on K ($\cong S^2 \times D^2$). The 1-manifold $\mathscr{I}_K \cap r_K^{-1}(\alpha_0, \alpha_1) = r_K^{-1}(\alpha_0, \alpha_1) \cap \mathscr{M}_{K,1}^0 \cap V_x \cap V_{y,K}$ gives a cobordism of the compact 0-manifolds $\mathscr{I}_K(\alpha_0)$ and $\mathscr{I}_K(\alpha_1)$. Thus the mod 2 intersection number $\#\mathscr{I}_K(\alpha)$ is independent of $\alpha \neq \pm 1$.

Similarly, given a generic 1-parameter family of asymptotically cylindrical metrics $\{g_{K,t} \mid t \in [0, 1]\}$ on K_+, the family $\{\mathscr{I}_{K,g_t}(\alpha)\}$ gives a homology between $\mathscr{I}_{K,g_0}(\alpha)$ and $\mathscr{I}_{K,g_1}(\alpha)$ when $\alpha \neq \pm 1$. Thus the mod 2 intersection number $N_K = \#\mathscr{I}_K(\alpha)$ is independent of $\alpha \neq \pm 1$ and choice of generic metric.

Theorem 5.11. $q_{\ell+1, u_1, x}(z_1, \dots, z_d, x, y) \equiv N_K \cdot q_{\ell, M}(z_1, \dots, z_d) \bmod 2$.

Proof. Since we have avoided multiplicity questions by assuming that $r_{M_0} \mid \mathscr{I}_{M_0}$ is 1-1, we have by Proposition 5.7 that $q_{\ell, M} \equiv \#r_{M_0}(\mathscr{I}_{M_0}) \bmod 2$. Also $\#\mathscr{I}_K(\alpha) = N_K$ for all $\alpha \neq \pm 1$; so the result follows from Proposition 5.10. If we remove the assumption on $r_{M_0} \mid \mathscr{I}_{M_0}$, we get the same result by keeping track of multiplicities. □

6. Calculation of N_K

In this section we shall calculate N_K by studying a specific example. Let Q denote the negative definite E_8 plumbing manifold. Its boundary $Y = \partial Q$ is the Poincaré homology 3-sphere $\Sigma(2, 3, 5)$ with its negative orientation. Let W be the result of performing surgery on any circle in the interior of Q. Since Q is simply connected, all such surgeries are trivial. Then W_+ is diffeomorphic to $Q_+ \# S^2 \times S^2$ and its intersection form is $-E_8 \oplus H$, where H is the hyperbolic form.

Consider the moduli space of charge 2, asymptotically trivial self-dual connections $\mathscr{M}_{W,2}(\vartheta)$. (Since Y is a Brieskorn homology 3-sphere, it follows from §3 that there is a single small δ such that we can base all our moduli spaces on connections with exponential δ-decay.) The moduli space $\mathscr{M}_{W,2}(\vartheta)$ is nontrivial [T1,2] and is a 10-dimensional manifold. Choose a pair of generic oriented surfaces Σ_1, Σ_2 in Q, representing classes z_1, z_2 with $z_1^2 = z_2^2 = -2$ and $z_1 \cdot z_2 = 1$, and let Σ_x, Σ_y be as in §5. We let $\mathscr{N}^2(\vartheta) = \mathscr{M}_{W,2}(\vartheta) \cap V_1 \cap V_2 \cap V_x \cap V_y$. As in the proof of Donaldson's Theorem B (see [D1] and [FS2]), $\mathscr{N}^2(\vartheta)$ is a 2-manifold with "internal" ends arising from instantons which bubble off at pairs of points of intersection of $\Sigma_1, \Sigma_2, \Sigma_x$, and Σ_y. In fact, Donaldson shows that there is exactly one such end of $\mathscr{N}^2(\vartheta)$ arising from each such intersection. Our choice of surfaces then gives us an odd number of these ends. Donaldson shows, furthermore, that each such end is a circle on which the class $u_1 \in H^1(\mathscr{B}_W^*; \mathbf{Z}_2)$ evaluates nontrivially. Thus, as in [FS2], there are an odd number of "asymptotic ends" of $\mathcal{N}^2(\vartheta)$ on which u_1 evaluates nontrivially. Dimension counting shows that the only possibility is for such ends to arise from nontrivial splittings of the form

$$\mathcal{M}_W(\rho) \times \mathcal{M}_Y(\rho, \vartheta) \to \mathcal{M}_{W,2}(\vartheta)$$

where $\rho \in \mathcal{R}(Y)$ and $\mathcal{M}_Y(\rho, \vartheta)$ is a moduli space of anti-self-dual connections on $Y \times \mathbf{R}$ with asymptotic conditions ρ at $-\infty$ and ϑ at $+\infty$. Furthermore $V_1 \cap V_2 \cap V_x \cap V_y$ intersects $\mathcal{M}_W(\rho)$ transversely and nontrivially; so $\dim \mathcal{M}_W(\rho) \geq 8$, and thus $0 \leq \dim \mathcal{M}_Y(\rho, \vartheta) \leq 2$. Furthermore, since the character variety $\boldsymbol{\chi}(Y)$ consists of isolated points, the only 0-dimensional anti-self-dual moduli space $\mathcal{M}_Y(\rho, \vartheta)$ consists of the singleton ϑ. The next proposition follows directly from computations in [FS1].

Proposition 6.1. *Let $Y = \Sigma(2, 3, 5)$ with its negative orientation. Up to conjugacy there are two nontrivial representations $\xi, \omega : \pi_1(Y) \to SU(2)$. The mod 8 dimensions of the corresponding moduli spaces of anti-self-dual connections on $\pm Y \times \mathbf{R}$ are $\dim \mathcal{M}_Y(\xi, \vartheta) \equiv 1$, $\dim \mathcal{M}_Y(\omega, \vartheta) \equiv 5$, $\dim \mathcal{M}_{-Y}(\xi, \vartheta) \equiv 4$, $\dim \mathcal{M}_{-Y}(\omega, \vartheta) \equiv 0$.* $\square$

It follows from this proposition that each asymptotic end of $\mathcal{N}^2(\vartheta)$ comes from a splitting $\mathcal{M}_W(\xi) \times \mathcal{M}_Y(\xi, \vartheta)$ where $\dim \mathcal{M}_Y(\xi, \vartheta) = 1$ and $\dim \mathcal{M}_W(\xi) = 9$. Since $\mathcal{M}_Y(\xi, \vartheta)$ is 1-dimensional, translational invariance of the anti-self-duality equation in the temporal gauge [F] implies that $\mathcal{M}_Y(\xi, \vartheta)$ is a disjoint union of copies of $\mathbf{R}$.

Viewed in the language of Theorem 4.3, given a family $\{g_t\}$ of generic asymptotically cylindrical metrics on W_+ which stretch a given segment $Y \times [-1, 1]$ of the end of W_+ to infinite length, we get for $t \geq$ some t_0, embeddings

$$\gamma_t : \mathcal{M}_W(\xi) \times \mathcal{M}_Y(\xi, \vartheta) \to \mathcal{M}_{W,2}(\vartheta; g_t),$$

and $\gamma_t(A, B) \to (A, B) \in \mathcal{M}_W(\xi) \times \mathcal{M}_Y(\xi, \vartheta)$ as $t \to \infty$. Now $V_1 \cap V_2 \cap V_x \cap V_y \cap (\mathcal{M}_W(\xi) \times \mathcal{M}_Y(\xi, \vartheta)) = \{V_1 \cap V_2 \cap V_x \cap V_y \cap \mathcal{M}_W(\xi)\} \times \mathcal{M}_Y(\xi, \vartheta) = N^1(\xi) \times \mathcal{M}_Y(\xi, \vartheta)$ for a 1-manifold $N^1(\xi)$. The standard dimension counting argument shows that $N^1(\xi)$ is compact.

Let $\widehat{\mathcal{M}_Y}(\xi, \vartheta)$ be a transversal to the $\mathbf{R}$-action on $\mathcal{M}_Y(\xi, \vartheta)$. Set $T_t = \gamma_t(N^1(\xi) \times \widehat{\mathcal{M}_Y}(\xi, \vartheta))$. The argument above using the proof of [D1, Theorem B] shows that $u_1[T_t] \neq 0$. Recall from §2 that $u_1 = w_1(\det \operatorname{Ind}_{\mathbf{R}} \not{D}_A)$. It follows that for any $A_1, A_2 \in \widehat{\mathcal{M}_Y}(\xi, \vartheta)$, $u_1[\gamma_t(N^1(\xi) \times A_1)] = u_1[\gamma_t(N^1(\xi) \times A_2)]$; so $\gamma_t^*(u_1) \in H^1(N^1(\xi) \times \widehat{\mathcal{M}_Y}(\xi, \vartheta); \mathbf{Z}_2) = \bigoplus H^1(N^1(\xi); \mathbf{Z}_2)$ is a diagonal class which restricts to the class $u \in H^1(N^1(\xi); \mathbf{Z}_2)$ in each component. The class u is obviously independent of t. We have

$$u_1[T_t] = \gamma_t^*(u_1)[N^1(\xi) \times \widehat{\mathcal{M}_Y}(\xi, \vartheta)] = \#\widehat{\mathcal{M}_Y}(\xi, \vartheta) \cdot u[N^1(\xi)].$$

Hence $[N^1(\xi)] \neq 0$ in $H_1(\mathcal{M}_W(\xi); \mathbf{Z}_2)$ since u evaluates nontrivially on it.

Suppose $N^1(\xi)$ was the boundary of $\mathbf{Z}_2$-chain C of $\mathcal{B}_W^*(\xi)$. Consider the map $\beta_t : \mathcal{M}_W(\xi) \times \mathcal{M}_Y(\xi, \vartheta) \to \mathcal{B}_{W,2}^*(\vartheta)$ described in §4. Its image is approximately anti-self-dual. The map β_t is deformed into γ_t; so in particular $\beta_t(N^1(\xi)\times\widehat{\mathscr{M}_Y}(\xi, \vartheta))$ is homologous to $\gamma_t(N^1(\xi)\times\widehat{\mathscr{M}_Y}(\xi, \vartheta))$ in $\mathscr{B}^*_{W,2}(\vartheta)$. The map β_t is constructed by a process involving truncation and matching overlaps. (See [Mr] and [F].) Thus β_t extends over all of $C \times \widehat{\mathscr{M}_Y}(\xi, \vartheta)$. This would imply that $T_t = \gamma_t(N^1(\xi) \times \widehat{\mathscr{M}_Y}(\xi, \vartheta))$ is nullhomologous, i.e. that $u_1[T_t] = 0$, a contradiction. Thus $[N^1(\xi)]$ is nonzero in $H_1(\mathscr{B}^*_W(\xi); \mathbf{Z}_2)$.

Lemma 6.2. $\pi_1(\mathscr{B}^*_W(\xi)) = \mathbf{Z}_2$.

Proof. We just apply the proof of Proposition 2.1 to our asymptotically flat end situation. The gauge group $\mathscr{G}_\delta$ consists of maps $g : X_+ \to SU(2)$ in $L^4_{1,\mathrm{loc}}$ such that $\| \nabla_\theta g \|_{\delta,3} < \infty$. Note that this group does not depend on the asymptotic condition ξ. Let $\widehat{\mathscr{G}} = \mathscr{G}/\mathbf{Z}_2$, so that $\pi_1(\mathscr{B}^*_W(\xi)) \cong \pi_0(\widehat{\mathscr{G}})$. Now $\pi_0(\mathscr{G}_\delta) = [(W_+, \partial), (S^3, 1)]$, which, since W_+ is spin, is the group $\mathbf{Z}_2$. One can now show that when one deforms the constant map $W_+ \to S^3$ with value -1 so that it sends ∂W_+ to 1, then the resulting map is homotopic rel boundary to the map to $1 \in S^3$. (Use the great circle connecting 1 to -1.) The result follows. □

Corollary 6.3. *The class $u \in H^1(N^1(\xi); \mathbf{Z}_2)$ is the restriction of the nontrivial element of $H^1(\mathscr{B}^*_W(\xi); \mathbf{Z}_2)$.* □

Next consider the moduli space $\mathscr{M}_{Q,1}(\vartheta)$. Again it follows from Taubes [T2] that $\mathscr{M}_{Q,1}(\vartheta)$ is a nonempty 5-manifold.

Proposition 6.4. *There is a 4-dimensional moduli space $\mathscr{M}_Q(\xi)$ which gives rise to a degree 2 relative Donaldson invariant, and $\#(\mathscr{M}_Q(\xi) \cap V_1 \cap V_2) \equiv 1 \bmod 2$.*

Proof. First notice that if we can find a 4-dimensional moduli space $\mathscr{M}_Q(\xi)$, then for any classes $a, b \in H_2(Q; \mathbf{Z})$ with corresponding divisors V_a and V_b in $\mathscr{B}^*_Q(\xi)$, the intersection $\mathscr{M}_Q(\xi) \cap V_a \cap V_b$ is compact—for there is not enough charge for an instanton bubble to occur (any moduli space left would have nonnegative dimension), and Proposition 6.1 implies that there is no splitting of the form

$$(\mathscr{M}_Q(\rho) \cap V_a \cap V_b) \times \mathscr{M}_Y(\rho, \xi) \to \mathscr{M}_Q(\xi) \cap V_a \cap V_b$$

for any ρ. Thus we would have a degree 2 Donaldson invariant.

The intersection $I = \mathscr{M}_{Q,1}(\vartheta) \cap V_1 \cap V_2$ is a 1-manifold in $\mathscr{B}^*_Q(\vartheta)$ which can have asymptotic ends as described above, ends coming from instanton bubbles occurring at points of intersection of Σ_1 and Σ_2 (there are an odd number of these by the choice of Σ_1 and Σ_2), and reducible connections. It follows from [D1] that the number of ends of $\mathscr{M}_{Q,1}(\vartheta)$ which are reducible is

$$\frac{1}{2}\sum\{(z_1 \cdot e)(z_2 \cdot e) \mid e \in H_2(Q_+; \mathbf{Z}), e^2 = -1\} = 0$$

since $H^2(Q_+; \mathbf{Z}) = -E_8$ is an even form. Now an argument similar to the one given above shows that there are an odd number of asymptotic ends of I

coming from splittings

$$(\mathcal{M}_Q(\xi) \cap V_1 \cap V_2) \times \mathcal{M}_Y(\xi, \vartheta) \quad \text{mod } 2.$$

Furthermore, there are no other asymptotic ends of I. Thus Proposition 6.1 implies that $\mathcal{M}_Q(\xi)$ is 4-dimensional, and it follows that $\#(\mathcal{M}_Q(\xi) \cap V_1 \cap V_2)$ is odd. □

Notice that this degree 2 relative Donaldson invariant $q_{Q,\xi}$ takes its values in the 1-dimensional Floer homology group generated by the class of ξ. Thus we can identify the values of $q_{Q,\xi}$ with the integers, and then $q_{Q,\xi}(z_1, z_2) \equiv 1$ mod 2 for the classes z_1, z_2 chosen above.

Digression 6.5. The basic technique of the proof of Proposition 6.4 can be used to prove the following "folk theorem".

> Let X be a compact simply connected negative definite 4-manifold with homology sphere boundary Y. Suppose that the intersection form of X is not diagonalizable over $\mathbf{Z}$. Then the 1-dimensional Floer instanton homology group $I_1(Y) \neq 0$.

The proof is basically given above. Counting ends provides a nonzero (in fact, odd) pairing of the Floer 1-cycle of Y given by the degree 2 relative Donaldson invariant with the Floer 1-cocycle (which is a Floer cycle of $-Y$) given by $\sum(\#\widehat{\mathcal{M}}^1_Y(\rho, \vartheta)) \cdot \rho$ where $\mathcal{M}^1_Y(\rho, \vartheta)$ consists of the 1-dimensional components of $\mathcal{M}_Y(\rho, \vartheta)$ and $\widehat{\mathcal{M}}^1_Y(\rho, \vartheta) = \mathcal{M}_Y(\rho, \vartheta)/\mathbf{R}$. A simple algebraic exercise shows that for a nondiagonal negative definite intersection form there always is a pair of classes z_1, z_2 such that $z_1 \cdot z_2 - \frac{1}{2}\sum\{(e \cdot z_1)(e \cdot z_2) \mid e^2 = -1\} \not\equiv 0$ mod 2. Thus both the Floer homology class given by the Donaldson invariant and the Floer cohomology class of the cocycle are nontrivial.

Theorem 6.6. $N_K \equiv 1$ mod 2.

Proof. We apply the arguments of §5 to $W_+ = Q_{+,0} \cup S^2 \times D^2 = Q_{+,0} \cup K$ where $Q_{+,0} = Q_+ \setminus (S^1 \times \mathbf{R}^3)$, and $S^1 \times \mathbf{R}^3$ is a tubular neighborhood of the circle on which surgery is performed to construct W. The proof of Theorem 5.11 applies to $V_1 \cap V_2 \cap V_x \cap V_y \cap \mathcal{M}_W(\xi) = N^1(\xi)$ without change because, by a dimension counting argument as above, no sequence of connections in $\mathcal{M}_W(\xi) \cap V_1 \cap V_2 \cap V_x \cap V_y$ can lose charge on the tube $Y \times \mathbf{R}^+$. Since $[N^1(\xi)] \neq 0$, we get $1 \equiv \#(\mathcal{M}_Q(\xi) \cap V_1 \cap V_2) \cdot N_K \equiv N_K$ mod 2. □

This together with Theorem 5.11 completes the proof of Theorem 1.1.

7. The u_2-invariant

In this section we shall study the invariant $q_{k,u_2,X}$ and prove Theorem 1.3. We shall rely heavily on the work already done in §5 and §6. Let N be a closed simply connected spin 4-manifold with b_N^+ even and ≥ 2 and let k be an even integer, $k > \frac{3}{4}(1 + b_N^+) + \frac{1}{4}$; so we have a $\mathbf{Z}_2$-polynomial invariant $q_{k,u_1,N}$ of degree $d = 4k - \frac{3}{2}(1 + b_N^+) - \frac{1}{2}$. Let $X = N\#S^2 \times S^2$ which is spin and $k + 1$

$> \frac{3}{4}(1 + b_X^+) + \frac{1}{2}$ is odd; so the $\mathbf{Z}_2$-polynomial invariant $q_{k+1, u_2, X}$ of degree $d+2$ is also defined. Let $z_1, \ldots, z_d \in H_2(N; \mathbf{Z}) \subset H_2(X; \mathbf{Z})$ be represented by generic oriented surfaces $\Sigma_1, \ldots, \Sigma_d$ as earlier and let $x = [S^2 \times 0]$ and $y = [0 \times S^2]$ with representative surfaces Σ_x and Σ_y as before. We must evaluate $q_{k+1, u_2, X}(z_1, \ldots, z_d, x, y)$.

Write $X = N_0 \cup_{S^2 \times S^1} K$ where N_0 is N with the tubular neighborhood of a circle removed and $K \cong S^2 \times D^2$. We adopt the notation and follow the arguments of §5.

Proposition 7.1. *The formal dimension of $\mathscr{M}_{N_0, m}(\alpha)$ is*

$$\dim \mathscr{M}_{N_0, m}(\alpha) = \begin{cases} 2d + 8(m-k), & \alpha \neq \pm 1, \\ 2d + 8(m-k) - 2, & \alpha = \pm 1. \end{cases}$$

Corollary 7.2. *If $2d + 8(m-k) \geq 0$ the moduli space $\mathscr{M}_{N_0, m}$ (with respect to the generic metric g_{N_0}) is a manifold whose formal dimension is $2d + 1 + 8(m-k)$.* □

Next consider the 1-dimensional intersection $\mathscr{I}_{N_0} = V_1 \cap \cdots \cap V_d \cap \mathscr{M}_{N_0, k}$.

Proposition 7.3. *If $A \in \mathscr{I}_{N_0}$ then $r_{N_0}(A) \neq \pm 1$.*

Proof. The subcomplex $r_{N_0}^{-1}(\pm 1)$ of $\mathscr{M}_{N_0, k}$ has dimension $2d-2$ and is met transversely by the codimension $2d$ submanifold $V_1 \cap \cdots \cap V_d$. □

Proposition 7.4. *The 1-dimensional intersection $\mathscr{I}_{N_0}$ is a compact manifold.*

Proof. This follows from Uhlenbeck weak compactness and from a counting argument as in Proposition 5.6. □

The cohomology class $u_1 \in H^1(\mathscr{B}_{N,k}; \mathbf{Z}_2)$ restricts to a class, which we shall still call u_1 in $H^1(\mathscr{B}_{N_0, k}; \mathbf{Z}_2) \cong \mathbf{Z}_2$ (cf. Lemma 6.2).

Proposition 7.5. *The intersection $\mathscr{I}_{N_0}$ consists of a finite number of circles; modulo 2 the number of these circles which are nontrivial in $\pi_1(\mathscr{B}^*_{N_0, k}) \cong \mathbf{Z}_2$ is $q_{k, u_1, N}(z_1, \ldots, z_d)$.*

Proof. As in the proof of Proposition 5.7, for large t we get maps: $\mathscr{N} \to \mathscr{M}_{N,k}(g_t)$ which are homeomorphisms onto their image, and $\mathscr{I}_{N_0} \subset \mathscr{N}$. If I is a component (a circle) of $\mathscr{I}_{N_0}$ then as $t \to \infty$, $\{\gamma_t(I)\}$ converges to $\{[A, r_{N_0}(A)] | A \in I\} \subset \mathscr{M}_{N,k} \times \boldsymbol{\chi}(\mathbf{R}^3 \times S^1)$. But I is contained in $V_1 \cap \cdots \cap V_d$ which intersects $\mathscr{M}_{N,k}(g_t)$ transversely, and so by Theorem 4.3(a) for large enough t there is a unique circle of $\mathscr{M}_{N,k}(g_t) \cap V_1 \cap \cdots \cap V_d$ close to $\gamma_t(I)$. Then as in Proposition 5.7 a counting argument shows that these circles comprise all of $\mathscr{I}_N(t) = \mathscr{M}_{N,k}(g_t) \cap V_1 \cap \cdots \cap V_d$. The invariant $q_{k, u_1, N}(z_1, \ldots, z_d)$ counts (modulo 2) the number of homotopically nontrivial components of $\mathscr{I}_N(t)$, but γ_t induces an isomorphism on π_1; so the proposition follows. □

Let $\mathcal{J}_X(t)$ denote $\mathcal{M}_{X,k}(g_t) \cap V_1 \cap \dots V_d \cap V_x \cap V_y$. The technique that proves Proposition 5.9 then gives

Proposition 7.6. *For large enough t, the image $\gamma_t(\mathcal{J}_{N_0} * \mathcal{J}_K)$ is homologous to $\mathcal{J}_X(t)$ in $\mathcal{B}^*_{X,k+1}$.* □

Lemma 7.7. *The inclusion induced homomorphism $\pi_1(\mathcal{M}_{K,1} \backslash \{\pm 1\}) \to \pi_1(\mathcal{B}^*_{K,1})$ is trivial.*

Proof. Let $\mathcal{N}_K = \mathcal{M}_{K,1} \setminus \{\pm 1\}$. Viewing S^4 as $K \cup D^3 \times S^1$, and referring again to Proposition 5.7, we have the pullback diagram

$$\begin{tikzcd} & \mathcal{N}^0 \arrow[dl] \arrow[dr] & \\ \mathcal{N}_K^0 \arrow[dr] & & \mathcal{R}(\mathbf{R}^3 \times S^1) \setminus \{\pm 1\} \arrow[dl] \\ & \mathcal{R}(S^2 \times S^1) \setminus \{\pm 1\} & \end{tikzcd}$$

and $\mathcal{N}^0 \to \mathcal{N}_K^0$ is a homeomorphism. Recall the embedding $\beta_t : \mathcal{N}_K \cong \mathcal{N} \to \mathcal{B}^*_{S^4,1}$, given by $\beta_t(A) = A \# r_K(A)$ over $K \cup (\mathbf{R}^3 \times S^1) = S^4$. For large enough t this map is homotopic to the embedding $\gamma_t : \mathcal{N}_K \cong \mathcal{N} \to \mathcal{M}_{S^4,1}(g_t)$. Let $r'_K : \mathcal{B}^*_{S^4,1} \to \mathcal{B}^*_{K,1}$ be the restriction. Then $r'_K \circ \beta_t$ is homotopic to the inclusion of $\mathcal{N}_K$ in $B^*_{K,1}$. But $\pi_1(\mathcal{B}^*_{S^4,1})$ is trivial. □

Before proceeding further we need to give a description of the cohomology class $u_2 \in H^2(\mathcal{B}^*_{X,k+1}; \mathbf{Z}_2)$ of [D1]. Since X is spin there is a family of real elliptic operators $\{\not{D}_A | A \in \mathcal{A}^*_{X,k+1}\}$ given by coupling the Dirac operator on X to connections. This family descends to $\mathcal{B}^{0,*}_{X,k+1}$; so for any compact subset T^0 of $\mathcal{B}^{0,*}_{X,k+1}$ we obtain $\mathrm{Ind}_{\mathbf{R}}(\not{D}, \{A\}) \in KO(T^0)$. This class does not descend directly to $\mathcal{B}^*_{X,k+1}$; however Donaldson shows that $\mathrm{Ind}_{\mathbf{R}}(\not{D}, \{A\}) \otimes \det(\mathrm{Ind}_{\mathbf{R}}(\not{D}, \{A\}))$ descends to $\mathcal{B}^*_{X,k+1}$ (since $k+1$ is odd). The class u_2 is defined to be $u_2 = w_2((\mathrm{Ind}_{\mathbf{R}}(\not{D}, \{A\}) \otimes \det(\mathrm{Ind}_{\mathbf{R}}(\not{D}, \{A\})))$.

Next we need to set the stage for the proof of Theorem 1.3. First we trivialize the projection

$$\mathcal{R}(S^2 \times S^1) \backslash \{\pm 1\} = SU(2) \backslash \{\pm 1\} = (-1, 1) \times S^2 \to \boldsymbol{\chi}(S^2 \times S^1) \backslash \{\pm 1\} = (-1, 1).$$

Then we can trivialize the projections $\mathcal{J}^0_{N_0} \to \mathcal{J}_{N_0}$ and $\mathcal{J}^0_K \to \mathcal{J}_K$ so that $\mathcal{J}^0_{N_0} = \mathcal{J}_{N_0} \times SO(3)$ and $r^0_{N_0} = (r_N, \pi)$, where $\pi : SO(3) \to S^2$ is the projection of the circle bundle, and similarly $\mathcal{J}^0_K = \mathcal{J}_K \times SO(3)$ where $r^0_K = (r_K, \pi)$. Now let $\mathcal{J}_{N_0} \odot \mathcal{J}_K$ denote the actual pullback of

$$\begin{tikzcd} \mathcal{J}_{N_0} \arrow[dr] & & \mathcal{J}_K \arrow[dl] \\ & (-1, 1) & \end{tikzcd}$$

(Compare with $\mathscr{I}_{N_0} * \mathscr{I}_K$ which is the quotient of the pullback $\mathscr{I}^0_{N_0} * \mathscr{I}^0_K$ by the action of $SO(3)$.) Working with based moduli spaces we have that the pullback

$$\begin{aligned}\mathscr{I}^0_{N_0} * \mathscr{I}^0_K &= \{((A, \xi), (B, \eta)) | r_{N_0}(A) = r_K(B), \pi(\xi) = \pi(\eta)\} \\ &= (\mathscr{I}_{N_0} \odot \mathscr{I}_K) \times (S^1 \times SO(3))\end{aligned}$$

where $S^1 \times SO(3)$ is the pullback of diagram (5.8).

Now $q_{k+1, u_2, X}(z_1, \dots, z_d, x, y)$ is given by evaluating u_2 on the class of the 2-cycle $\mathscr{I}_X(t) \in H_2(\mathscr{B}^*_{X, k+1}; \mathbf{Z}_2)$. So by Proposition 7.6 we need to compute the value of u_2 on $\gamma_t(\mathscr{I}_{N_0} * \mathscr{I}_K)$. This will be done as in Proposition 5.10 by using the excision property for indices. Fixing a basepoint $0 \in S^2$, we have $(-1, 1) \times \{0\}$ contained in $SU(2) \setminus \{\pm 1\}$ as a transversal to the projection to $(-1, 1)$ and there is the pullback diagram

$$\begin{tikzcd} & (\mathscr{I}_{N_0} \times S^1) * (\mathscr{I}_K \times S^1) \arrow[dl] \arrow[dr] & \\ \mathscr{I}_{N_0} \times S^1 \arrow[dr] & & \mathscr{I}_K \times S^1 \arrow[dl] \\ & (-1, 1) \times \{0\} & \end{tikzcd}$$

Under the projection map from based equivalence classes to ordinary equivalence classes of connections, the pullback $(\mathscr{I}_{N_0} \times S^1) * (\mathscr{I}_K \times S^1) = (\mathscr{I}_{N_0} \odot \mathscr{I}_K) \times (S^1 \times S^1)$ maps onto $\mathscr{I}_{N_0} * \mathscr{I}_K$.

Consider the family of real elliptic operators $(\not{D}, (\mathscr{I}_{N_0} \times S^1) * (\mathscr{I}_K \times \{1\}))$ which we may view as a family over $(\mathscr{I}_{N_0} \times S^1) * (\mathscr{I}_K \times S^1)$ which is constant in the direction of the final S^1. By the excision principle we have

$$\begin{aligned}&\mathrm{Ind}_{\mathbf{R}}(\not{D}, (\mathscr{I}_{N_0} \times S^1) * (\mathscr{I}_K \times S^1)) - \mathrm{Ind}_{\mathbf{R}}(\not{D}, (\mathscr{I}_{N_0} \times S^1) * (\mathscr{I}_K \times \{1\})) \\ &\quad = \mathrm{Ind}_{\mathbf{R}}(\not{D}, \mathscr{I}_K \times S^1) - \mathrm{Ind}_{\mathbf{R}}(\not{D}, \mathscr{I}_K \times \{1\}),\end{aligned}$$

where this last difference must be pulled back over $(\mathscr{I}_{N_0} \times S^1) * (\mathscr{I}_K \times S^1)$. Since $\mathscr{I}_K$ consists of arcs and (by Lemma 7.7) nullhomotopic circles, we have that $\mathrm{Ind}_{\mathbf{R}}(\not{D}, \mathscr{I}_K \times \{1\})$ is trivial, i.e. $\mathrm{Ind}_{\mathbf{R}}(\not{D}, \mathscr{I}_K \times \{1\}) = m_1 \cdot 1$, where $m_1 = 2\,\mathrm{ind}\not{D}_K + 1$ is the numerical index. Each component of $\mathscr{I}_K \times S^1$ is homotopic to $\{\text{point}\} \times S^1$ for a point in $\mathscr{B}^*_{K,1}$. It thus suffices to compute the index of the Dirac operator twisted by the family $\{B' \#_\rho I_{y_0} | \rho \in S^1 \subset SO(3)\}$, where B' is a fixed connection in $\mathscr{B}^{0,*}_{K,0}$ which is flat near the point $y_0 \in K$, I_{y_0} is an instanton at y_0, and the grafted connections have fixed postive scale. Then it follows as in [D1] that

$$\mathrm{Ind}_{\mathbf{R}}(\not{D}, \mathscr{I}_K \times S^1) = \eta_1 + (m_1 - 1) \cdot 1,$$

where η_1 is the restriction of the Hopf real line bundle. Thus

$$\mathrm{Ind}_{\mathbf{R}}(\not{D}, \mathscr{I}_K \times S^1) - \mathrm{Ind}_{\mathbf{R}}(\not{D}, \mathscr{I}_K \times \{1\}) = \eta_1 - 1.$$

In the same way

$$\begin{aligned}&\operatorname{Ind}_{\mathbf{R}}(\not{D}, (\mathcal{I}_{N_0} \times S^1) * (\mathcal{I}_K \times \{1\})) - \operatorname{Ind}_{\mathbf{R}}(\not{D}, (\mathcal{I}_{N_0} \times \{1\}) * (\mathcal{I}_K \times \{1\}))\\ &\quad = \operatorname{Ind}_{\mathbf{R}}(\not{D}, \mathcal{I}_{N_0} \times S^1) - \operatorname{Ind}_{\mathbf{R}}(\not{D}, \mathcal{I}_{N_0} \times \{1\}).\end{aligned}$$

For a fixed $A' \in \mathcal{B}^{0,*}_{N_0, k-2}$ consider the family

$$\{A' \#_{\rho_0} I_{x_0} \#_{\rho_1} I_{x_1} | \rho_i \in S^1 \subset SO(3)\}.$$

The image of this family in $\mathcal{B}^*_{N_0, k}$ is a circle which generates $\pi_1(\mathcal{B}^*_{N_0, k}) \cong H_1(\mathcal{B}^*_{N_0, k}, \mathbf{Z}) = \mathbf{Z}_2$. (Cf. [D1].) We have

$$\operatorname{Ind}_{\mathbf{R}}(\not{D}, \{A' \#_{\rho_0} I_{x_0} \#_{\rho_1} I_{x_1}\}) = \eta_2 + \eta_3 + (m_2 - 2) \cdot 1$$

where $m_2 = 2 \operatorname{ind} \not{D}_{N_0} + k$. A connected component I of $\mathcal{I}_{N_0}$ which is homotopically nontrivial is homotopic to the image of the above family in $\mathcal{B}^*_{N_0, k}$, and for such a component

$$\operatorname{Ind}_{\mathbf{R}}(\not{D}, I \times S^1) = \eta_2 + \eta_3 + (m_2 - 2) \cdot 1.$$

Similarly,

$$\operatorname{Ind}_{\mathbf{R}}(\not{D}, I \times \{1\}) = \operatorname{Ind}_{\mathbf{R}}(\not{D}, \{A' \#_{\rho_0} I_{x_0} \#_{\rho_1} I_{x_1} | \rho_1 \in S^1,\ \rho_0 \text{ fixed}\}) = \eta_3 + (m_2 - 1) \cdot 1.$$

So

$$\operatorname{Ind}_{\mathbf{R}}(\not{D}, I \times S^1) - \operatorname{Ind}_{\mathbf{R}}(\not{D}, I \times \{1\}) = \eta_2 - 1.$$

For a connected component J of $\mathcal{I}_{N_0}$ which is homotopically trivial, we have

$$\operatorname{Ind}_{\mathbf{R}}(\not{D}, J \times S^1) - \operatorname{Ind}_{\mathbf{R}}(\not{D}, J \times \{1\}) = (\eta_2 + (m_2 - 1) \cdot 1) - m_2 \cdot 1 = \eta_2 - 1;$$

so we get the same result in either case.

Finally, consider

$$\operatorname{Ind}_{\mathbf{R}}(\not{D}, (\mathcal{I}_{N_0} \times \{1\}) * (\mathcal{I}_K \times \{1\})) = \operatorname{Ind}_{\mathbf{R}}(\not{D}, (\mathcal{I}_{N_0} \odot \mathcal{I}_K) \times \{(1, 1)\})$$

where $(1, 1) \in S^1 \times S^1$. Now $r_K : \mathcal{I}_K \to (-1, 1)$ is a finite-to-one map which by Theorem 6.7 has odd degree. Hence as a homology class in $\mathcal{B}^*_{K,1}$ rel $(r_K^{-1}(\pm 1))$, $\mathcal{I}_K$ is homologous to an odd number of arcs each mapping homeomorphically onto $(-1, 1)$ via r_K. (This uses Lemma 7.7.) It is easy to see that components of $\mathcal{I}_K$ which are homologically trivial rel $(r_K^{-1}(\pm 1))$ cannot contribute nontrivially to $\operatorname{Ind}_{\mathbf{R}}(\not{D}, (\mathcal{I}_{N_0} \odot \mathcal{I}_K) \times \{(1, 1)\})$. I.e. if J' is such a component, then

$$\operatorname{Ind}_{\mathbf{R}}(\not{D}, (\mathcal{I}_{N_0} \odot J') \times \{(1, 1)\}) = m_3 \cdot 1$$

where $m_3 = 2\widehat{A}(X) + k + 1$. There are an odd number of components which are essential rel $r_K^{-1}(\pm 1)$. Working with homology, we may assume that such a component is a single arc, I', and such that $r_K : I' \cap r_K^{-1}(-1, 1) \to (-1, 1)$ is a homeomorphism. Then $\mathcal{I}_{N_0} \odot I' \cong \mathcal{I}_{N_0}$. If I is a component of $\mathcal{I}_{N_0}$ then

$\operatorname{Ind}_{\mathbf{R}}(\not{D}, (I \odot I') \times \{(1, 1)\})$ is $\operatorname{Ind}_{\mathbf{R}}(\not{D}, I)$ pulled back over $I \odot I' \times S^1 \times S^1$. Thus if I is trivial in $\pi_1(\mathscr{B}^*_{N_0, k})$,

$$\operatorname{Ind}_{\mathbf{R}}(\not{D}, (I \odot I') \times \{(1, 1)\}) = m_3 \cdot 1,$$

and if I is nontrivial then

$$\operatorname{Ind}_{\mathbf{R}}(\not{D}, (I \odot I') \times \{(1, 1)\}) = \eta_3 + (m_3 - 1) \cdot 1.$$

Adding the above expressions we obtain

Lemma 7.8. (a) *For each component I of $\mathscr{I}_{N_0}$ which is nontrivial in $\pi_1(\mathscr{B}^*_{N_0, k})$ and component I' of $\mathscr{I}_K$ which is essential rel $r_K^{-1}(\pm 1)$,*

$$\operatorname{Ind}_{\mathbf{R}}(\not{D}, (I \times S^1) * (I' \times S^1)) = \eta_1 + \eta_2 + \eta_3 + (m_3 - 3) \cdot 1.$$

(b) *For each component J of $\mathscr{I}_{N_0}$ which is trivial in $\pi_1(\mathscr{B}^*_{N_0, k})$,*

$$\operatorname{Ind}_{\mathbf{R}}(\not{D}, (J \times S^1) * (I' \times S^1)) = \eta_1 + \eta_2 + (m_3 - 2) \cdot 1.$$

(c) *For each component J' of $\mathscr{I}_K$ which is inessential rel $r_K^{-1}(\pm 1)$,*

$$\operatorname{Ind}_{\mathbf{R}}(\not{D}, (\mathscr{I}_{N_0} \times S^1) * (J' \times S^1)) = \eta_1 + \eta_2 + (m_3 - 2) \cdot 1. \quad \square$$

Proof of Theorem 1.3. A transversal of $(\mathscr{I}_{N_0} \times S^1) * (\mathscr{I}_K \times S^1)$ can be obtained by fixing the final S^1, say. For components I of $\mathscr{I}_{N_0}$ which is nontrivial in $\pi_1(\mathscr{B}^*_{N_0, k})$ and I' of $\mathscr{I}_K$ which is essential rel $r_K^{-1}(\pm 1)$, let T_I be the corresponding transversal of $(I \times S^1) * (I' \times S^1)$. Then

$$\operatorname{Ind}_{\mathbf{R}}(\not{D}, T_I) \otimes \det \operatorname{Ind}_{\mathbf{R}}(\not{D}, T_I) = (\eta_2 + \eta_3 + (m_3 - 2) \cdot 1)(\eta_2 \eta_3).$$

Let the cohomology generators of the two S^1-factors be t_2 and t_1, respectively, and let t_3 be the cohomology generator of I. The total Stiefel-Whitney class of $\operatorname{Ind}_{\mathbf{R}}(\not{D}, T_I) \cdot \det \operatorname{Ind}_{\mathbf{R}}(\not{D}, T_I)$ is $w = (t_2 + t_3 + (m_3 - 2))(t_2 t_3)$; so $u_2([T_I]) = (m_3 - 2)(t_2 t_3)([T_I]) = m_3 - 2 \equiv 1 \bmod 2$. If instead we evaluate on a homotopically trivial component J of $\mathscr{I}_{N_0}$, we obtain

$$\operatorname{Ind}_{\mathbf{R}}(\not{D}, T_J) \otimes \det \operatorname{Ind}_{\mathbf{R}}(\not{D}, T_J) = (\eta_2 + (m_3 - 1) \cdot 1)\eta_2.$$

So $w = (t_2 + (m_3 - 1))t_2$, and $u_2([T_J]) = 0$. Similarly $u_2([T]) = 0$ for a transversal of $(\mathscr{I}_{N_0} \times S^1) * (J' \times S^1)$ for any component J' of $\mathscr{I}_K$ which is essential rel $r_K^{-1}(\pm 1)$. Since the number of components I' is odd, this means that on the one hand, $u_2([\mathscr{I}_{N_0} * \mathscr{I}_K])$ is (modulo 2) the number of homotopically nontrivial components of $\mathscr{I}_{N_0}$, and by Proposition 7.5 this is just $q_{k, u_1, N}(z_1, \dots, z_d)$. On the other hand, by definition, $u_2([\mathscr{I}_{N_0} * \mathscr{I}_K]) = q_{k+1, u_2, X}(z_1, \dots, z_d, x, y)$. $\square$

8. Invariant theory and the Donaldson polynomial mod 2

For M a closed simply connected 4-manifold, let $\operatorname{Sym}^d_R(H_2(M; \mathbf{Z}))$ be the ring of d-linear symmetric functions on $H_2(M; \mathbf{Z})$ with values in a ring R.

The symmetric product $\gamma_1\gamma_2 \in \mathrm{Sym}_R^{d_1+d_2}(H_2(M;\mathbf{Z}))$ of $\gamma_1 \in \mathrm{Sym}_R^{d_1}(H_2(M;\mathbf{Z}))$ and $\gamma_2 \in \mathrm{Sym}_R^{d_2}(H_2(M;\mathbf{Z}))$ is defined by the rule

$$\gamma_1\gamma_2(x_1, \dots, x_{d_1+d_2}) = \frac{1}{d_1!d_2!}\sum_{\sigma\in S_{d_1+d_2}} \gamma_1(x_{\sigma(1)}, \dots, x_{\sigma(d_1)})\gamma_2(x_{\sigma(d_1+1)}, \dots, x_{\sigma(d_1+d_2)})$$

where $S_{d_1+d_2}$ denotes the symmetric group on $d_1 + d_2$ letters. The intersection form Q_M of M is an element of $\mathrm{Sym}_{\mathbf{Z}}^2(H_2(M;\mathbf{Z}))$, and the degree d Donaldson invariant $q_{\ell,M}$ is an element of $\mathrm{Sym}_{\mathbf{Z}}^d(H_2(M;\mathbf{Z}))$. Define $Q_M^{(p)} \in \mathrm{Sym}_{\mathbf{Z}}^{2p}(H_2(M;\mathbf{Z}))$ by

$$Q_M^{(p)} = \frac{1}{p!}Q_M^p.$$

It is interesting to note that if the homology classes $z_1, \dots, z_{2p}$ are represented by surfaces $\Sigma_1, \dots, \Sigma_{2p}$ in general position, then $Q_M^{(p)}(z_1, \dots, z_{2p})$ is the number of ways of placing p points on the intersections of pairs of these surfaces (counted with suitable signs) such that each surface contains a point.

The results of Wall [W1] mentioned in the introduction imply that the diffeomorphism group of $X = M\#S^2\times S^2$ maps onto the orthogonal group O_X of automorphisms of $H_2(X;\mathbf{Z})$ which preserve Q_X. In this case, classical invariant theory implies that if γ is a nonzero element of $\mathrm{Sym}_{\mathbf{Z}}^d(H_2(X;\mathbf{Z}))$ which is a diffeomorphism invariant (hence is left invariant by O_X), then $d = 2p$ and γ is a multiple of $Q_X^{(p)}$. Such a result is false for $\mathrm{Sym}_{\mathbf{Z}_2}^d(H_2(X;\mathbf{Z}))$; so we are not able to prove Theorem 1.5 by an appeal to algebra. Instead, our proof combines algebra with specific knowledge of the invariant $q_{\ell+1,u_1,X}$ deduced via gauge theory. We begin with the gauge-theoretic arguments. The next proposition is a version of Donaldson's connected sum theorem [D2] in the context of the invariant $q_{\ell+1,u_1,X}$. Our argument follows the lines of a proof of Donaldson's theorem given by John Morgan.

Theorem 8.1. *Let M be a closed simply connected spin 4-manifold with a degree d Donaldson invariant $q_{\ell,M}$ with ℓ odd. Let $X = M\#S^2\times S^2$ and let x and y denote the homology classes $[S^2\times 0]$ and $[0\times S^2]$. Let $z_1, \dots, z_r \in H^2(M;\mathbf{Z})$ and $w_j = x$ or y, $j = 1, \dots, d+2-r$. Suppose $r \neq 0$ or $d+2$. Then the mod 2 invariant $q_{\ell+1,u_1,X}(z_1, \dots, z_r, w_1, \dots, w_{d+2-r}) = 0$ unless $r = d$ and $\{w_{d+1}, w_{d+2}\} = \{x, y\}$.*

Proof. We apply an argument derived from considering a sequence of metrics $\{g_\nu\}$ on X whose limit is the one-point union $(M \vee S^2\times S^2, g_M \vee g_{S^2\times S^2})$ where g_M and $g_{S^2\times S^2}$ are generic. Fix points $p_0 \in M$ and $q_0 \in S^2\times S^2$ and embedded geodesic balls $B_M(p_0, \nu)$ and $B_{S^2\times S^2}(q_0, \nu)$ of radius ν. Then $(M\#S^2\times S^2, g_\nu)$ is obtained by identifying boundary collars in $M_0(\nu) = M\backslash B_M(p_0, \nu)$ and $S^2\times S^2\backslash B_{S^2\times S^2}(q_0, \nu)$ in such a way that outside of a small neighborhood of the neck where the identification is made, g_ν agrees with g_M and $g_{S^2\times S^2}$. (See [D2,§IV].) If $q_{\ell+1,u_1,X}(z_1,\dots,z_r,w_1,\dots,w_{d+2-r}) \neq 0$, then for each ν there is an $A_\nu \in \mathscr{M}_{X,\ell+1}(g_\nu)\cap V_1\cap\cdots\cap V_{d+2}$,where the V_i's are the divisors corresponding to good surfaces representing the z_i and w_j. As usual, Uhlenbeck's theorems on compactness and removability of singularities imply that there are connections $A_M \in \mathscr{M}_{M,m}(g_M)$ and $A_{S^2\times S^2} \in \mathscr{M}_{S^2\times S^2,n}(g_{S^2\times S^2})$ such that A_ν converges to $A_M \vee A_{S^2\times S^2}$ together with instantons at ρ points in M and at σ points in $S^2\times S^2$.

At first we suppose that $0<r<d$. Suppose that $m>0$ and $n>0$. Since surfaces representing the z_j and w_k are chosen in general position, no instanton point lies on more than two of these surfaces. Thus A_M must lie on at least $r-2\rho$ of $V_1,\dots,V_r$; hence $2d-8(\ell-m)\geq 2(r-2\rho)$. Similarly, from $A_{S^2\times S^2}$ we deduce that $8n-6\geq 2(d+2-r-2\sigma)$. Combining these inequalities with the charge count $m+n+\rho+\sigma\leq \ell+1$ leads to a contradiction. If $m>0$, $n=0$, then $2d-8(\ell-m)\geq 2(r-2\rho)$ and $2\sigma\geq d+2-r$. So $\ell+1\geq m+\rho+\sigma\geq \ell+1-\frac{r}{4}+\frac{\rho}{2}+\frac{d}{4}$. This contradicts the assumption $r<d$. If $m=0$, $n=0$ then $2\rho\geq r$ and $2\sigma\geq d+2-r$; and then the charge count $\ell+1\geq\rho+\sigma$ contradicts the basic inequality $\ell>\frac{3}{4}(1+b_M^+)$.

To complete the proof in the case when $0<r<d$, we need to consider the situation where all the connections in the 1-dimensional intersection

$$\mathscr{I}_X(\nu)=\mathscr{M}_{X,\ell+1}(g_\nu)\cap V_1\cap\cdots\cap V_{d+2}$$

limit weakly to the trivial connection Θ_M on M and none limit weakly to the trivial connection on $S^2\times S^2$. A counting argument does not suffice here. This is precisely the situation encountered in the proof of the connected sum theorem.

As in Donaldson's proof, for large ν, we need to consider an open subset U of $\mathscr{B}^*_{X,\ell+1}$ consisting of connections whose restrictions to $M_0(\nu)$ are close to Θ_M off a finite number of small balls where the charge is concentrated. To define U, fix $\varepsilon>0$. Then $A\in\mathscr{B}^*_{X,\ell+1}$ is in U if there are a finite number of disjoint balls B_i in $M_0(\nu)$ with centers p_i and radii λ_i such that

(1) $\int_{M_0(\nu)\setminus\cup B_i}|F_A|^2<\varepsilon$,

(2) $\sum\lambda_i^2<\varepsilon$,

(3) $|\int_{B_i}|F_A^-|^2-8\pi^2 m_i|<\varepsilon$ for some positive integers m_i.

The upshot of the counting argument given above is that if we fix ε, then for small enough ν, the intersection $\mathscr{I}_X(\nu)$ is contained in U. The subset of U consisting of connections A for which some m_i is greater than one is of codimension at least 4 in U. Since $\mathscr{I}_{X,p}(\nu)$ is 1-dimensional, we can modify the third condition defining U:

(3′) $|\int_{B_i}|F_A^-|^2-8\pi^2|<\varepsilon$ for each i.

Note that U breaks up into connected components such that all connections in any given component have approximately the same charge $\sum m_i$. To show that $\mathscr{I}_X(\nu)$ is homologically trivial in $\mathscr{B}^*_{X,\ell+1}$, it suffices to work with the piece $\mathcal{I}_{X,\rho}(\nu)$ lying in, say, U_ρ, the union of the components of U with charge approximately constant and equal to $\sum m_i = \rho$. We are assuming that $r \geq 1$; so $\rho \geq \frac{r}{2} > 0$. Also, we may as well assume that $\mathcal{I}_{X,\rho}(\nu)$ is nonempty.

Consider now $U_{\rho,M} \subset \mathscr{B}^*_{M,\rho}$ which consists of connections in $\mathscr{B}^*_{M,\rho}$ satisfying the defining conditions for U. According to the theory of Taubes and Donaldson, for small enough ε, the anti-self-dual connections in $U_{\rho,M}$ can be described as follows (cf. [T1, D1]). Let F^+_M denote the bundle oriented orthonormal frames of the space of self-dual 2-forms on M, and let $S^\rho(F^+_M \times \mathbf{R}^+)$ denote the complement in the symmetric product of ρ copies of $F^+_M \times \mathbf{R}^+$ of the preimage of the "fat diagonal" Δ^ρ_M under the obvious projection. (Here $\Delta^\rho_M = \{\{x_i\} \in \mathrm{Symm}^\rho(M) \mid x_i = x_j, \text{ some } i \neq j\}$.) We consider the diagonal action of $SO(3)$ on $S^\rho(F^+_M \times \mathbf{R}^+)$ (where $SO(3)$ acts trivially on the $\mathbf{R}^+$ factor). Then there is an $SO(3)$-equivariant embedding

$$\gamma^0 : S^\rho(F^+_M \times \mathbf{R}^+) \to \mathscr{B}^{*,0}_{M,\rho}$$

and an $SO(3)$-equivariant map

$$\psi^0 : S^\rho(F^+_M \times \mathbf{R}^+) \to \mathbf{R}^{3b^+_M}$$

such that $\{\gamma^0\{(f_i, \lambda_i)\} | \sum \lambda_i^2 < \varepsilon\}$ is equal to $U^0_{\rho,M}$ and such that $\gamma^0((\psi^0)^{-1}(0)) = U_{\rho,M} \cap \mathscr{M}^0_{M,\rho}$. Taking the quotient by $SO(3)$, this descends to

$$\gamma : S^\rho(F^+_M \times \mathbf{R}^+)/SO(3) \to \mathscr{B}^*_{M,\rho}$$

and to a section ψ of the rank $3b^+_M$ vector bundle

$$\begin{array}{c} \eta_M = \bigoplus_{b^+_M} [S^\rho(F^+_M \times \mathbf{R}^+) \times_{SO(3)} \mathbf{R}^3] \\ \downarrow \\ S^\rho(F^+_M \times \mathbf{R}^+)/SO(3) \end{array}$$

such that $\gamma(\psi^{-1}(0)) = U_{\rho,M} \cap \mathscr{M}_{M,\rho}$. By letting the scales λ_i assume the value 0, this set-up extends naturally to the Uhlenbeck compactification $\bar{\mathscr{M}}_{M,\rho}$, where the section gives obstructions to the lower charge problem with points of concentration (instanton points). (See [D1,5.6] and [DK,§7.2.8].)

Likewise, there is an "obstruction bundle" description of $U_\rho \cap \mathscr{M}_{X,\ell+1}(g_\nu)$ given by an $SO(3)$-equivariant map

$$\sigma^0 : S^\rho(F^+_M \times \mathbf{R}^+) \times \mathscr{M}^0_{S^2\times S^2,n} \to \mathbf{R}^{3b^+_M}$$

(again, diagonal action) where $n = \ell + 1 - \rho$. Thus U_ρ can be identified with

$$S^\rho(F^+_M \times \mathbf{R}^+) \times \mathscr{M}^0_{S^2\times S^2,n}/SO(3),$$

and $U_\rho \cap \mathscr{M}_{X,\ell+1}(g_\nu)$ can be identified with the zeros of a section $\sigma(\nu)$ of

$$\begin{array}{c} \bigoplus_{b^+_M} [(S^\rho(F^+_M \times \mathbf{R}^+) \times \mathscr{M}^0_{S^2\times S^2,n}) \times_{SO(3)} \mathbf{R}^3] \\ \downarrow \\ [S^\rho(F^+_M \times \mathbf{R}^+) \times \mathscr{M}^0_{S^2\times S^2,n}]/SO(3) \end{array}$$

Again, this description extends to $\bar{\mathscr{M}}_{\ell+1,X}(g_\nu)$ where we let some λ_i's equal 0.

The open set U has two distinct ends. The first consists of connections which are concentrated; this corresponds to the situation where some of the $\lambda_i = 0$. The second end, which we shall denote $\mathrm{Fr}(U)$, consists of those $[\{(f_i, \lambda_i)\}; (A, \xi)] \in U$ such that $\sum \lambda_i^2 = \varepsilon$. Since $\mathscr{I}_X(\nu) \subset \mathrm{Int}\, U$, the section σ restricted over $\mathrm{Fr}(U) \cap V_1 \cap \cdots \cap V_{d+2}$ has no zeros.

It follows from the description in [T1] that as $\nu \to 0$, the section $\sigma(\nu)$ "decouples", that is, it limits to the sum of sections $\sigma_M = \psi$ of η_M and $\sigma_{S^2\times S^2}$ of

$$\begin{array}{c} \eta_{S^2\times S^2} = \bigoplus_{b_M^+}(\mathscr{M}^0_{S^2\times S^2,n} \times_{SO(3)} \mathbf{R}^3) \\ \downarrow \\ \mathscr{M}_{S^2\times S^2,n} \end{array}$$

Since $\mathscr{M}_{S^2\times S^2,n}$ consists of anti-self-dual connections, the limiting section $\sigma_{S^2\times S^2} = 0$. Thus for small enough neck radius ν, the corresponding section $\sigma(\nu)$ is almost a sum $\sigma(\nu) = \sigma_M(\nu) + \sigma_{S^2\times S^2}(\nu)$, and $\sigma_{S^2\times S^2}(\nu) \to 0$ as $\nu \to 0$. Furthermore, for fixed ν, $\sigma_M(\nu)\{(f_i, \lambda_i)\} \to 0$ as $\sum \lambda_i^2 \to 0$.

In particular, this means that for small ν, $\sigma(\nu)$ is asymptotically $\sigma_M(\nu)$ as $\sum \lambda_i^2 \to \varepsilon$. We fix such a small ν and now drop it from the notation. Then we have our neighborhood U_ρ, and $\sigma \sim \sigma_M$ near $\mathrm{Fr}(U_\rho)$. The intersection $\mathscr{I}_{X,\rho}$ is cut out by the zero set of σ restricted to $U_\rho \cap V_1 \cap \cdots \cap V_{d+2} = W_\rho$. Thus on $\mathrm{Fr}(W_\rho)$ the section $\sigma \sim \sigma_M$ is nonvanishing. Since W_ρ consists of connections which are almost anti-self-dual (more precisely, which satisfy conditions (1)–(3) above) the notion of Uhlenbeck compactification $\bar{W}_\rho$ makes sense. Note also that the compactness of $\mathscr{I}_X$ implies that σ has no zeros on the singular set of $\bar{W}_\rho$.

The formal situation is this—we have a compact singular space $\bar{W}_\rho$ with singular set (corresponding to the lower strata of the Uhlenbeck compactification) of codimension ≥ 3. Over W_ρ there is a vector bundle η of rank $3b_M^+$ with a section σ, nonvanishing over the singular set, and over the boundary $\mathrm{Fr}(W_\rho)$, σ is pulled back from the section σ_M of $\eta_M|_{\mathrm{Fr}(W_{\rho,M})}$. This set-up gives us a relative Euler class $e \in H^{3b_M^+}(\bar{W}_\rho, \mathrm{Fr}(W_\rho); \mathbf{Z})$. Clearly, $\dim \bar{W}_\rho = 3b_M^+ + 1$, and if S denotes the singular set of $\bar{W}_\rho$ then Poincaré duality gives

$$H_1(\bar{W}_\rho; \mathbf{Z}) \cong H_c^{3b^+}(\bar{W}_\rho, \mathrm{Fr}(W_\rho) \cup S; \mathbf{Z}) \cong H^{3b^+}(\bar{W}_\rho, \mathrm{Fr}(W_\rho); \mathbf{Z})$$

because of the codimension of S. The Poincaré dual of e in $H_1(\bar{W}_\rho; \mathbf{Z})$ is represented by $\mathscr{I}_{X,\rho}$. Thus it will suffice to show that $e = 0$.

Since $\mathscr{I}_{X,\rho}$ is nonempty, the usual counting argument shows that $\rho \geq \frac{r}{2}$ and $8n - 6 \geq 2(d + 2 - r)$. By counting parameters in the base spaces of the obstruction bundles η_M and η, we see that $\dim \bar{W}_{\rho,M} = 8\rho - 3 - 2r$ and $\dim \bar{W}_\rho = 8\rho + 8n - 6 - 2(d+2)$. Thus $3b_M^+ + 1 = \dim \bar{W}_\rho = \dim W_{\rho,M} + [8n - 3 - 2(d + 2 - r)] \geq \dim \bar{W}_{\rho,M} + 3$. So $\sigma|_{\mathrm{Fr}(W_\rho)}$ is pulled back from the section $\sigma_M|_{\mathrm{Fr}(W_{\rho,M})}$, and the base $W_{\rho,M}$ has dimension at least 2 less than the rank $3b_M^+$ of η_M. Note that any nonvanishing section of $\eta_M|_{\mathrm{Fr}(W_{\rho,M})}$ is homotopic through nonvanishing sections to $\sigma_M|_{\mathrm{Fr}(W_{\rho,M})}$ since the obstructions to such homotopies lie in $H^i(\mathrm{Fr}(W_{\rho,M}); \pi_i(S^{3b_M^+-1})) = 0$.

We now need to separate the argument into two cases. Suppose first that $3b_M^+ + 1 > \dim \bar{W}_{\rho,M} + 3 = 8\rho - 2r$. In this case our plan is to construct nonvanishing sections τ_M of η_M over $\bar{W}_{\rho,M}$ and $\tau_{S^2\times S^2}$ of $\eta_{S^2\times S^2}$ over $Y = \mathscr{M}_{S^2\times S^2} \cap V_{r+1} \cap \cdots \cap V_{d+2}$ and combine them to get a nonvanishing section τ of η over $\bar{W}_\rho$. This section will be nonvanishing as well over the singular set of $\bar{W}_\rho$ and over $\mathrm{Fr}(W_\rho)$ it will be pulled back from τ_M. Since $\tau_M|_{\mathrm{Fr}(W_{\rho,M})}$ is homotopic to $\sigma_M|_{\mathrm{Fr}(W_{\rho,M})}$ through nonvanishing sections of $\eta_M|_{\mathrm{Fr}(W_{\rho,M})}$, the Poincaré dual of the Euler class e will be represented by the (empty) zero set of τ; hence $e = 0$.

We first construct τ_M. Let k be the unique integer such that

$$8\rho - 2r - 3 \leq 3k < 8\rho - 2r \leq 3b_M^+,$$

and consider the rank $3k$ vector bundle

$$\begin{array}{c}\bigoplus_{i=1}^{k}(W_{\rho,M}^0 \times_{SO(3)} \mathbf{R}^3)\\ \downarrow\\ W_{\rho,M}\end{array}$$

This is a subbundle of $\eta_M|_{W_{\rho,M}}$. Obstructions to the existence of a nonvanishing section of this bundle lie in $H^i(W_{\rho,M}; \pi_{i-1}(S^{3k-1}))$. Since $H^{3k}(W_{\rho,M}; \mathbf{Z}) = 0$, all obstructions vanish and there is such a section τ_M. Note that the total space $\bigoplus_{i=1}^k (W_{\rho,M}^0 \times_{SO(3)} \mathbf{R}^3)$ is the quotient by $SO(3)$ of $\bigoplus_k (W_{\rho,M}^0 \times \mathbf{R}^3) \cong W_{\rho,M}^0 \times \mathbf{R}^{3k}$; so we equivalently have a nonvanishing $SO(3)$-equivariant map $\tau_M^0 : W_{\rho,M}^0 \to \mathbf{R}^{3k}$.

Next, consider the situation on $Y = \mathscr{M}_{S^2\times S^2, n} \cap V_{r+1} \cap \cdots \cap V_{d+2}$ which has dimension $\dim Y = 8n - 6 - 2(d+2-r) = 3b_M^+ + 1 - (8\rho - 2r) < 3b_M^+ + 1 - 3k$. Thus $\dim Y \leq 3(b_M^+ - k)$. We now wish to find a nonvanishing section $\tau_{S^2\times S^2}$ of the rank $3(b_M^+ - k)$ vector bundle $\bigoplus_{i=k+1}^{b_M^+}(Y^0 \times_{SO(3)} \mathbf{R}^3)$ over Y. (The point of the case we are considering is that $b_M^+ - k > 0$.) The only possible obstruction arises in case $\dim Y = 3(b_M^+ - k)$. In this case the obstruction lies in $H^{3(b_M^+-k)}(Y; \mathbf{Z})$ and is $c^{b_M^+-k}$ where $c \in H^3(Y; \mathbf{Z})$ is the Euler class of the vector bundle $Y^0 \times_{SO(3)} \mathbf{R}^3$ associated to the basepoint fibration over Y. Since this bundle has odd rank, its Euler class c is 2-torsion; so $c^{b_M^+-k}$ is 2-torsion as well. However, $H^{3(b_M^+-k)}(Y; \mathbf{Z})$ is torsion-free, and so the obstruction vanishes. Again, since $\bigoplus_{i=k+1}^{b_M^+}(Y^0 \times_{SO(3)} \mathbf{R}^3) = \bigoplus_{b_M^+-k}(Y^0 \times \mathbf{R}^3)/SO(3)$, we have a nonvanishing $SO(3)$-equivariant map $\tau^0_{S^2\times S^2} : Y^0 \to \mathbf{R}^{3(b^+_M-k)}$.

To define $\tau^0 : W^0_\rho \to \eta|W^0_\rho \cong (W^0_{\rho,M}\times Y^0)\times \mathbf{R}^{3b^+_M}$, consider a typical point

$$Q = [\{(f_i,\lambda_i)\};(A,\xi)] \in W^0_{\rho,M} \subset S^\rho(F^+_M\times \mathbf{R}^+)\times \mathscr{M}^0_{S^2\times s^2,n}.$$

Let $\lambda(Q) = \sum \lambda_i^2$, and define

$$\tau^0(Q) = \lambda(Q)\cdot \tau^0_M\{(f_i,\lambda_i)\} + (\varepsilon-\lambda(Q))\cdot \tau^0_{S^2\times S^2}(A,\xi).$$

(Here the image of τ^0_M lies in the first k of the $\mathbf{R}^3$-summands of $\mathbf{R}^{3b^+_M}$ and the image of $\tau^0_{S^2\times S^2}$ lies in the last b^+_M-k of them.) The section τ^0 is $SO(3)$-equivariant, nonvanishing, and nonvanishing when extended to the singular set of W^0_ρ (where some (or all) of the $\lambda_i = 0$). As we described above, this completes the proof in the case where $3b^+_M+1 > 8\rho-2r$.

Next, we need to consider the case where $\dim W^0_{\rho,M} = 8\rho - 2r = 3b^+_M+1$. Then we have $\dim W_\rho = \dim W^0_{\rho,M} + \dim Y^0 - 3 = 3b^+_M + 1 = \dim W^0_{\rho,M}$; so $\dim Y = \dim Y^0 - 3 = 0$. Thus the cut-down moduli space $Y = \mathscr{M}_{S^2\times S^2}\cap V_{r+1}\cap \cdots \cap V_{d+2}$ consists of a finite number of points. This means that $\mathscr{I}_{X,\rho}$ breaks up into unions of connected components corresponding to these points. It thus suffices to assume that Y is a single connection and show that the homology class of $\mathscr{I}_{X,\rho}$ vanishes. Then Y^0 is a copy of $SO(3)$; so $W^0_\rho = W^0_{\rho,M}\times SO(3)$, and the obstruction bundle is:

$$\begin{array}{c} \eta = \bigoplus_{b^+_M}(\bar W^0_\rho \times_{SO(3)} \mathbf{R}^3) \cong \bigoplus_{b^+_M}(\bar W^0_{\rho,M}\times \mathbf{R}^3) \cong \bar W^0_{\rho,M}\times \mathbf{R}^{3b^+_M} \\ \downarrow \\ \bar W_\rho = (\bar W^0_\rho \times SO(3))/SO(3) \cong \bar W^0_{\rho,M} \end{array}$$

The obstruction bundle is trivial; however this is not enough to deduce that the homology class $[\mathscr{I}_{X,\rho}]$ is trivial because $[\mathscr{I}_{X,\rho}]$ is Poincaré dual to a *relative* Euler class in $H^{3b^+_M}(\bar W^0_{\rho,M}, \mathrm{Fr}(W^0_{\rho,M});\mathbf{Z})$.

Recall that the condition on a section which defines the relative class is that over $\mathrm{Fr}(W^0_{\rho,M})$ it must be homotopic to the pullback of σ_M, a section of $\eta_M|_{\mathrm{Fr}(W_{\rho,M})}$. Since $\mathrm{Fr}(W_{\rho,M})$ is a manifold of dimension $3(b^+_M-1)$, the Euler class argument given in the construction of $\tau_{S^2\times S^2}$ works here as well to show that that there is a nonvanishing section τ_∂ of η_M over $\mathrm{Fr}(W_{\rho,M})$ such that τ_∂ takes its values in a subbundle of rank $3(b^+_M-1)$. (This argument needs $3(b^+_M-1)>0$ to work, for if $\mathrm{Fr}(W_{\rho,M})$ were 0-dimensional, τ_∂ would still need to take values in a bundle of positive rank. Of course we have the standard assumption that $b^+_M > 1$.)

Now return to this relative Euler class problem of extending a nonvanishing section of

$$\begin{array}{c} \bar W^0_{\rho,M}\times \mathbf{R}^{3b^+_M} \\ \downarrow \\ \bar W^0_{\rho,M} \end{array}$$

from $\mathrm{Fr}(W^0_{\rho,M})$ over all of $\bar{W}^0_{\rho,M}$. As in the previous case, this problem is independent of the section over $\mathrm{Fr}(W^0_{\rho,M})$ as long as it is pulled back from a section of η_M—so we work with the pullback of τ_∂. Up to homotopy through nonvanishing sections, the pullback is the same as a map

$$\tau'_\partial : \mathrm{Fr}(W^0_{\rho,M}) \to S^{3(b^+_M-1)-1} \subset S^{3b^+_M-1}.$$

Since τ'_∂ is not a surjection onto $S^{3b^+_M-1}$, it is nullhomotopic and therefore it extends to a map $\bar{W}^0_{\rho,M} \to S^{3b^+_M-1}$. This means that the relative Euler class vanishes and our proof of Theorem 8.1 is completed for the case where $0 < r < d$.

Before continuing with the other cases, we wish to make a few comments on the argument above. First of all, $SO(3)$-equivariance forces us to work with whole $\mathbf{R}^3$-summands of $\mathbf{R}^{3b^+_M}$. It would not be sufficient, for example, to get a nonvanishing section of η_M over $\mathrm{Fr}(W_{\rho,M})$ which takes its values in a subbundle of rank $3b^+_M - 2$, because the pullback over $\mathrm{Fr}(W^0_{\rho,M})$ would not take its values in a proper subbundle of η.

Second, this last argument comes close to filling in the details in the alternate proof of Theorem 1.1 which was sketched in the introduction. In that case we have a 0-dimensional cut-down moduli space over M, and on $S^2 \times S^2$ there is a single point instanton with center $(S^2 \times 0) \cap (0 \times S^2)$. Thus $\rho = 1$, $\bar{W}^0_{\rho,S^2\times S^2} \cong \mathrm{c}SO(3)$, Y^0 is a finite union of copies of $SO(3)$, and $b^+_{S^2\times S^2} = 1$. For simplicity we work with a single copy of $SO(3)$. Then we get $\bar{W}_\rho \cong \bar{W}^0_{\rho,S^2\times S^2} \cong \mathrm{c}SO(3)$ and η is the trivial bundle over $\bar{W}_\rho$ with fiber $\mathbf{R}^3$. The relative Euler class lives in $H^3(\bar{W}_\rho, \mathrm{Fr}(W_\rho); \mathbf{Z}) \cong H^3(\mathrm{c}SO(3), SO(3); \mathbf{Z}) \cong \mathbf{Z}_2$, and it is the obstruction to the nonvanishing extension of a map $\mathrm{Fr}(W_\rho) \cong SO(3) \to \mathbf{R}^3$ pulled back from $\tau_\partial : \{\text{point}\} \to \mathbf{R}^3$. In fact, if $\tau_\partial = v \in \mathbf{R}^3$ then the pullback $SO(3) \to \mathbf{R}^3$ is given by $g \to g(v)$, and this realizes the nontrivial cohomology class.

To finish the proof of Theorem 8.1, we still need to dispense with the rest of the cases. Suppose that $q_{\ell+1,u_1,X}(z_1, \dots, z_{d+1}, x) \neq 0$ (so $r = d+1$). Arguing as above in case $m > 0$, $n > 0$, we obtain inequalities $8m \geq 8\ell + 2(1-2\rho)$ and $8n - 6 \geq 2(1-2\sigma)$. These inequalities are again incompatible with the charge count. We similarly rule out the cases $m = 0$ and/or $n = 0$.

Finally, we need to compute $q_{\ell+1,u_1,X}(z_1, \dots, z_d, x, x)$. This calculation follows §§5 and 6, except that in Theorem 5.11 we obtain

$$q_{\ell+1,u_1,X}(z_1, \dots, z_d, x, x) \equiv N(x, x) \cdot q_{\ell,M}(z_1, \dots, z_d) \mod 2$$

where $N(x, x)$ is the mod 2 intersection number

$$N(x, x) \equiv \#(\mathscr{M}_{K,1} \cap V_x \cap V'_x \cap r_K^{-1}(\alpha)) \quad \text{for } \alpha \in r_{M_0}(\mathscr{I}_{M_0}).$$

(The divisor V'_x corresponds to a second surface representing x.) Refer now to §6. The proof preceding Lemma 6.2, which uses an argument based on [D1, Theorem B], shows that in this case $[N^1(\alpha)] = 0 \in H_1(\mathscr{B}^*_{W'}(\alpha); \mathbf{Z}_2)$ since there are an even number of "internal" ends of $\mathscr{N}^2(\vartheta)$. Then the proof of Theorem 6.7 shows $N(x, x) \equiv 0 \bmod 2$. Similarly we can show

$$q_{\ell+1, u_1, X}(z_1, \dots, z_d, y, y) = 0.$$

(Alternatively we can prove that $q_{\ell+1, u_1, X}(z_1, \dots, z_d, y, y) = 0$ by applying the transformation S_0 defined below to the equation

$$q_{\ell+1, u_1, X}(z_1, \dots, z_d, x, x) = 0$$

and then using the invariance of $q_{\ell+1, u_1, X}$ under diffeomorphisms.) □

We continue with the hypothesis that M is a closed simply connected spin 4-manifold with a Donaldson invariant $q_{\ell, M}$ of degree d with ℓ odd. In particular $b_M^+ \geq 3$. Now $H_2(M; \mathbf{Z}) \cong 2aE_8 \oplus bH$ where $a, b \in \mathbf{Z}$ and $b \geq 0$. It follows from [D1] that $a = 0$ if $b \leq 2$; so $b \geq 3$. Let $X = M \# S^2 \times S^2$ and let s_X denote the symplectic form on $H_2(X; \mathbf{Z}_2)$ given by $s_X(u, v) \equiv Q_X(\bar{u}, \bar{v})$ mod 2 where $\bar{u}$ and $\bar{v}$ are any lifts of u, v to $H_2(X; \mathbf{Z})$. (Here we are using the fact that $H_1(X; \mathbf{Z}) = 0$.) Let $\Phi_X : H_2(X; \mathbf{Z}) \to \mathbf{Z}_2$ be given by $\Phi_X(w) \equiv \frac{1}{2} Q_X(w, w) \bmod 2$, and let $\varphi_X : H_2(X; \mathbf{Z}_2) \to \mathbf{Z}_2$ be the induced quadratic form. Note that $\varphi_X(u+v) + \varphi_X(u) + \varphi_X(v) = s_X(u, v)$ and that the orthogonal group O_X leaves Φ_X invariant. For $u, v \in H_2(X; \mathbf{Z})$ write $u \equiv v$ mod 2 provided $v = u + 2w$ for some $w \in H_2(X; \mathbf{Z})$.

Lemma 8.2. *Let $u, v \in H_2(X; \mathbf{Z})$ be primitive vectors satisfying $\Phi_X(u) = \Phi_X(v)$. Then there is an $h \in O_X$ such that $h(u) \equiv v \bmod 2$.*

Proof. This is a consequence of a result of Wall [W3, Theorem 6] that since $b_X^+ \geq 2$ (in fact ≥ 4 in our case) and X is spin, O_X is transitive on primitive vectors with a given norm $Q_X(u, u)$. Suppose $\Phi_X(u) = \Phi_X(v)$. Then $Q_X(v, v) = Q_X(u, u) + 4r$ for some $r \in \mathbf{Z}$. We wish to show that there is a primitive vector $\bar{v} \equiv v \bmod 2$ with $Q_X(\bar{v}, \bar{v}) = Q_X(u, u)$, for then Wall's theorem provides an $h \in O_X$ with $h(u) = \bar{v} \equiv v \bmod 2$. Since $H_2(X; \mathbf{Z}) \cong 2aE_8 \oplus (b+1)H$, we can find $e_1, f_1, e_2, f_2 \in H_2(X; \mathbf{Z})$ such that $Q_X(e_i, e_j) = Q_X(f_i, f_j) = 0$, and $Q_X(e_i, f_j) = \delta_{i,j}$. Let $\alpha = \frac{1}{2} Q_X(v, v)$. Applying Wall's theorem we obtain a $g \in O_X$ with $g(v) = \alpha e_1 + f_1$. If $r = 2m$ define z by

$$g(z) = e_2 - mf_2.$$

If $r = 2(m + \alpha) + 1$, define z by

$$g(z) = e_2 - mf_2 - (1+\alpha)e_1 + f_1.$$

Now set $\bar{v} = v - 2z$. Its norm $Q_X(\bar{v}, \bar{v}) = Q_X(u, u)$. Furthermore, $\bar{v}$ is primitive since if $r = 2m$ then $Q_X(g(\bar{v}), e_1) = 1$, and in the other case $Q_X(g(\bar{v}), e_1 + f_2) = 1$. Thus Lemma 8.2 follows. □

Let Sp_X denote the group of linear transformations of $H_2(X; \mathbf{Z}_2)$ which preserve the symplectic form s_X, and let $O_{+,X}$ be the subgroup which preserves the quadratic form φ_X. There is a homomorphism $\Gamma : O_X \to O_{+,X}$ induced by reduction mod 2.

Lemma 8.3. *The image* $\Gamma(O_X)$ *acts transitively on* $\varphi_X^{-1}(0)\setminus\{0\}$ *and on* $\varphi_X^{-1}(1)$.

Proof. If $u, v \in H_2(X; \mathbf{Z}_2)$ are nonzero vectors such that $\varphi_X(u) = \varphi_X(v)$ we can find primitive vectors $\bar{u}, \bar{v} \in H_2(X; \mathbf{Z})$ which reduce mod 2 to u and v. Apply Lemma 8.2 to $\bar{u}$ and $\bar{v}$. □

Before continuing, let us recall that $\mathbf{Z}_2$- quadratic spaces which correspond to a symplectic form are classified by their dimension and Arf invariant. (See [S].) Now $H_2(X; \mathbf{Z}) \equiv 2aE_8 \oplus (b+1)H$; so its Arf invariant is 0 and $(H_2(X; \mathbf{Z}_2), \varphi_X) \cong (8|a| + b + 1)H$ as a $\mathbf{Z}_2$- quadratic space. (Note that $2E_8$ has Arf invariant 0, and so is isomorphic to $8H$ over $\mathbf{Z}_2$.)

Lemma 8.4. *The homomorphism* $\Gamma : O_X \to O_{+,X}$ *is surjective.*

Proof. The main theorem of [McL] states that the image of O_X in Sp_X is either all of $O_{+,X}$ or is a symmetric group on $2n+1$ or $2n+2$ letters, where $2n = \dim(H_2(X; \mathbf{Z}_2))$. To see that it is all of $O_{+,X}$, we need to describe the embeddings of S_{2n+1} and S_{2n+2} in $O_{+,X}$. (See [LPS, p. 34].) For any m the symmetric group S_m fixes the natural symplectic form

$$\langle(\zeta_i), (\eta_i)\rangle = \sum \zeta_i \eta_i$$

on $S(m) = \{(\zeta_i) \in (\mathbf{Z}_2)^m | \sum \zeta_i = 0\}$, and S_m acts on $S(m)$ by permutation of the coordinates. Define $P_m : S(m) \to \mathbf{Z}_2$ by

$$P_m(\zeta_1, \dots, \zeta_m) = \begin{cases} 1, & \text{if } \#\{i|\zeta_i = 1\} \equiv 2 \bmod 4, \\ 0, & \text{if } \#\{i|\zeta_i = 1\} \equiv 0 \bmod 4. \end{cases}$$

Then P_m is a quadratic form on $S(m)$ with associated symplectic form $\langle\ ,\ \rangle$. When $n \equiv 0 \bmod 4$, the Arf invariant of the $\mathbf{Z}_2$- quadratic space $(S(2n+1), P_{2n+1})$ is 0; so $(S(2n+1), P_{2n+1})$ and $(H_2(X; \mathbf{Z}_2), \varphi_X)$ are isomorphic. The symmetric group S_{2n+1} only embeds in $O_{+,X}$ when $n \equiv 0 \bmod 4$, and is then given by the action of S_{2n+1} on $S(2n+1)$.

For the case of S_{2n+2}, again start with the action of S_{2n+2} on $S(2n+2)$. The diagonal, $d = (1, \dots, 1) \in S(2n+2)$ is fixed by the action of S_{2n+2}. Thus $S(2n+2)/\mathrm{span}(d)$ is a S_{2n+2}-space. If $n \equiv 3 \bmod 4$ the Arf invariant of $(S(2n+2)/\mathrm{span}(d), P_{2n+2})$ vanishes, and so $(S(2n+2)/\mathrm{span}(d), P_{2n+2})$ is isormorphic to $(H_2(X; \mathbf{Z}_2), \varphi_X)$. This gives the embedding of S_{2n+2} in $O_{+,X}$. (If $n \not\equiv 3 \bmod 4$ then S_{2n+2} does not embed in $O_{+,X}$.)

Let $c = (1, 1, 0, \dots, 0)$ and $c' = (1, 1, 1, 1, 1, 1, 0, \dots, 0) \in (\mathbf{Z}_2)^{2n+1}$. Then $c, c' \in S(2n+1)$, and $P_{2n+1}(c) = 1 = P_{2n+1}(c')$. There is no $\sigma \in S_{2n+1}$ such that $\sigma(c) = c'$. However by Lemma 8.3, $\Gamma(O_X)$ is transitive on P_{2n+1}^{-1}. Thus $\Gamma(O_X)$ is strictly larger than S_{2n+1}. A similar argument shows that the image of O_X is also strictly larger than S_{2n+2}. □

Let $2m = \dim H_2(M; \mathbf{Z}_2)$; so

$$H_2(X; \mathbf{Z}_2) = H_2(M\#S^2 \times S^2; \mathbf{Z}_2) \cong (m+1)H$$

as a $\mathbf{Z}_2$-quadratic space. Thus there is a basis $\{e_i, f_i | i = 0, \dots, m\}$ of $H_2(X; \mathbf{Z}_2)$ such that $\varphi_X(e_i) = 0 = \varphi_X(f_i)$, and $s_X(e_i, e_j) = s_X(f_i, f_j) = 0$,

$s_X(e_i, f_j) = \delta_{i,j}$. We choose this basis so that e_0 is the mod 2 reduction of $x \in H_2(S^2 \times S^2; \mathbf{Z})$ and f_0 is the mod 2 reduction of y. Let $\{\alpha_i, \beta_i\}$ be a dual basis. Define the families $R_{i,j}$ and S_i in $O_{+,X}$ by

$$\begin{aligned} &R_{i,j}(e_i) = e_j, && R_{i,j}(e_j) = e_i, \\ &R_{i,j}(f_i) = f_j && R_{i,j}(f_j) = f_i, \\ &R_{i,j}(e_k) = e_k, && R_{i,j}(f_k) = f_k \quad \text{if } k \neq i, j \end{aligned}$$

and

$$\begin{aligned} &S_i(e_i) = f_i, && S_i(f_i) = e_i, \\ &S_i(e_k) = e_k, && S_i(f_k) = f_k \quad \text{if } k \neq i. \end{aligned}$$

It follows from Lemma 8.4 that there are transformations $\widetilde{R}_{i,j}$ and $\widetilde{S}_i$ in O_X such that $\Gamma(\widetilde{R}_{i,j}) = R_{i,j}$ and $\Gamma(\widetilde{S}_i) = S_i$. By [W2, Theorem 2] each transformation in O_X is induced from a diffeomorphism of X. Hence the $R_{i,j}$ and S_i are induced from diffeomorphisms.

We will be working with $\mathrm{Sym}^*_{\mathbf{Z}_2}(H_2(X; \mathbf{Z}_2))$. Since $H_1(X; \mathbf{Z}) = 0$, the mod 2 Donaldson polynomial invariant $q_{\ell+1, u_1, X}$ can be viewed as an element of $\mathrm{Sym}^{d+2}_{\mathbf{Z}_2}(H_2(X; \mathbf{Z}_2))$, and the dual basis vectors α_i and β_j belong to $\mathrm{Sym}^1_{\mathbf{Z}_2}(H_2(X; \mathbf{Z}_2))$. Define $\alpha_i^k \in \mathrm{Sym}^k_{\mathbf{Z}_2}(H_2(X; \mathbf{Z}_2))$, given by $\alpha_i^k(e_i, \dots, e_i) = 1$, and $\alpha_i^k = 0$ on all other sets of k basis vectors. Similarly define β_i^k. The power α_i^k is not the same as the symmetric product of k copies of α_i (which is 0 if $k \geq 2$). As is pointed out by Ruan [R]

$$\alpha_i \alpha_i^k = \begin{cases} \alpha_i^{k+1}, & \text{if } i \text{ is even}, \\ 0, & \text{if } i \text{ is odd}. \end{cases}$$

Ruan further points out that $\mathrm{Sym}^p_{\mathbf{Z}_2}(H_2(X; \mathbf{Z}_2))$ is generated by monomials

$$\lambda_{R,S} = \alpha_0^{r_0} \cdots \alpha_m^{r_m} \beta_0^{s_0} \cdots \beta_m^{s_m}$$

where $R = (r_0, \dots, r_m)$ and $S = (s_0, \dots, s_m)$ with $\sum(r_i + s_i) = p$.

Theorem 8.5. *Let M be a closed simply connected spin 4-manifold with a Donaldson polynomial invariant $q_{\ell, M}$ of degree d where ℓ is odd. Let $X = M \# S^2 \times S^2$. Then*

$$q_{\ell+1, u_1, X} = \begin{cases} \epsilon_{\ell+1, X} Q_X^{(p+1)} + P_{\ell+1, X} \bmod 2 & \text{if } d = 2p, \\ P_{\ell+1, X} \bmod 2 & \text{if } d \text{ is odd}, \end{cases}$$

where $\epsilon_{\ell+1, X} \in \mathbf{Z}_2$, and $P_{\ell+1, X}$ is a sum (mod 2) *of monomials of the form $\alpha_i^{r_i} \beta_i^{s_i}$ with $r_i + s_i = d + 2$.*

Proof. Write

$$(*) \qquad q_{\ell+1, u_1, X} = \sum \epsilon_{R,S} \lambda_{R,S} \in \mathrm{Sym}^{d+2}_{\mathbf{Z}_2}(H_2(X; \mathbf{Z}_2))$$

where $\lambda_{R,S}$ are monomials as above and $\epsilon_{R,S} = 0$ or 1. Suppose that a monomial $\lambda_{R,S}$ in this expression has some $r_j \geq 2$ and $\epsilon_{R,S} = 1$. Then there are vectors $v_1, \ldots, v_{d+2-r_j}$ such that each v_k is a basis vector e_i or f_i but not e_j, and $\lambda_{R,S}(e_j, \ldots, e_j, v_1, \ldots, v_{d+2-r_j}) = 1$. Since all of the other monomials $\lambda_{R',S'}$ vanish on $(e_j, \ldots, e_j, v_1, \ldots, v_{d+2-r_j})$, we have

$$q_{\ell+1,u_1,X}(e_j, \ldots, e_j, v_1, \ldots, v_{d+2-r_j}) = 1.$$

Now $q_{\ell+1,u_1,X}$ is invariant under diffeomorphisms of X, and the transformations $R_{i,j}$ are all induced from diffeomorphisms. Applying $R_{0,j}$ we get

$$1 = q_{\ell+1,u_1,X}(e_0, \ldots, e_0, R_{0,j}(v_1), \ldots, R_{0,j}(v_{d+2-r_j})).$$

Since $e_0 = x$, unless all the $R_{0,j}(v_i) = f_0 = y$, Theorem 8.1 implies that this equals 0. The exceptional case where $R_{0,j}(v_i) = f_0$, $i = 1, \ldots, d+2-r_j$, occurs when $v_i = f_j$, $i = 1, \ldots, d+2-r_j$, in other words when $\lambda_{R,S} = \alpha_i^{r_i}\beta_i^{s_i}$ with $r_i + s_i = d+2$. This means that in $(*)$ no monomial $\lambda_{R,S}$ for which any $r_j \geq 2$ can have a nonzero coefficient, except for those monomials of the type which occur in $P_{\ell+1,X}$. In particular, if any $\lambda_{R,S}$ with nonzero coefficient is not of the special type $\alpha_i^{r_i}\beta_i^{s_i}$, then counting its exponents we have

$$d + 2 = \sum(r_i + s_i) \leq 2m + 2 = \dim H_2(X; \mathbf{Z}_2).$$

Thus without loss we may assume that $d \leq 2m$.

Suppose that there is a monomial $\lambda_{R,S}$ with nonzero coefficient in $q_{\ell+1,u_1,X}$ for which $r_j = 1$ but $s_j = 0$. Then, as above, there are standard basis vectors $v_1, \ldots, v_{d+1}$ in $H_2(X; \mathbf{Z}_2)$, none of which are e_j or f_j and such that

$$q_{\ell+1,u_1,X}(e_j, v_1, \ldots, v_{d+1}) = \lambda_{R,S}(e_j, v_1, \ldots, v_{d+1}) = 1.$$

Again applying $R_{0,j}$ we get

$$1 = q_{\ell+1,u_1,X}(e_0, R_{0,j}(v_1), \ldots, R_{0,j}(v_{d+1})).$$

But $e_0 = x$ and $R_{0,j}(v_k) \neq y$ for $k = 1, \ldots, d+1$; so Theorem 8.1 implies that

$$q_{\ell+1,u_1,X}(e_0, R_{0,j}(v_1), \ldots, R_{0,j}(v_{d+1})) = 0,$$

a contradiction. Thus for each monomial $\lambda_{R,S}$ not of the form $\alpha_i^{r_i}\beta_i^{s_i}$ appearing with nonzero coefficient in $q_{\ell+1,u_1,X}$, if $r_j \neq 0$, then also $s_j \neq 0$. Such a $\lambda_{R,S}$ has the form

$$\alpha_{i_1}\beta_{i_1}\cdots\alpha_{i_k}\beta_{i_k}$$

where $2k = d+2$. This already shows that d must be even if $q_{\ell+1,u_1,X} \neq P_{\ell+1,X}$. Let $d = 2p$; so $k = p+1$.

Let $\sigma = \tau_t \cdots \tau_1 \in S_{m+1}$ be a permutation, written as a product of transpositions. For any transposition $t = (i, j)$ let $R_\tau = R_{i,j}$. Let $R = R_{\tau_t} \circ \cdots \circ R_{\tau_1}$. If $\alpha_{i_1}\beta_{i_1}\cdots\alpha_{i_k}\beta_{i_k}$ appears with nonzero coefficient in $q_{\ell+1,u_1,X}$ then

$$\begin{aligned}
1 &= q_{\ell+1,u_1,X}(e_{i_1}, f_{i_1}, \ldots, e_{i_k}, f_{i_k}) \\
&= q_{\ell+1,u_1,X}(R(e_{i_1}), R(f_{i_1}), \ldots, R(e_{i_k}), R(f_{i_k})) \\
&= q_{\ell+1,u_1,X}(e_{\sigma(i_1)}, f_{\sigma(i_1)}, \ldots, e_{\sigma(i_k)}, f_{\sigma(i_k)}).
\end{aligned}$$

Thus $\alpha_{\sigma(i_1)}\beta_{\sigma(i_1)}\cdots\alpha_{\sigma(i_k)}\beta_{\sigma(i_k)}$ also appears with nonzero coefficient. Since the α_i and β_i commute, if we sum these expressions over $\sigma \in S_{m+1}$ then each expression appears $k! = (p+1)!$ times. Thus if $q_{\ell+1,u_1,X} \neq P_{\ell+1,X}$, then

$$q_{\ell+1,u_1,X} = P_{\ell+1,X} + \frac{1}{k!}\sum_{\sigma\in S_{m+1}} \alpha_{\sigma(1)}\beta_{\sigma(1)}\cdots\alpha_{\sigma(k)}\beta_{\sigma(k)} \equiv P_{\ell+1,X} + Q_X^{(p+1)} \mod 2.$$

We have already shown that if $d > 2m$ then $q_{\ell+1,u_1,X} = P_{\ell+1,X}$, and furthermore, $Q_X^{(p+1)} \equiv 0 \mod 2$ in that case. □

Proof of Theorem 1.5. If $z_1, \dots, z_d \in H_2(M;\mathbf{Z})$, then Theorems 1.1 and 8.5 imply

$$\begin{aligned} q_{\ell,M}(z_1,\dots,z_d) &\equiv q_{\ell+1,u_1,X}(z_1,\dots,z_d,x,y) \\ &\equiv (\epsilon_{\ell+1,X}Q_X^{(p+1)} + P_{\ell+1,X})(z_1,\dots,z_d,x,y) \\ &\equiv \epsilon Q_M^{(p)}(z_1,\dots,z_d) \mod 2, \end{aligned}$$

since $P_{\ell+1,X}(z_1,\dots,z_d,x,y) \equiv 0$. □

Since the form $Q_M^{(p)}$ lies in $\mathrm{Sym}^{2p}_{\mathbf{Z}}(H_2(M;\mathbf{Z}))$ and since $Q_M^{(p)} \equiv 0 \bmod 2$ for $2p > \mathrm{rank}(H_2(M;\mathbf{Z}))$, we have

Theorem 8.6. *Suppose that M is a closed simply connected spin 4-manifold with a Donaldson polynomial $q_{\ell,M}$ of degree d, where ℓ is odd. If $b_M^+ \equiv 1 \bmod 4$, or if $d > \mathrm{rank}(H_2(M;\mathbf{Z}))$ then $q_{\ell,M} \equiv 0 \bmod 2$.* □

We now need to refer to a recent theorem of Y. Ruan [R] which, when combined with Theorem 1.5, will prove Theorem 1.6.

(8.7) **Theorem** (Ruan). *Let M be a closed simply connected spin 4-manifold with a Donaldson polynomial $q_{\ell,M}$ of degree d, where ℓ is odd. Then the symmetric product $Q_M q_{\ell,M} \equiv 0 \bmod 2$.*

If $b_M^+ \equiv 1 \bmod 4$, then Theorem 8.7 follows as well from our Theorem 8.6. Furthermore, note that

$$Q_M Q_M^{(p)} = (p+1)Q_M^{(p+1)} \mod 2.$$

Also note that when b_M^+ is odd and M is spin, the mod 2 intersection form on $H_2(M;\mathbf{Z}_2)$ is just the standard form on the direct sum of an odd number, say k, hyperbolic pairs. Thus mod 2, $Q^{(j)}$ will take on nonzero values for $j \le k$ and will be identically 0 (mod 2) for $j \ge k+1$.

Theorem 8.8. *Let M be a simply connected spin 4-manifold with a Donaldson invariant $q_{\ell,M}$, ℓ odd and with $b_M^+ \equiv 7 \bmod 8$. Then, $q_{\ell,M} \equiv 0 \bmod 2$.*

Proof. If $b_M^+ \equiv 7 \bmod 8$, then the Donaldson invariant $q_{\ell,M}$, ℓ odd has degree $4p$ for some p. If $Q_M^{(2p)} \equiv 0 \bmod 2$, then Theorem 1.5 implies $q_{\ell,M} \equiv 0 \bmod 2$. If $Q_M^{(2p)} \not\equiv 0 \bmod 2$, then our comments above imply that also

$Q_M^{(2p+1)} \neq 0 \bmod 2$. Theorem 1.5 implies $q_{\ell,M} \equiv \epsilon_{\ell,M} Q_M^{(2p)} \bmod 2$. Then by Ruan's result (Theorem 8.7)

$$0 \equiv Q_M q_{\ell,M} \equiv \epsilon_{\ell,M} Q_M Q_M^{(2p)} = (2p+1)\,\epsilon_{\ell,M} Q_M^{(2p+1)} \mod 2$$

so that $\epsilon_{\ell,M} = 0$ and again $q_{\ell,M} \equiv 0 \bmod 2$. □

Theorem 1.6 now follows from Theorems 8.6 and 8.8.

In summary, the mod 2 Donaldson polynomials vanish except possibly when either ℓ is even or when ℓ is odd, $b_M^+ \equiv 3 \bmod 8$, and the degree $d \leq \operatorname{rank}(H_2(M; \mathbf{Z}))$. In [DK, p. 417] it is pointed out that if M is the $K3$ surface (so $b_M^+ = 3$) then $q_{5,M} = Q_M^{(7)}$. Known examples lead to the following conjecture.

Conjecture 8.9. *Let M be a simply connected spin 4-manifold with a Donaldson invariant $q_{\ell,M}$, ℓ odd. Then $q_{\ell,M} \not\equiv 0 \bmod 2$ if and only if $b_+ = 3$ and $\ell = 5$.*

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Department of Mathematics, Michigan State University, East Lansing, Michigan 48824

E-mail address: ronfint@math.msu.edu

Department of Mathematics, University of California, Irvine, California 92717

E-mail address: rstern@math.uci.edu