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IMMERSED 2-SPHERES IN 4-MANIFOLDS: LECTURES FOR THE XI ELAM AUGUST 1993

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ABSTRACT. The purpose of these lectures is to highlight what is currently known about simply-connected smooth 4-manifolds and to discuss very recent advances in the computations of the Donaldson polynomial invariants. In particular we will focus on the work of Fintushel-Stern where they attempt to understand the behavior of the Donaldson invariants in the presence of immersed spheres. There are other techniques that can be used to broaden the understanding of smooth 4-manifolds; most importantly the beautiful recent results of Kronheimer-Mrowka [KM1, KM2, KM3] which are derived from a study of singular connections. However we will concentrate on our more elementary techniques and ideas and, at the end, discover that they too lead us to a better understanding of smooth 4-manifolds.

1. OVERVIEW

Our problem is to classify closed smooth 4-manifolds. To avoid the group theoretic problems arising from the fact that any finitely presented group can occur as the fundamental group of a smooth closed 4-manifold, we assume our manifolds (unless otherwise specified) are simply-connected. Most of the classical invariants for 4-manifolds are encoded by the intersection form Q_X. This form is an integral unimodular symmetric bilinear pairing

$$Q_X : H_2(X;Z) \otimes H_2(X;Z) \to Z$$

obtained by representing homology classes as oriented embedded submanifolds and counting intersections with signs; it is Poincaré dual to the pairing given by cup product. From the intersection form one can determine its **rank** (which is the second Betti number $b_2(X) = \mathrm{rank} H_2(X;Z) = b^+(X) + b^-(X)$), its **signature** $= \sigma(X) = b^+(X) - b^-(X)$ (where $b^\pm(X)$ are the dimensions of the $\pm$-eigenspaces of Q_X) and its **type** (which is **even** if $Q_X(x,x) \equiv 0$ mod 2 for all x, and **odd** otherwise). A form is **definite** provided one of $b^\pm(X)$ vanishes, and

The author thanks ELAM, UNAM and CIMAT for their kind and generous hospitality during this XI ELAM. Not all the material presented in these printed lectures was orally presented. In particular the material in Section 5 is included for the sake of completeness.

1.4. Wall's Stability Theorem [W2, W3]. *If two simply-connected closed smooth 4-manifolds have isomorphic intersection forms, then they are "stably diffeomorphic", that is, they become diffeomorphic after connect summing with a number of copies of $S^2 \times S^2$. (This number is unspecified.)*

1.5. Wall's Diffeomorphism Theorem [W2, W3]. *If X is a simply connected smooth 4-manifold with Q_X indefinite, then the homology induced map*

$$Diff(X\#S^2 \times S^2) \to O < H_2(X\#S^2 \times S^2; Z), Q_X >$$

is an isomorphism.

Given two homeomorphic simply-connected closed smooth 4-manifolds X and Y, it is then an interesting problem to determine the minimal integer $k(X,Y)$ for which $X\#_{k(X,Y)}S^2 \times S^2$ is diffeomorphic to $Y\#_{k(X,Y)}S^2 \times S^2$. Despite some heavy input from gauge theory ([D7, FS5]) it is still reasonable to conjecture that $k(X,Y) \leq 1$

The fundamental questions concerning 4-manifolds still remain.

1.6. Existence. *Determine the unimodular, symmetric, bilinear integral forms that are realized as the intersection form of a closed, simply-connected smooth 4-manifold.*

1.7. Uniqueness. *Given a closed, simply-connected smooth 4-manifold M, determine the distinct smooth structures on M.*

These questions are the subject of active current research.

1.1. Existence. The spectacular work of Donaldson and its derivatives [D1, D3, FS1, FS2] has shown that there are serious restrictions on the intersection form of a closed, simply-connected 4-manifold and has made progress towards a verification of the following existence conjecture:

1.8. 11/8 Conjecture. *The intersection form of a closed, simply-connected smooth 4-manifold must be either*

- *diagonalizable (over Z), or*
- *even and $\frac{b_2(X)}{|\sigma(X)|} \geq \frac{11}{8}$.*

All complex surfaces satisfy these two conditions. Also, all diagonalizable forms are realized by the connected sums of $\pm CP^2$. Although the classification of definite integral forms is an active area of research and far from an accomplished feat, nature has a way of taking care of all this. Donaldson's first monumental piece of work [D1] shows that no non-diagonalizable definite form can be realized as the intersection form of a closed smooth 4-manifold; his

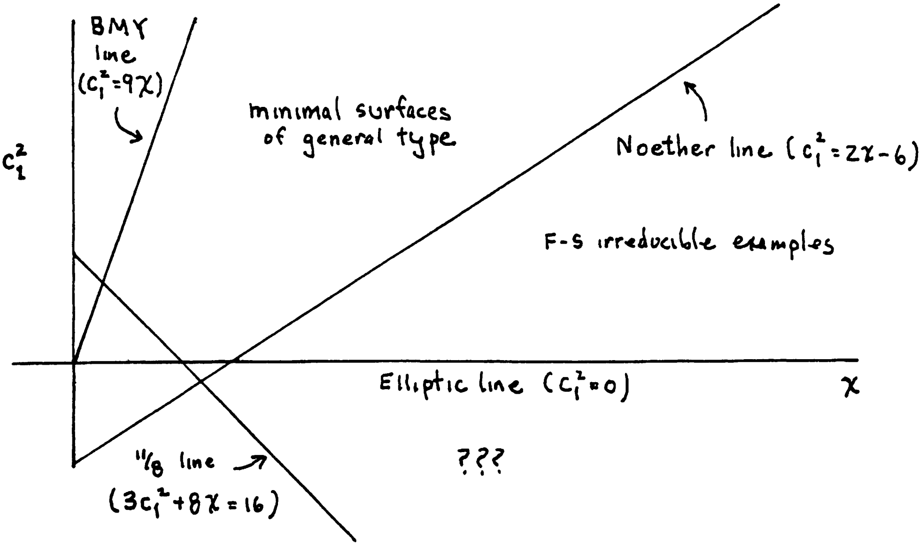

FIGURE 1. Geography of 4-manifolds

$Sym^2(H_2(X;\mathbb{Z}))$, the Donaldson polynomials $q_{X,d}$ are defined for manifolds with odd $b^+ > 1$ and are symmetric polynomials of degree $d \equiv \frac{1}{2}(b^+ + 1)$ mod 4 in the 2-dimensional homology of X, i.e. $q_{X,d} \in Sym^d(H_2(X;\mathbb{Z}))$. Equivalently, these can be regarded as homogeneous polynomial functions

$$q_{X,d} : H_2(X,\mathbb{Z}) \to \mathbb{Z}.$$

Formally, one can define the polynomial q_d by the formula

$$q_{X,d}(h) = \langle\, \mu(h)^d,\ [\mathcal{M}_d]\,\rangle,$$

where $\mathcal{M}_d$ is the instanton moduli space of dimension $2d$ (depending on a choice of a Riemannian metric on X) and μ is a natural map from $H_2(X)$ to $H^2(\mathcal{M}_d)$. Because $\mathcal{M}_d$ is usually non-compact, this pairing needs to be more carefully interpreted before it can be regarded as well-defined: a prescription for evaluation was given in [D6], subject to the constraint $2d > 3(b^+ + 1)$, and by various tricks the construction has since been extended [FM2, MM1] so as to remove this restriction. It was the idea of Kronheimer and Mrowka [KM3] to extend this defintion to all degrees $d \equiv \frac{1}{2}(b^+ + 1)$ mod 2 by defining

$$2q_{X,d-2}(h) = \langle\, \mu(h)^{d-2}\nu,\ [\mathcal{M}_d]\,\rangle,$$

During the later half of 1993 the paucity of computations took a dramatic shift on two different fronts.

First, Kronheimer and Mrowka announced in [KM3] a very general Structure Theorem for the analytic function q_X.

Theorem 1.10. *If X is a simply-connected 4-manifold of simple type, then there exist finitely many cohomology classes $K_1, \ldots, K_p \in H^2(X, \mathbb{Z})$ and non-zero rational numbers $a_1, \ldots, a_p$ such that*

$$q_X = \exp\left(\frac{Q}{2}\right) \sum_{s=1}^{p} a_s e^{K_s}$$

as analytic functions on $H_2(X, \mathbb{R})$. Here Q is the intersection form, regarded as a quadratic function. Each of the 'basic classes' K_s is an integral lift of $w_2(X)$.

Theorem 1.11. *Let X be a simply-connected 4-manifold of simple type and let $\{K_s\}$ be the set of basic classes given by Theorem 1.10. If Σ is any smoothly embedded, essential connected surface in X with normal bundle of non-negative degree, then the genus of Σ satisfies the lower bound*

$$2g - 2 \geq \Sigma \cdot \Sigma + \max_s K_s \cdot \Sigma.$$

These outstanding results are an outgrowth of their earlier work concerning singular connections [KM1, KM2]

Second, Fintushel and Stern introduced the notion of a **rational blow-down** of a smooth 4-manifold and announced results concerning the relationship between the Donaldson polynomials of a manifold and its rational blow-downs which allowed the computation of the Donaldson series for many classes of 4-manifolds. At first this construction was introduced to study regular elliptic surface $E(\chi; m_1, \ldots m_r)$ with holomorphic Euler characteristic $\chi \geq 2$ and r multiple fibers of multiplicities $m_1, \ldots, m_r$. To insure that X is simply connected we must have $r \leq 2$, and in the case $r = 2$ the two multiplicities must be coprime. First, starting only from the computation of the Donaldson series for the $K3$ surface, they showed

$$q_{E(\chi)} = \exp\left(\frac{Q}{2}\right) (\sinh F)^{\chi - 2}$$

and

$$q_{E(\chi)\#\overline{\mathbb{CP}}^2} = \exp\left(\frac{Q}{2}\right) (\sinh F)^{\chi - 2} \cosh E$$

where F is the cohomology class of the fiber and E is the cohomology class of the exceptional divisor. From this later formula follows a general blow-up formula for all elliptic (and other) to prove (independent of (but motivated by) the work [KM3]) Theorem 1.10. However, their version of Theorem 1.11 differs. In particular, they prove

Theorem 1.12. *Let X be a simply-connected 4-manifold as above and let $\{K_s\}$ be the set of basic classes given by Theorem 1.10. If Σ is any immersed 2-sphere with $p \neq 0$ positive double points representing a non-trivial homology class in X (of any square), then*

$$2p - 2 \;\geq\; \Sigma \cdot \Sigma \;+\; \max_s K_s \cdot \Sigma.$$

Further, if $p = 0$ and Σ represents a homology class which is either K_i or $K_i + K_j$ for some $i \neq j$, then

$$0 \;\geq\; \Sigma \cdot \Sigma \;+\; \max_s K_s \cdot \Sigma.$$

Otherwise,

$$-2 \;\geq\; \Sigma \cdot \Sigma \;+\; \max_s K_s \cdot \Sigma.$$

Thus, on the one hand, for homology classes with positive self-intersection Kronheimer-Mrowka's Theorem 1.11 is stronger than Fintushel-Stern's Theorem 1.12 since Kronheimer-Mrowka deal with genus of representatives and Fintushel-Stern deal with the number of positive double points. However, Theorem 1.12 also covers homology classes with negative self-intersection.

In the remaining lectures we will introduce the ideas that go into this latter work of Fintushel-Stern. Neither will the statements of theorems be their most general nor will the proofs be complete. We refer the interested attendee to the original articles [FS8, FS9](which are, at this very moment, being written). Some of these results are also obtained by Kronheimer-Mrowka [KM3] through their study of singular connections. In particular, one important piece of their proof of Theorem 1.10 is a computation of the Donaldson series for elliptic surfaces without multiple fibers and their blowups (Theorem 2.1). They too prove a recursion formula, but for surfaces with large positive self-intersection.

2. Three Gauge-Theoretic Results

The whole theory developed by Fintushel-Stern depends upon three seemingly innocuous gauge theoretic results.

The first result generalizes the computation of the Donaldson series $q_{K3} = \exp\left(\frac{Q}{2}\right)$ for the $K3$ surface to the elliptic surfaces $E(n)$, $n \geq 2$, with holomorphic Euler characteristic n and with no multiple fibers. Recall that the $K3$ surface is diffeomorphic to $E(2)$. It turns out that this is rather easy and could have been accomplished shortly after the computation for the $K3$ surface. Simultaneous to this computation we will compute the Donaldson series for the blow-up of these elliptic surfaces, i.e. for $E(n)\#\overline{\mathbb{CP}}^2$.

and if $p = 4k$ or $p = 4k + 2$, then

$$q_{X(4k),d}(x_1, \ldots, x_d) = q_{X,d}(x_1, \ldots, x_d) + 2\sum_{t=1}^{k} \frac{1}{(2t)!} q_{X,c_{2t},d+2t}(x_1, \ldots, x_d, z_{1,2t}, \ldots, z_{2t,2t}) + q_{X,c_0,d}(x_1, \ldots, x_d),$$

where $z_{2s-1,t} = u_0 - (t-2s+1)u_t$, $z_{s,t} = u_0 + (t-2s+1)u_t$, $w_{2t+1} = PD(\sum_{j=0}^{2t+1} u_j + u_{2t+3} + u_{2t+5} + \cdots + u_{p-2})$, $w_{2t} = -PD(u_1 + u_3 + \cdots u_{2t-1})$, *and* $c_{2t} = -PD(\sum_{j=0}^{2t} u_j + u_{2t+2} + u_{2t+4} + \cdots + u_{p-2})$.

Note that the embedding of $B(p)$ in X is important as can be seen through the $SO(3)$ Donaldson invariants $q_{X,w,d}$.

Putting the facts that $b^+(X(p)) = b^+(X)$ and every odd $b^+ = 2k - 1$ can be realized by the elliptic $E(k)$ together with the experimental fact that these elliptic surfaces have the largest Euler characteristic amongst all irreducible 4-manifolds with a given b^+, it is reasonable to conjecture that all irreducible simply-connected smooth 4-manifolds can be obtained from the $E(k)$ by a sequence of blow-ups and rational blow downs. In any event, it would be interesting to characterize those manifolds so obtained for their Donaldson series can be computed by using 2.1 and 2.2.

The last gauge theoretic result has the most powerful consequences on the structure of the Donaldson series and on the the minimial number of positive double points of an immersed surface representing a given homology class. In particular, the following result concerning embedded spheres of negative self-intersection, together with Theorem 2.1, **formally** implies the structure theorems and the theorems concerning the number of positive double points given in the first lecture. We will discuss these formalities in our fourth lecture. It seems rather amazing that such a result concerning embedded spheres implies the existence of the so-called basic classes for simply-connected smooth 4-manifolds.

Theorem 2.3. *Let X be a simply-connected 4-manifold with simple type which contains an embedded 2-sphere S representing a homology class $[S]$ with $Q(S,S)$ negative. Then there exists constants $A_{i,k}$ and $B_{j,k}$ depending only on $Q([S],[S])$ such that if $Q([S],[S]) = -(2k+1)$, then*

$$\begin{aligned} q_{X,d+2k-1}(x_1, \ldots, x_d, [S]^{2k-1}) = \\ A_{0,k} q_{X,[S],d}(x_1, \ldots, x_d) + A_{1,k} q_{X,d+1}(x_1, \ldots, x_d, [S]) \\ + A_{2,k} q_{X,d+3}(x_1, \ldots, x_d, [S]^3) + A_{3,k} q_{X,d+5}(x_1, \ldots, x_d, [S]^5) \\ + \ldots A_{k-1,k} q_{X,d+(2k-3)}(x_1, \ldots, x_d, [S]^{2k-3}) \end{aligned}$$

and if $n \equiv 1 \mod 2$ we can write

$$\frac{1}{(2d+1)!}q_{X,2d+1} =$$

$$c_1\frac{Q^d}{2^d d!}F + c_3\frac{Q^{d-1}}{2^{d-1}(d-1)!}F^3 + \cdots + c_{2t+1}\frac{Q^{d-t}}{2^{d-t}(d-t)!}F^{2t+1} + \cdots + c_{2d+1}F^{2d+1}.$$

What is important here is that the coefficients c_j are independent of the degree of the Donaldson polynomial. This can be seen in several different ways and was first observed by Peter Kronheimer. Here we outline a proof of this fact more in the spirit of these lectures.

A very special case of our Theorem 2.3 applies to a 4-manifold with an embedded 2 sphere T with $Q([T],[T]) = -2$ (of which there are many in surfaces with deformations, in particular the $E(n)$). This is in fact a theorem originally due to Danny Rubermann and began our interest in spheres of negative square.

Theorem 3.1. *Suppose that T is an embedded sphere with $Q([T],[T]) = -2$. Let*

$$x_1, \ldots, x_d \in H_2(X;\mathbb{Z})$$

be homology classes that are orthogonal to $[T]$. Then

$$q_{X,d+2}(x_1,\ldots x_d,[T],[T]) = 2q_{X,[T],d}(x_1,\ldots,x_d)$$

To see the utility of this theorem in verifying that the coefficients c_j above are independent of the degree of the Donaldson polynomials, assume that $n \equiv 0 \mod 2$ and take a T with $F\cdot[T] = 0$ and write

$$\frac{1}{(2d-2)!}q_{X,[T],2d-2} = c_0'\frac{Q^{d-1}}{2^{d-1}(d-1)!} + c_2'\frac{Q^{d-2}}{2^{d-2}(d-2)!}F^2 + \ldots$$

Now suppose $v_{2d-2} \in [T]^\perp \cap F^\perp$ is a vector of length $2d-2$ consisting of elements of $H_2(X;\mathbb{Z})$. Then

$$\begin{aligned} q_{X,2d}(v_{2d-2},[T]^2) &= (2d)!c_0\frac{Q^d}{2^d d!}(v_{2d-2},[T]^2) \\ &= \frac{(2d)!c_0}{2^d d!}\frac{d\cdot 2!\cdot(2d-2)!}{(2d)!}Q^{d-1}(v_{2d-2})Q([T]^2) \\ &= \frac{-2c_0(2d-2)!}{2^{d-1}(d-1)!}Q^{d-1}(v_{2d-2}) \end{aligned}$$

But, by Theorem 3.1

$$q_{X,2d}(v_{2d-2},[T]^2) = 2q_{X,[T],2d-2}(v_{2d-2}) = 2(2d-2)!\frac{c_0'}{2^{d-1}(d-1)!}Q^{d-1}(v_{2d-2}).$$

Thus $c_0 = c_0''$. In a similar fashion we get that $c_{2j} = c_{2j}''$ for all j which was what we were after.

This technique of course generalizes in several directions. But for our purposes we now have that

$$\begin{aligned} q_{E(2n)} &= \sum \frac{q_{E(2n),2d}}{(2d)!} \\ &= c_0 \sum \frac{Q^d}{2^d d!} + c_2 F^2 \sum \frac{Q^{d-1}}{2^{d-1}(d-1)!} + c_4 F^2 \sum \frac{Q^{d-2}}{2^{d-2}(d-2)!} + \cdots \end{aligned}$$

so that

$$q_{E(2n)} = \exp(\frac{Q}{2}) \sum c_{2i} F^{2i}.$$

Similarly,

$$q_{E(2n-1)} = \exp(\frac{Q}{2}) \sum c_{2i-1} F^{2i-1}.$$

It is here that the beauty of forming the Donaldson series exposes itself; it takes care of all the excess combinatorial factors.

3.2. The leading coefficient for $E(3)$. To begin an induction, we need a beginning. We now know that

(1) $$q_{E(3)} = \exp(\frac{Q}{2}) \sum c_{2i-1} F^{2i-1}.$$

To begin, we claim that $c_1 = 1$. First, it is shown in [FS6] that if $c \in H^2(E(3); \mathbb{Z})$ satisfies $< c, f > \neq 0$ (where f is the homology class of the fiber), then the 0-degree Donaldson invariant is

(2) $$q_{E(3),c,0} = (-1)^{\frac{c^2+F\cdot c}{2}}$$

Secondly, another easy special case of our Theorem 2.3 applies to the section of $E(3)$ which is an embedded sphere S in $E(3)$ with $Q([S],[S]) = -3$.

Theorem 3.2. *Suppose that S is an embedded sphere with $Q([S],[S]) = -3$. Let $x_1, \ldots x_d \in H_2(X;\mathbb{Z})$ be homology classes that are orthogonal to $[S]$. Then*

$$q_{X,d+1}(x_1, \ldots x_d, [S]) = -q_{X,[S],d}(x_1, \ldots x_d)$$

Together, (1),(2), and 3.2 imply that $c_1 = 1$. So

(3) $$q_{E(3)} = \exp(\frac{Q}{2})(F + c_3 \frac{F^3}{3!} + c_5 \frac{F^5}{5!} + \cdots + c_{2t+1} \frac{F^{2t+1}}{(2t+1)!} + \cdots)$$

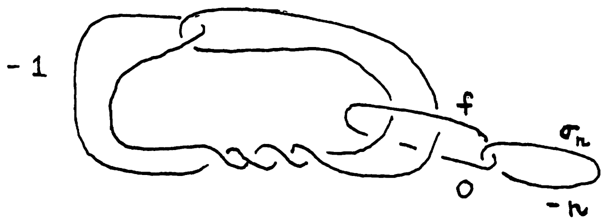

FIGURE 3. $C(n)$

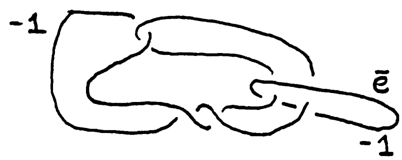

FIGURE 4. $C(2,3,11)$

which is obtained from $\Sigma(2,3,11)$ by attaching two 2-handles along the framed link in Figure 3. These two 2-handles represent the homology class f of the fiber and the homology class s_n of a section.

The third piece is the Milnor fiber $B(2,3,6n-13)$. The fact that $B(2,3,11) \cup C(n) \cup B(2,3,6n-13)$ is diffeomorphic to $E(n)$ can be seen via Kirby calculus and will not be discussed here.

Let $W(n) = C(n) \cup B(2,3,6n-13)$ with $\partial W(n) = -\Sigma(2,3,11)$, so that

$$E(n) = B(2,3,11) \cup W(n).$$

Now $\partial B(2,3,11)$ bounds another interesting manifold $C(2,3,11)$ obtained as the union $C(2,3,11) = B(2,3,5) \cup D$, where D is constructed by attaching one 2-handle to $\Sigma(2,3,5)$ along the framed link in Figure 4.

The claim is that

$$C(2,3,11) \cup W(n) = E(n-1)\#\overline{\mathbb{C}}P^2.$$

Again, this can be seen via Kirby-calculus and will not be dicsussed here.

We need to keep track of some important homology classes in our various pieces. First, let $f \in H_2(E(n);\mathbb{Z})$ denote the homology class of the fiber and let $s_n \in H_2(E(n);\mathbb{Z})$ denote the homology class of a section. Let $\bar{e} \in H^2(D;\mathbb{Z})$ denote the generator for the homology class of D so that $Q(\bar{e},\bar{e}) = -1$. It is a fact that $s_{n-1} = s_n + e$ and that $\bar{e} = e - f$.

Proof. Now

$$q_{E(2)\#\overline{\mathbb{CP}}^2,2}(\overline{e},\sigma_3) = 2!\{(1+\frac{Q}{2})(1+\frac{E^2}{2})\}(\overline{e},\sigma_3) = (Q+E^2)(\overline{e},\sigma_3) = -2$$

and

$$q_{E(3),3}(x_1,x_2,\sigma_3) = 3!\frac{Q}{2}F(x_1,x_2,\sigma_3) = \frac{3!}{2}\frac{2}{3!}Q(x_1,x_2)F(\sigma_3) = 2$$

Thus

$$q_{C(2,3,11),1}(\overline{e})\cdot q_{W(3),1}(\sigma_3) = q_{E(2)\#\overline{\mathbb{CP}}^2,2}(\overline{e},\sigma_3) = -2$$

and

$$q_{B(2,3,11),2}(x_1,x_2)\cdot q_{W(3),1}(\sigma_3) = q_{E(3),3}(x_1,x_2,\sigma_3) = +2.$$

However $B(2,3,11)\subset E(2)$, so that since $q_{E(2)} = \exp(\frac{Q}{2})$, then

$$q_{B(2,3,11),2}(x_1,x_2) = q_{E(2),2}(x_1,x_2) = \alpha_3.$$

Thus $q_{C(2,3,11),1}(\overline{e}) = -\alpha_3$ $\square$

If $\alpha\in FH_*(\Sigma)$, let $\overline{\alpha}\in FH_{5-*}(-\Sigma)$ be its dual.

Lemma 3.4. $q_{W(3),4\ell-1}(\sigma_3^{4\ell-1}) = 2^{4\ell-1}c_{4\ell-1}\cdot\overline{\alpha}_7$.

Proof.

$$\begin{aligned}
&q_{W(3),4\ell-1}(\sigma_3^{4\ell-1})\\
&\quad= q_{W(3),4\ell-1}(\sigma_3^{4\ell-1})\cdot q_{B(2,3,11),2}(x_1,x_2) = q_{E(3),4\ell+1}(x_1,x_2,\sigma_3^{4\ell-1})\\
&\quad= (4\ell+1)!c_{4\ell-1}\frac{Q}{2}\frac{F^{4\ell-1}}{(4\ell-1)!}Q(x_1,x_2,\sigma_3^{4\ell-1})\\
&\quad= (4\ell+1)!c_{4\ell-1}\frac{1}{2(4\ell-1)!}\frac{2!(4\ell-1)!}{(4\ell+1)!)}Q(x_1,x_2)2^{4\ell-1}\\
&\quad= 2^{4\ell-1}c_{4\ell-1}.
\end{aligned}$$

$\square$

Lemma 3.5. $q_{C(2,3,11),3}(\overline{e}^3) = 2\cdot\alpha_7$.

Proof.

$$\begin{aligned}
&q_{E(2)\#\overline{\mathbb{CP}}^2,4}(\overline{e}^3,\sigma_3)\\
&= 4!(a_2\frac{Q}{2}\frac{E^2}{2}+a_4\frac{E^4}{4!})(\overline{e},\sigma_3) = 6a_2\frac{3!2}{4!}Q(\overline{e}^2)Q(e,\overline{e})Q(e,\sigma_3)+a_4(-1)^3(2)\\
&= 6a_2-2a_4 = 4.
\end{aligned}$$

Proof.

$$q_{W(3),4\ell+1}(\sigma_3^{4\ell-1}, pt)\cdot q_{B(2,3,11),1}(\overline{e}) = q_{E(2)\#\overline{\mathbb{C}P}^2,4\ell+2}(\overline{e},\sigma_3^{4\ell-1},pt)$$
$$= 2q_{E(2)\#\overline{\mathbb{C}P}^2,4\ell}(\overline{e},\sigma_3^{4\ell-1}) = -2^{4\ell}a_{4\ell}$$

But also

$$q_{E(2)\#\overline{\mathbb{C}P}^2,4\ell+2}(\overline{e},\sigma_3^{4\ell-1},pt)$$
$$= q_{C(2,3,11),1}(\overline{e})\cdot q_{W(3),4\ell+1}(\sigma_3^{4\ell-1},pt) = -\alpha_3\cdot q_{W(3),4\ell+1}(\sigma_3^{4\ell-1},pt)$$
$$= -2^{4\ell}c_{4\ell-1}$$

Thus $c_{4\ell-1} = a_{4\ell}$.

Similarly,

$$q_{W(3),4\ell+1}(\sigma_3^{4\ell+1})\cdot q_{B(2,3,11),1}(\overline{e}) = q_{E(2)\#\overline{\mathbb{C}P}^2,4\ell+2}(\overline{e},\sigma_3^{4\ell+1}) = -2^{4\ell+1}a_{4\ell+2}$$

and this also equals $(-1)2^{4\ell+1}c_{4\ell+1}$. Thus $c_{4\ell+1} = a_{4\ell+2}$.

Further,

$$q_{W(3),4\ell+1}(\sigma_3^{4\ell-1},pt)\cdot q_{B(2,3,11),3}(\overline{e}^3) =$$
$$= 2q_{E(2)\#\overline{\mathbb{C}P}^2,4\ell+2}(\overline{e},\sigma_3^{4\ell-1})$$
$$= 2^{4\ell}(3a_{4\ell} - a_{4\ell+2})$$

and

$$q_{W(3),4\ell+1}(\sigma_3^{4\ell+1},pt)\cdot q_{B(2,3,11),3}(\overline{e}^3) = (2)(2^{4\ell}c_{4\ell-1}) = 2^{4\ell+1}c_{4\ell-1} = 2^{4\ell+1}a_{4\ell}$$

so that $2a_{4\ell} = 3a_{4\ell} - a_{4\ell+2}$ and $a_{4\ell} = a_{4\ell+2}$.

Similarly,

$$q_{W(3),4\ell+3}(\sigma_3^{4\ell+1},pt)\cdot q_{B(2,3,11),3}(\overline{e}^3) =$$
$$= 2q_{E(2)\#\overline{\mathbb{C}P}^2,4\ell+4}(\overline{e}^3,\sigma_3^{4\ell+1})$$
$$= 2^{4\ell+2}(3a_{4\ell+2} - a_{4\ell+4})$$

and

$$q_{W(3),4\ell+3}(\sigma_3^{4\ell+1},pt)\cdot q_{B(2,3,11),3}(\overline{e}^3) = (2)(2^{4\ell+2}c_{4\ell+1}) = 2^{4\ell+3}a_{4\ell}$$

so that $2a_{4\ell+2} = 3a_{4\ell+2} - a_{4\ell+4}$ and $a_{4\ell+2} = a_{4\ell+4}$. □

Now that we have a blow-up formula for $E(2)$, i.e. a computation for $q_{E(2)\#\overline{\mathbb{C}P}^2}$, and since blowing up is an operation performed at a single point, we have a blow-up formula for any manifold that, say, contains the fiber and section of $E(2)$, i.e. a so-called nucleus for $E(2)$. This configuration, denoted G_2, has boundary the homology sphere $\Sigma(2,3,11)$ which, as we have already used, has sparse Floer homology so that simple cut-and-paste

$q_{W(n)}$ of degree congruent to n mod 2. One checks that these relative invariants indeed have the boundary conditions dual to α_3 and α_7. Now inductively,

$$\begin{aligned} q_{E(n-1)\#\overline{\mathbb{C}P}^2} &= \exp(\frac{Q}{2})\sinh^{n-3}(F)\cosh(E) \\ &= \exp(\frac{Q}{2})[\sinh^{n-3}(F)\cosh(F)\cosh(\overline{E}) + \sinh^{n-3}(F)\sinh(F)\sinh(\overline{E})] \end{aligned}$$

so that the degree congruent to n relative invariants for $W(n)$ are given by

$$q_{W(n)} = \exp(\frac{Q_{W(n)}}{2})\sinh^{n-3}(F)\sinh(F)(\overline{\alpha}_3 + \overline{alpha}_7).$$

Hence

$$q_{E(n)} = q_{B(2,3,11)}q_{W(n)} = \exp(\frac{Q_B}{2})\exp(\frac{Q_{W(n)}}{2})\sinh^{n-2}(F) = \exp(\frac{Q_{E(n)}}{2})\sinh^{n-2}(F).$$

This inductive proof points out the utility of the multiplicative property of the Donaldson series. (To be PC I should at this point mention TQFT's).

The relative invariants for $C(2,3,11)$ involving the other parity of degrees are not required for this computation. Indeed, these relative invariants must be treated carefully, for their boundary conditions need not be irreducible. It is here that the fact that $C(2,3,11)$ is negative definite brings up the possibility of reducible connections. These do occur and play an important role in [FKS].

These arguments also show that if we consider $SO(3)$ Donaldson series we get the same answers if the bundles restrict trivially to fiber. However, if they do not, then let $c \in H^2(E(n);\mathbb{Z})$ be an integral lift of the Poincare dual of the second Stiefel-Whitney class of these bundles. Similar arguments show that

$$q_{E(n),c} = (-1)^{\frac{c^2+F\cdot c}{2}}\exp\left(\frac{Q_{E(n)}}{2}\right)\cosh^{n-2}(F)$$

Further, if the bundles restrict nontrivially to the exceptional curve in the blow-up and, say, trivially to F, then

$$q_{E(n)\#\overline{\mathbb{C}P}^2,c} = (-1)^{\frac{c^2+E\cdot c}{2}}\exp\left(\frac{Q_{E(n)\#\overline{\mathbb{C}P}^2}}{2}\right)\sinh^{n-2}(F)\sinh(E)$$

To conclude, we thank Tom Mrowka for giving us the courage to show that our Theorem 2.3 implies a blow-up formula under a weaker hypothesis, namely

Proposition 3.11. *If X has simple type, and if $q_X = \exp(\frac{Q_X}{2})C$, then*

$$q_{X\#\overline{\mathbb{C}P}^2} = \exp(\frac{Q_{X\#\overline{\mathbb{C}P}^2}}{2})C\cosh(E).$$

The *multiplicity* of the surgery is the absolute value of the degree of

$$\mathrm{pr}_{\partial D^2} \circ \varphi : \mathrm{pt} \times \partial D^2 \to \partial D^2.$$

Let $E(n)_\varphi$ denote the result of this operation on $E(n)$. Note that multiplicity 0 is a possibility. It follows from work of Gompf [G1, Prop.2.1], which uses a construction of Moishezon [Mn1], that if φ and φ' have the same multiplicity, there is a diffeomorphism, fixing the boundary, from $E(n)_\varphi$ to $E(n)_{\varphi'}$. Thus we may use the notation $E(n;p)$ to denote any $E(n)_\varphi$ where the multiplicity of φ is p.

In $E(n;p)$ there is again a copy of the fiber F, but there is also a new torus fiber, the *multiple fiber*. We shall denote its homology class by f_p; so in $H_2(E(n;p);\mathbf{Z})$ we have $f = p\, f_p$. We can continue this process on other fibers, but to insure that the resulting manifold is simply-connected we can take at most two log-transforms with multiplicities that are pairwise relatively prime. So, if we take two log-transforms of order p and q respectively, denote the result by $E(n;p,q)$.

The homology class f of the fiber of $E(n)$ can be represented by an immersed sphere with one positive double point. Blow-up this double point and take the proper transform of f so that the class $f - 2e_1$ (where e_1 is the homology class of the exceptional divisor) is represented by an embedded sphere with square -4. This is just the configuration $B(2)$. Now the exceptional divisor intersects this sphere in two positive points. Blow-up one of these points and take proper transforms. There results the homology classes $u_0 = f - 2e_1 - e_2$ and $u_1 = e_1 - e_2$ which is just the configuration $B(3)$. Continuing in this fashion we see that $B(p)$ naturally embeds in $E(n)\#_{p-1}\overline{\mathbf{C}P}^2$. The key observation is:

Proposition 4.1. *$E(n;p)$ is obtained from $E(n)\#_{p-1}\overline{\mathbf{C}P}^2$ by rationally blowing down $B(p)$.*

Proof. To see that $E(n;p)$ results from blowing down $B(p)$, we offer the sequence of Kirby calculus moves in Figure 6 and Figure 7. For this recall that the neighborhood of the immersed sphere with one positive double point representing f is just a neighborhood of a cusp fiber (see [FS6]). This neighborhood is just 0-framed surgery on the right-handed trefoil knot. The first picture is then a handlebody picture for the cusp neighborhood blown-up $p-1$ times with the configuration $B(p)$ represented by the homology classes (which are represented by spheres) $u_0 = f - 2e_1 - e_2 - \cdots - e_{p-1}$, $u_1 = e_1 - e_2$, $u_2 = e_2 - e_3$, $\ldots$, $u_{p-1} = e_{p-2} - e_{p-1}$. In the second picture we add the handle from Figure 5 and then blow down, keeping track of the dual 2-handle which is labelled with 0-framing. If in the final picture one replaces the handle with a dot on it by a 1-handle, there results the handlebody picture for the log-transformed cusp neighborhood (cf. [G1]). $\square$

FIGURE 7. Logarithmic Transform=Rational Blowdown

We now easily compute $q_{E(2;2)}$ on $X \setminus B(2)$.

$$\begin{aligned} q_{E(2;2)} &= q_{E(2)\#\overline{\mathbb{C}P}^2} - q_{E(2)\#\overline{\mathbb{C}P}^2, f-2e} \\ &= \exp(\frac{Q}{2})\cosh(E) - (-1)^{\frac{(f-2e)^2 + E\cdot(f-2e)}{2}} \exp(\frac{Q}{2})\cosh(E) \\ &= 2\exp(\frac{Q}{2})\cosh(E). \end{aligned}$$

While E is not a cohomolgy class in $E(2;2)$, the appropriate classe is $E + \frac{1}{2}(F - 2E) = \frac{F}{2}$ (since we can add multiples of classes in the neighborhood of the -4 sphere), i.e. the multiple fiber. So

$$q_{E(2;2)} = 2\exp(\frac{Q}{2})\cosh(\frac{F}{2}) = \exp(\frac{Q}{2})\frac{\sinh(F)}{\sinh(\frac{F}{2})}$$

4.3. Recursion Relations as O.D.E.'s for q_X. To compute $q_{X(p)}$ for large p requires an important preliminary observation about the Donaldson series. If $S \in H_2(X;\mathbb{Z})$, then the interior product $\iota_S(q_{X,d}) \in Sym^{d-1}(H_2(X))$ given by

$$(\iota_S(q_{X,d}))(x_1, \ldots, x_{d-1}) = d \cdot q_{X,d}(x_1, \ldots, x_{d-1}, S)$$

is a derivation. Further

$$\frac{(\iota_S(q_{X,d}))(\cdot)}{d!} = \frac{q_{X,d}(\cdot, S)}{(d-1)!}$$

so that ι_S acts as a derivation on the Donaldson series. We then define

$$\frac{\partial}{\partial S} q_X = \sum_d \frac{\iota_S(q_{X,d})}{d!}$$

An easy induction argument shows that

$$\frac{\partial^{2k}}{\partial S^{2k}} \exp(\frac{Q}{2}) = \exp(\frac{Q}{2}) \sum_{t=0}^{k} Q^{k-t}(S,S)\tilde{S}^{2t}\binom{2k}{2t}\frac{(2k-2t)!}{2^{k-t}(k-t)!} \tag{5}$$

and

$$\frac{\partial^{2k+1}}{\partial S^{2k+1}} \exp(\frac{Q}{2}) = \exp(\frac{Q}{2}) \sum_{t=0}^{k} Q^{k-t}(S,S)\tilde{S}^{2t+1}\binom{2k+1}{2t+1}\frac{(2k-2t)!}{2^{k-t}(k-t)!} \tag{6}$$

where $\tilde{S}$ is the Hom dual of S. Also

$$\frac{\partial}{\partial S}\cosh(\kappa) = \sinh(\kappa)(S \cdot \kappa)$$

and

$$\frac{\partial}{\partial S}\sinh(\kappa) = \cosh(\kappa)(S \cdot \kappa).$$

These are indispensable formula for all of our later work.

$$+(E_2\cdot v)(E_4\cdot w)+(E_4\cdot v)(E_1\cdot w)+(E_4\cdot v)(E_4\cdot w)]\sinh^2(u)\cosh^2(u)$$
$$-\frac{1}{2}(E_2\cdot v)(E_1\cdot w)\cosh^4(u)-\frac{1}{2}(E_4\cdot v)(E_3\cdot w)\sinh^4(u)\}$$
$$=\exp(\frac{Q}{2})\{\cosh^4(u)+[2\cosh^4(u)+3\sinh^2(u)\cosh^2(u)]$$
$$+\frac{1}{2}[4\cosh^4(u)+14\sinh^2(u)\cosh^2(u)+2\sinh^4(u)]\}$$
$$=\exp(\frac{Q}{2})[5\cosh^4(u)+10\sinh^2(u)\cosh^2(u)+\sinh^4(u)]$$
$$=\exp(\frac{Q}{2})[2\cosh(4u)+2\cosh(2u)+1]=\exp(\frac{Q}{2})\frac{\sinh(5u)}{\sinh(u)}.$$

Since $u\cdot\frac{f}{5}=1$, $u\cdot\tau=-1$, and cosh is an even function, we then have

$$q_{E(2;5)}=\exp(\frac{Q}{2})\frac{\sinh(F)}{\sinh(\frac{F}{5})}$$

The computation of the Donaldson series for $E(n;p,q)$ is now a matter of careful differentiation, using Theorem 2.2 to determine which derivatives to take. We refer the listener to [FS8] for the detailed combinatorics.

4.5. Horikowa Surfaces. To further illustrate the power of Theorem 2.2 we compute a few more examples. Here the goal is to find, for $n\geq 4$, a pair of disjoint $B(n-2)$ in $E(n)$ with the u_0 as sections of $E(n)$ and for $j>0$, $u_j\in f^{\perp}$. If we let $E_{n,1}$ be $E(n)$ with one $B(n-2)$ blown down, an application of Theorem 2.2 shows that if n is odd, then

$$q_{E_{n,1}}=\exp(\frac{Q}{2})\sinh(\kappa)$$

and if n is even, then

$$q_{E_{n,1}}=\exp(\frac{Q}{2})\cosh(\kappa)$$

with $\kappa^2=n-3$. If $E_{n,1}$ were an algebraic surface, this computation shows that it is minimal. However, it is easy to check that this violates the Noether inequality, so the $E_{n,1}$ are not homeomorphic to any complex surface.

Let $E_{n,2}$ be $E(n)$ with both $B(n-2)$ blown down. Another application of Theorem 2.2 shows that if n is odd, then

$$q_{E_{n,2}}=2^{n-3}\exp(\frac{Q}{2})\sinh(\kappa)$$

and if n is even, then

$$q_{E_{n,2}}=2^{n-3}\exp(\frac{Q}{2})\cosh(\kappa)$$

with $\kappa^2=2n-6$. However, from our construction it will be clear that $E_{n,2}$ is in fact a Horikowa surface. Note this generalizes our earlier computations where $n=4$.

(3.10, 3.11) state that $q_{X\#\overline{\mathbb{C}P}^2} = \exp\left(\frac{Q_{X\#\overline{\mathbb{C}P}^2}}{2}\right) C\cosh(E)$. Let $x_1, \ldots x_b \in H_2(X;\mathbb{Z})$ be a basis so that each x_t can be represented by an immersed sphere with p_t positive double points and n_t negative double points (if X is simply-connected any class can be so represented). Choose $p,n \in \mathbb{Z}$ so that $p \geq b$ and $p \geq p_t$, $n \geq n_t$ for all $1 \leq t \leq b$. Then in $\hat{X} = X\#_{b(p+n)}\overline{\mathbb{C}P}^2$ the classes $\hat{x}_t = x - 2e_{t,1} - 2e_{t,2} - \cdots - 2e_{t,p}$, $1 \leq t \leq b$, can be represented by embedded spheres with $Q(\hat{x}_t, \hat{x}_t) = Q(x_t, x_t) - 4p = r(x_t, p)$, which is negative if $q_X \neq 0$. Now apply Theorem 2.3 to these classes. Using Section 4.3, these recurrence relations for $q_{\hat{X}}$ (valid for classes orthogonal to the $\hat{x}_t$) translates, if $r(x_t, p) = -(2k+1)$, to the O.D.E.

$$(7)\quad \frac{\partial^{2k-1}}{\partial \hat{x}_t^{2k-1}} q_{\hat{X}} = \{A_{1,k}\frac{\partial^{2k-3}}{\partial \hat{x}_t^{2k-3}} + \cdots + A_{t,k}\frac{\partial^{2k-(2t+1)}}{\partial \hat{x}_t^{2k-(2t+1)}} + \cdots A_{k-1,k}\frac{\partial}{\partial \hat{x}_t}\}q_{\hat{X}} + A_{k,k}q_{\hat{X},\hat{x}_t}$$

and if $r(x,p) = 2k$, to the O.D.E.

$$(8)\quad \frac{\partial^{2k}}{\partial \hat{x}_t^{2k}} q_{\hat{X}} = \{A'_{1,k}\frac{\partial^{2k-2}}{\partial \hat{x}_t^{2k-2}} + \cdots + A'_{r,k}\frac{\partial^{2(k-r)}}{\partial \hat{x}_t^{2(k-r)}} + \cdots A'_{k-1,k}\frac{\partial^2}{\partial \hat{x}_t^2}\}q_{\hat{X}} + A'_{k,k}q_{\hat{X},\hat{x}_t}.$$

It is important to note that these coefficients only depend upon k, and hence they only depend upon $Q(x_t, x_t)$ and p. The differentiation formulas (5) and (6) for $\exp(\frac{Q}{2})$ now imply that if we write $q_{\hat{X}} = \exp\left(\frac{Q_{\hat{X}}}{2}\right)\hat{C}$, then $\hat{C}$ also satisfies constant coefficient O.D.E.'s which, if $r(x_t, p) = -(2k+1)$, are given by

$$(9)\quad \frac{\partial^{2k-1}}{\partial \hat{x}_t^{2k-1}}\hat{C} = \{a_{1,k}\frac{\partial^{2k-3}}{\partial \hat{x}_t^{2k-3}} + \cdots + a_{r,k}\frac{\partial^{2k-(2r+1)}}{\partial \hat{x}_t^{2k-(2r+1)}} + \cdots a_{k-1,k}\frac{\partial}{\partial \hat{x}_t}\}\hat{C} + a_{k,k}\hat{D}_{\hat{x}_t}$$

and if $r(x_t, p) = 2k$, are given by:

$$(10)\quad \frac{\partial^{2k}}{\partial \hat{x}_t^{2k}}\hat{C} = \{a'_{1,k}\frac{\partial^{2k-2}}{\partial \hat{x}_t^{2k-2}} + \cdots + a'_{r,k}\frac{\partial^{2(k-r)}}{\partial \hat{x}_t^{2(k-r)}} + \cdots a'_{k-1,k}\frac{\partial^2}{\partial \hat{x}_t^2} + a'_{k,k}\}\hat{C} + a'_{k+1,k}\hat{D}_{\hat{x}_t}.$$

There are two difficulties. First, these are not constant coefficient O.D.E.'s because of the appearance of the $\hat{D}_{\hat{x}_t}$; second these O.D.E's are only valid for classes orthogonal to the $\hat{x}_t$. To remedy both of these problems we blow-up one more time. For $1 \leq t \leq b$, let $\acute{x}_t = \hat{x}_t - e$ in $\hat{X}\#\overline{\mathbb{C}P}^2$. These classes are again represented by spheres with self-intersection one less than before and for which we have O.D.E's as above. Here is where we first explicitly utilize a blowup formula. This formula (Proposition 3.11) shows that the new C is $\hat{C}\cosh(E)$ and the new D is $\hat{D}\sinh(E)$. Now differentiate (using the fact that $\frac{\partial}{\partial(x-e)} = \frac{\partial}{\partial x} - \frac{\partial}{\partial e}$) and equate the coefficients of $\cosh(E)$ to obtain new O.D.E.'s for the original $\hat{C}$ that do not involve $\hat{D}$ and are valid on all of $\hat{C}$ (since $\hat{x}_t + Q(\hat{x}_t, \hat{x}_l)e \in <\hat{x}_l>^\perp$).

A similar trick (i.e. using the blow-up formula, differentiating, and equating the coefficients of the various products of the $\cosh(E_{t,j})$) shows that for each basis vector x_t there is a constant coefficient O.D.E of degree $r(x_t, p)+1$, if $r(x_t, p)$ is odd, and of degree $r(x_t, p) - 1$ if $r(x_t, p)$ is even. Further, the coefficients only depend upon $r(x_t, p)$; in particular they

$$-\sum_{r=1}^{k-1} a_{r,k} \sum_{j=1}^{n} b_j \sinh(K_j)(K_j \cdot [S])^{2k-(2r+1)} - a_{k,k} \sum_{j=1}^{n} \pm b_j \sinh(K_j)$$

on $[S]^{\perp}$. (This uses the fact that $q_{X,[S]} = \exp(\frac{Q_X}{2}) \sum_{j=1}^{n} \pm b_j \sinh(K_j)$.) Unless these equations are trivial on $[S]^{\perp}$ and $K_j \cdot [S] \neq 0$ for some j, then we can factor out the $b_j \sinh(K_j)$ and have that the $K_j \cdot [S]$ are characteristic roots for the ODE (9). Since these are universal ODE's, by plugging the Donaldson series for $E(2k+1)$ into (9), we see that these characteristic roots all satisfy the inequality

$$-2 \geq Q([S],[S]) + max_s\{|K_s \cdot [S]|\}.$$

More generally, suppose $x \in H_2(X;\mathbf{Z})$ is represented by an immersed sphere with p positive double points and n negative double points. Then, in $\hat{X} = X\#_p\overline{\mathbb{C}P}^2\#_n\overline{\mathbb{C}P}^2$ the homology class $x = x - 2e_1 - \cdots - 2e_p$ is represented by a sphere S with $Q([S],[S]) = Q(x,x) - 4p$, which is negative if $q_X \neq 0$. Now the blowup formula Proposition 3.11 implies that the basic classes $\hat{K}_r$ for $\hat{X}$ are of the form $K_j \pm e_1 \cdots \pm e_{p+n}$. The simple inequality above asserts that

$$-2 \geq Q([S],[S]) + max_s\{|\hat{K}_s \cdot [S]|\},$$

i.e

$$-2 \geq Q([S],[S]) + max_j\{|(K_j \pm e_1 \cdots \pm e_{p+n}) \cdot (x - 2e_1 - \cdots - 2e_p)|\}.$$

This also satisfies the inequality

$$2p - 2 \geq Q([S],[S]) + max_j\{K_j \cdot [S]\}.$$

The exceptions, i.e. when the equations are trivial on $[S]^{\perp}$ and $K_j \cdot [S] \neq 0$ for some j, are handled by the exceptions in the statement of Theorem 1.12.

5.2. The unknotting number of torus knots. Let $T(p,q)$ denote the (p,q) torus knot. Consider the Brieskorn homology sphere $\Sigma(p,q,2pq-1)$ which bounds the Milnor fiber $B(p,q,2pq-1)$. Further, $-\Sigma(p,q,2pq-1)$ bounds the even manifold $S(p,q,2pq-1)$ with intersection form $\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$ with respect to the basis of 2-handles obtained by doing a 0-framed surgery on $T(p,q)$ and then -2-surgery on its meridian. Let H_0 denote the class with 0 self-intersection and let S denote the class with self-intersection -2. Since S is represented by a 2-sphere, $X(p,q,2pq-1) = B(p,q,2pq-1) \cup S(p,q,2pq-1)$ is an algebraic surface with large diffeomorphism group with respect to its canonical class κ (whose homology class resides in $S(p,q,2pq-1)$). Again, since S is represented by a 2-sphere, $\kappa = 2aH_0 + aS$ for some $a \in \mathbf{Z}$. Further,

$$e(X) = (p-1)(q-1)(2pq-2) + 4$$

facilitates these evluations of the Donaldson polynomials at points of X. Although the resulting coefficients are not initially computed, since they are universal, we can use our computations for the $E(n)$ to compute them. First some preliminaries.

Let P be a principal $SU(2)$ or $SO(3)$ bundle over X. Since we need to alternate between the cases of $SU(2)$ and $SO(3)$ bundles, it will be convenient throughout to use $SO(3)$ as structure group and to identify an $SU(2)$ bundle with its associated (adjoint) $SO(3)$ bundle with $w_2 = 0$. This causes no loss in generality. Suppose that the moduli space $\mathcal{M}_X(P)$ of anti-self-dual connections on P has dimension

$$\dim \mathcal{M}_X(P) = -2p_1(P) - 3(1 + b_X^+) = 2d.$$

Then there is a Donaldson invariant $q_{X,P} \in \mathrm{Sym}^d_{\mathbf{Z}}(H_2(X;\mathbf{Z}))$. It is obtained from a homomorphism $\mu : H_2(X;\mathbf{Z}) \to H^2(\mathcal{B}^*_X(P);\mathbf{Z})$ which is described in [D6]. ($\mathcal{B}^*_X(P)$ is the space of gauge equivalence classes of irreducible connections on P.) For each choice of an oriented surface Σ_i representing the homology class $z_i \in H_2(X;\mathbf{Z})$, there is a codimension 2 submanifold V_i of $\mathcal{B}^*_X(P)$ which is a cocycle representative of $\mu(z_i)$. These cocycle representatives can be chosen so that the intersection

$$\mathcal{M}_X(P) \cap V_1 \cap \cdots \cap V_d$$

is transverse (hence 0-dimensional), compact, and oriented. Donaldson's polynomial invariant $q_{X,P}(z_1, \ldots, z_d)$ is the algebraic number of points in the above intersection, and depends (up to sign) only on the classes z_i and the diffeomorphism type of X. The sign is determined by the orientation of $\mathcal{M}_X(P)$, and this is in turn determined by the orientation of X and a choice of orientation on $H^2_+(X;\mathbf{R})$. (See [D5].)

Now suppose $X = Z \cup N$. Consider a collar neighborhood (neck) $\partial N \times [-1,1]$ in X, and suppose that we have a sequence of generic metrics $\{g_n\}$ on X which stretch the length of the neck to infinity and whose limit is the disjoint union of generic metrics on N_+ and Z_+. It follows from Uhlenbeck's weak compactness theorem [U] that any sequence $A_n \in \mathcal{M}_X(P, g_n) \cap V_1 \cap \cdots \cap V_d$ has a weak limit $A_N \amalg A_Z$ in the disjoint union of moduli spaces for N_+ and Z_+. Both A_N and A_Z limit asymptotically to flat connections on ∂N. (These limiting anti-self-dual connections need not necessarily have exponential decay.) In situations where the limit of $\{A_n\}$ is necessarily a strong limit, i.e. where there is no bubbling or loss of energy in a tube $\partial N \times \mathbf{R}$, this process can be reversed to obtain all of $\mathcal{M}_X(P, g_n) \cap V_1 \cap \cdots \cap V_d$ for large enough n. (See [Mr], [T], [MMR].) This is what we mean by the "Mayer-Vietoris Principle". Roughly, the Donaldson invariant of X can be computed as a sum of products $q_N[\lambda] \cdot q_Z[\lambda]$ of relative invariants where $[\lambda]$ runs over the character variety $\chi(T_0)$.

Suppose, in reverse, that we are trying to glue together anti-self-dual connections over Z and N to obtain anti-self-dual connections over X. A first problem which arises concerns where $e(B)$ is the Euler number of B ([APS1], [FS3]). Note that since $X = M \cup_L B$ then

$$\dim \mathcal{M}_X = \dim \mathcal{M}_B[\lambda] + h_\lambda(L) + \dim \mathcal{M}_M[\lambda].$$

To indicate some of the ideas that go into the proofs of Theorem 2.2 and Theorem 2.3 we conclude by proving the simplest version of Theorem 2.3, namely Rubermann's result;

Theorem 6.2. *Suppose that T is an embedded sphere with $Q([T],[T]) = -2$. Let $x_1, \dots x_d \in H_2(X;\mathbb{Z})$ be homology classes that are orthogonal to $[T]$. Then*

$$q_{X,d}(x_1, \dots x_{d-2}, [T], [T]) = 2q_{X,[T],d-2}(x_1, \dots x_d)$$

Proof. Write $X = Q \cup N$. Note that $\partial N = L(2,-1) = \mathbb{R}P^3$. Since $b^+(Z) > 0$, generically there are no reducible anti-self-dual (ASD) connections on Q. However, since $b^+(N) = 0$ there are indeed nontrivial reducible ASD connections in complex line bundles $\mathbb{L}^m$, $m \in \mathbb{Z}$, with $c_1(\mathbb{L}^m) = m$ represented by a harmonic 2-form and with $\mathbb{L}^m|_{\partial N}$ the flat line bundle with holonomy -1 on the meridian curve. In particular $\mathbb{L}$ is determined by the non-trivial representation $\lambda = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of $\pi_1(\partial N) = \mathbb{Z}_2$ into $SO(3)$ so that $ad(\mathbb{L}) = \mathbb{L} \oplus \overline{\mathbb{L}}$ is a trivial bundle. Thus $h_\lambda = h_\theta = 3$. Let $\mathbb{C} \cong \mathbb{L}^0$ be the trivial bundle. Note that for reducible connections on N, $c_1 \in H^2(N;\mathbb{Z}) \cong H_2(N;\partial N;\mathbb{Z}) \cong \mathbb{Z}$ and that $c_1(\mathbb{L}) = \gamma$ is a generator. Then $-4p_1(\mathbb{L}) = c_2(\mathbb{L} \oplus \overline{\mathbb{L}}) = -c_1^2(\mathbb{L}) = \frac{1}{2}$

Also, on Q

$$\dim \mathcal{M}_{c_Q}(Q) = 8c_Q - \frac{3}{2}(e(Q) + \sigma(Q)) - \frac{h_\alpha}{2} - \frac{\rho_\alpha}{2} = 8c_Q - 3(1 + b^+(X))$$

and on N

$$\dim \mathcal{M}_{c_N}(N) = 8c_N - \frac{3}{2}(e(N) + \sigma(N)) - \frac{h_\alpha}{2} + \frac{\rho_\alpha}{2} = 8c_N - 3.$$

If we have $A_i \in \mathcal{M}(X, g_i) \cap V_1 \cap \dots V_{d-2} \cap V_{T,1} \cap V_{T,2}$, then stretching the neck we get that in the Uhlenbeck limit $A_i \mapsto A_Q \amalg A_N + r$ bubbles on $Q + s$ bubbles on N, where $A_Q \in \mathcal{M}_Q$ $A_N \in \mathcal{M}_N$. Further $dim\mathcal{M}_Q + dim\mathcal{M}_N + 8r + 8s + 3 \leq 2d$.

The next step is a sequence of dimension counting arguments to determine r, s, and the irreducibility of A_Q and A_N.

If A_Q and A_N are both irreducible, then $dim\mathcal{M}_Q \geq 2(d-2-2r)$ and $dim\mathcal{M}_N \geq 2(2-2s)$ so that $(2d-4-4r) + (4-4s) + 8r + 8s + 3 \leq 2d$ which is impossible.

If A_Q is flat and A_N is irreducible, then $2r \geq d-2$ and $dim\mathcal{M}_Q = 0 - 3(1+b^+(Q))$ so that $-3(1+b^+(Q)) + (4-4s) + (4d-8) + 8s + 3 \leq 2d$; i.e. $2d - 3(1+b^+(X)) - 1 + 4s \leq 0$. But to guarantee compactness we need that $k > \frac{3}{4}(1+b^+(X)) + \frac{1}{2}$, so that $2d - 3(1+b^+(X)) > 4$. So we cannot have A_Q flat and A_N irreducible.

$c_1 = -2$ complex line bundle over S^2 and $\tilde{V}_{T,2}$ is just another section. Thus $\tilde{V}_{T,2} \cdot \Delta_p = -2$ and

$$q_{X,d}(x_1, \ldots x_d, [T], [T]) = -2 \cdot \mathcal{I}_Q.$$

This again is a general phenemena we will have in the proofs of our more general theorems. We will have that our original $q_{X,d}$ can be written as a product of a universal constant coming from a reducible computation on N times a relative invariant on Q. The final task is to identify this relative invariant with an absolute (but different) invariant of X. For the case at hand we will just attach the trivial connection to $\mathcal{M}_{k-\frac{1}{2}}(Q)$ and claim that we get a computation for $q_{X,[T],d-2}(x_1, \ldots x_{d-2})$.

To see this, we pass to $SO(3)$ bundles over Q and note that $\mathcal{M}_{k-\frac{1}{2},0}(Q) \cong \mathcal{M}_{k-\frac{1}{2}}(Q)$ and that the boundary value for these $SO(3)$-connections is trivial since $\mathrm{ad}\lambda = \vartheta$, the trivial connection. So glue on the trivial connection from N (there is no obstruction). This gives us an $SO(3)$-bundle over X with $w_2 \neq 0$ since there is a twist in the gluing (afterall we are ending with a bundle with fractional c_2). This w_2 is the unique non-trivial element $c \in H^2(X; \mathbb{Z}_2)$ which restrict trivially to both Q and N. The Poincare dual of w_2 is just the mod 2 reduction of $[T]$. Thus

$$q_{X,d}(x_1, \ldots x_{d-2}, [T], [T]) = \pm 2 q_{X,[T],d-2}(x_1, \ldots x_d).$$

Since this ± 2 is universal, a test on one example shows that it must be $+2$. $\square$

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